PHYSICAL REVIEW D VOLUME 27, NUMBER 2 15 JANUARY 1983

Loop expansion in massless three-dimensional QED

E. I. Guendelman and Z. M. Radulovic
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 2 September 1982)

It is shown how the loop expansion in massless three-dimensional QED can be made finite, up to three loops, by absorbing the infrared divergences in a gauge-fixing term. The same method removes leading and first subleading singularities to all orders of perturbation theory, and all singularities of the fermion self-energy to four loops.

I. INTRODUCTION

Three-dimensional field theories are of physical interest for finite-temperature four-dimensional field theories which are, in the limit of infinite temperature, governed by corresponding (Euclidean) three-dimensional theories.[1,5] Also, they are relevant by themselves as toy models (in 2 + 1 dimensions) for studying confinement.[2]

Theories that are renormalizable in four dimensions become superrenormalizable in three dimensions, where the coupling constant e^2 acquires the dimension of a mass. In the case of massless fields, as in gauge theories or scalar theories at their critical point, this fact is responsible for infrared (IR) divergences in the naive loop expansion of various amplitudes. That is, for dimensional reasons, the higher-order diagrams must involve a high power of momentum variable in the denominators of loop integrals which become divergent at certain orders.

The IR divergences can be avoided by a resummation of a perturbation series such as the $1/N$ expansion in three-dimensional QED with N fermion flavors. However, for $N=1$ it is inapplicable; in SU(N) gauge theories the leading term in $1/N$ involves all planar diagrams, and no summation techniques are known.

Here we study the loop expansion of two- and three-point functions with large external momenta in massless QED_3. The IR behavior of this theory is similar to the IR behavior of three-dimensional Yang-Mills theory because both theories contain charged massless particles. The latter are of immediate physical interest for high-temperature perturbative studies of the thermodynamics of Yang-Mills fields.[1,3,5]

We regularize the loop expansion using dimensional regularization.[7] The dimensionality of the coupling constant in D dimensions is fixed to be $\mu^{2-D/2}$, where μ has units of mass. The IR divergences of Feynman diagrams appear as poles in the complex D plane at $D=3$. Since there are no ultraviolet divergences, all the poles are of IR origin. We find that the loop expansion can be made finite, up to the three-loop level, by absorbing the IR singularities in a gauge-fixing term. We also show that in this way the leading and the first subleading singularities can be eliminated to all orders in the loop expansion. This procedure of elimination of singularities corresponds to a renormalization of a gauge parameter and as all renormalization procedures it gives a prescription-dependent finite part.

It is shown that up to four loops the fermion propagator can be made finite.

Previous studies of the IR problem in three-dimensional gauge theories have found that logarithms of the coupling constant appear in the two-point functions.[4-6] We find that in QED_3 logarithms of the coupling constant can appear in gauge-variant Green's functions. In the present approach, once the gauge fixing is specified completely, then to three loops everything is calculable and well defined in contrast to previous treatments,[4,8] where only leading and first subleading logarithms were calculable. Also by appropriate gauge fixing, the coupling constant, which may appear as a scale for a logarithm of the momentum variable in some gauges, can be replaced by another scale in other gauges, making the resulting loop expansion analytic in the coupling constant to three loops.

For higher loop orders leading and first subleading logarithms also contain a gauge-dependent scale.

Beyond three loops, unless unexpected cancellations occur, singularities that are not removable by a gauge transformation will appear and they will have to be canceled by introducing counterterms that will produce uncalculable analytic contributions and logarithms of momentum.

Section II of this paper explains the method of extracting the singularity from the two-loop fermion self-energy. Cancellation of singularities by a gauge transformation is the subject of Sec. III. Section IV shows the renormalization procedure for three-loop diagrams and discusses singularities at higher orders. Appendix A contains proof of the gauge-fixing procedure of Sec. III, and Appendix B shows the renormalization of the fermion self-energy at four loops.

II. ONE- AND TWO-LOOP DIAGRAMS

To regularize IR singular diagrams, we use dimensional regularization with the usual conventions,[7] except for the trace where we choose $\mathrm{Tr}1=2^{D/2-1/2}$ in order to have the correct limit at $D=3$, because γ matrices in three dimensions are 2×2 Pauli matrices. Final results are, of course, independent of continuation conventions [see discussion after Eq. (2.8)].

The theory is defined by

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\bar{\psi}\partial\!\!\!/\psi-e\bar{\psi}A\!\!\!/\psi\ ,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu \tag{2.1}$$

with the coupling constant

$$e=\mu^{2-D/2}=\mu^{(1+\eta)/2},\quad \eta=3-D\ . \tag{2.2}$$

In covariant gauges, with gauge parameter α, the photon and fermion propagators are

$$D_{\mu\nu}(p)=\frac{-iP_{\mu\nu}}{p^2+i\epsilon-\Pi(p^2)}-\frac{i\alpha p_\mu p_\nu}{(p^2+i\epsilon)^2}\ , \tag{2.3}$$

$$S(p)=\frac{i}{p\!\!\!/-\Sigma(p)}\ , \tag{2.4}$$

where

$$P_{\mu\nu}=g_{\mu\nu}-p_\mu p_\nu/p^2,\quad \Pi_{\mu\nu}=P_{\mu\nu}\Pi(p^2)\ .$$

One-loop self-energies, Fig. 1, are finite in the limit $D\to3$,

$$\Pi^{(1)}(p^2)=-p^2\left[\frac{\mu}{\sqrt{-p^2}}\right]^{1+\eta}\frac{\Gamma(\frac{1}{2}+\eta/2)\Gamma(\frac{3}{2}+\eta/2)}{(8\pi)^{1-\eta/2}\Gamma(2-\eta)} \tag{2.5a}$$

$$=-\frac{p^2}{16}\frac{\mu}{(-p^2-i\epsilon)^{1/2}}+O(\eta)\ , \tag{2.5b}$$

$$\Sigma^{(1)}(p)=-\alpha p\!\!\!/\left[\frac{\mu}{\sqrt{-p^2}}\right]^{1+\eta}\frac{\Gamma(\frac{1}{2}+\eta/2)\Gamma(\frac{1}{2}-\eta/2)\Gamma(\frac{3}{2}-\eta/2)}{(4\pi)^{3/2-\eta/2}\Gamma(1-\eta)} \tag{2.6a}$$

$$=-\frac{\alpha p\!\!\!/}{16}\frac{\mu}{(-p^2-i\epsilon)^{1/2}}+O(\eta)\ . \tag{2.6b}$$

It follows that the dimensionless expansion parameter is

$$\lambda(p)=\frac{\mu}{\sqrt{-p^2}} \tag{2.7}$$

and the convergent loop expansion exists for $\lambda(p)<1$, i.e., at large (Euclidean) momenta. By inserting (2.5) and (2.6) into higher-order diagrams and carrying out the loop integrations, the singularities will be encountered, in general, as a consequence of soft momenta flowing through subdiagrams which are themselves reliable only for large momenta.

The first divergence appears in the fermion self-energy at two loops. All diagrams that contribute to $\Sigma^{(2)}$ are shown in Fig. 2, but only Fig. 2(a) is divergent, and straightforward calculation gives

$$\Sigma^{2(a)}(p)=-i\mu^{1+\eta}\int\frac{d^Dq}{(2\pi)^D}\gamma^\mu\frac{1}{p\!\!\!/+q\!\!\!/}\gamma^\nu\times\frac{P_{\mu\nu}}{q^4}\Pi^{(1)}(q^2)\ , \tag{2.8a}$$

where $\Pi^{(1)}(q^2)$ is given by (2.5a),

$$\Sigma^{2(a)}(p)=\frac{p\!\!\!/}{96\pi^2}\left[\frac{\mu}{\sqrt{-p^2}}\right]^{2(1+\eta)}[\Gamma(-\eta)+\mathrm{const}] \tag{2.8b}$$

$$=-\frac{p\!\!\!/}{96\pi^2}\left[\frac{\mu}{\sqrt{-p^2}}\right]^2\left[\frac{1}{\eta}+2\ln\left[\frac{\mu}{\sqrt{-p^2}}\right]+C\right]+O(\eta)\ . \tag{2.8c}$$

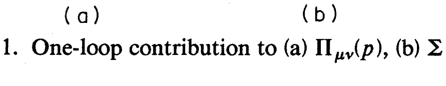

FIG. 1. One-loop contribution to (a) $\Pi_{\mu\nu}(p)$, (b) $\Sigma(p)$.

This result consists of three characteristic terms: the pole, the corresponding $\ln\lambda(p)$, and the last term C

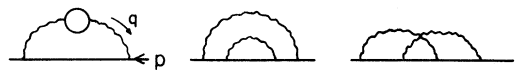

FIG. 2. Two-loop contributions to $\Sigma(p)$.

is a finite constant. First we notice that the $\ln\lambda(p)$ term has the same coefficient as has been previously found by resummation of the perturbation series.[4] It is clear from (2.8b) that the coefficients of $\ln\lambda(p)$ and the pole $1/\eta$ is fixed, while C depends on continuation conventions. In Sec. III, we show how the pole term can be subtracted by a gauge transformation that gives a finite part which redefines C, i.e., absorbs its continuation convention dependence. Thus the total two-loop contribution to Σ can be written in the form

$$\Sigma^{(2)}=\not p\lambda^2(p)[A^{(2)}(\ln\lambda(p)+C^{(2)})+F^{(2)}] , \quad (2.9)$$

where $A^{(2)}=-1/48\pi^2$, $F^{(2)}$ is a finite constant obtained by evaluation of finite diagrams Figs. 2(b) and 2(c), and $C^{(2)}$ is a gauge-dependent constant.

III. METHOD OF ELIMINATION OF DIVERGENCES

A. Definition of a gauge-fixing term

We are going to define a gauge transformation, starting from an α gauge as used in Sec. II, so that the infrared divergences can be eliminated from the loop expansion.

QED in an α gauge is defined by the following generating functional:

$$Z[J_\mu,\xi,\bar\xi,\alpha]=\int \mathscr{D}A_\mu\mathscr{D}\psi\,\mathscr{D}\bar\psi\, e^{iS(A_\mu,\psi,\bar\psi,J_\mu,\xi,\bar\xi,\alpha)} ,$$

$$S(A_\mu,\psi,\bar\psi,J_\mu,\xi,\bar\xi,\alpha)=\int d^Dx\left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\bar\psi\not\partial\psi-e\bar\psi\not A\psi-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2+J_\mu A^\mu+\bar\xi\psi+\bar\psi\xi\right] .$$

The gauge is still not completely specified by this generating functional because the functional integral needs some boundary conditions. We can fix those boundary conditions by canonically quantizing the Lagrangian

$$\mathscr{L}_{\rm eff}=\mathscr{L}-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2 ,$$

with the conditions

$$\langle 0|\psi|0\rangle=\langle 0|\bar\psi|0\rangle=\langle 0|A_\mu|0\rangle=0 . \quad (3.1)$$

For a more careful treatment of these conditions see Appendix A. Now we are interested in performing the following c-number gauge transformation starting from this gauge,

$$\psi(x)\to\psi^B_{(x)}=e^{-ig_0B^\mu x_\mu}\psi(x) ,$$
$$A_\mu(x)\to A^B_\mu(x)=A_\mu(x)+\frac{g_0}{e}B_\mu , \quad (3.2)$$

where B_μ is a dimensionless vector and g_0 is a constant with dimensions of mass. QED in this new gauge can be formulated in different but equivalent ways: One can keep the same form for $\mathscr{L}_{\rm eff}$ and quantize it canonically, but instead of the condition $\langle 0|A_\mu(x)|0\rangle=0$ one will have $\langle 0|A^B_\mu(x)|0\rangle=(g_0/e)B_\mu$, while the condition $\langle 0|\psi^B|0\rangle=\langle 0|\bar\psi^B|0\rangle=0$ holds.

Equivalently we can write $\mathscr{L}_{\rm eff}(A^B_\mu,\psi^B,\bar\psi^B)$ in terms of A_μ, $\bar\psi^B$, and ψ^B, dynamical variables which have zero vacuum expectation value, and are therefore more convenient to work with,

$$\mathscr{L}_{\rm eff}(A^B_\mu,\psi^B,\bar\psi^B)=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar\psi^B(i\not\partial-e\not A)\psi^B-g_0\bar\psi^B\not B\psi^B-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2 , \quad (3.3)$$

that is, we treat the $(g_0/e)B_\mu$ part of A^B_μ as an "external" field.

The generating functional with these dynamical variables is

$$Z[J_\mu,\xi,\bar\xi,\alpha,g_0B_\mu]=\int \mathscr{D}A_\mu\mathscr{D}\psi\,\mathscr{D}\bar\psi\, e^{iS(A_\mu,\psi,\bar\psi,J_\mu,\xi,\bar\xi,\alpha,g_0B_\mu)} ,$$

$$S(A_\mu,\psi,\bar\psi,J_\mu,\xi,\bar\xi,\alpha,g_0B_\mu)=\int d^Dx\left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar\psi(i\not\partial-e\not A)\psi-g_0\bar\psi\not B\psi-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2+J_\mu A^\mu+\bar\xi\psi+\bar\psi\xi\right] . \quad (3.4)$$

The index B in ψ^B and in $\bar\psi^B$ has been suppressed since in the functional integral they appear as dummy variables. This generating functional has to be complemented with the information that all dynamical variables used here have zero vacuum expectation value, and that defines completely our formalism.

When one calculates gauge-invariant amplitudes using $Z[J_\mu,\xi,\bar\xi,\alpha,g_0B_\mu]$, they will not depend on B_μ, and they will not be affected if, instead of using

$Z[J_\mu,\xi,\bar{\xi},\alpha,g_0B_\mu]$, we use

$$Z[J_\mu,\xi,\bar{\xi},\alpha,g_0]=\int d^DB\, e^{iB^2/2}Z[J_\mu,\xi,\bar{\xi},\alpha,g_0B_\mu] \tag{3.5}$$

[this follows closely the procedure used in defining the α gauges where an average over all possible $\partial_\mu A^\mu=f(x)$ gauges is taken].

We will show next how gauge-dependent quantities can be made finite by means of this gauge-fixing procedure.

From (3.5) we can read off the Feynman rules for the B photons. They are shown in Fig. 3(a). Notice that the B photons do not carry momentum and no additional loop integration has to be introduced when one diagram contains an internal B photon. In addition, we have photons described by the A_μ field that are quantized in the usual way (as used in Sec. II). One can see explicitly that the contribution of these zero-momentum photons to gauge-invariant amplitudes is zero, which is consistent with the fact that they represent the effect of a gauge transformation.

B. Cancellation of Singularities

In order to cancel the divergences of the two-loop fermion propagator it is enough that the diagrams in Fig. 3(b) cancel their divergences.

This can be achieved if [see Eq. (2.8c)]

$$g_0{}^2=\frac{\mu^2}{96\pi^2}\frac{1}{\eta}+g^2\ , \tag{3.6}$$

where g^2 is some finite, gauge-dependent constant. Now the renormalization prescription amounts to fixing the additional gauge freedom, i.e., choosing g^2, or choosing $C^{(2)}$ in Eq. (2.9) where $C^{(2)}$ contains both finite parts of the diagram and the counterterm.

The constant $C^{(2)}$ can be chosen to modify the argument of $\ln\lambda(p)$ by writing it as $C^{(2)}=\ln(\Lambda/\mu)$. Therefore the scale that appears in the logarithm of

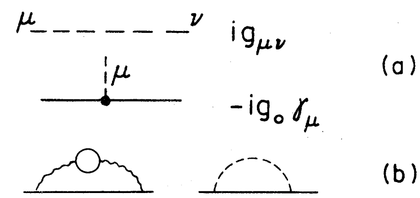

FIG. 3. (a) Feynman rules for B photons. (b) Divergent two-loop diagram and its corresponding counterterm.

p^2 is completely arbitrary and independent of μ, because Λ is a gauge-dependent quantity. Therefore, with appropriate gauge fixing, there are no nonanalytic terms of the coupling constant. The same consideration holds for all contributions up to three loops and for all the leading and the first subleading singularities (see Sec. IV).

Notice that the contributions to the fermion self-energy illustrated in Fig. 4 are identically zero in the dimensional regularization method. Because B photons carry no momentum, we have

$$\propto \int \frac{d^Dp}{p^2}=0\ .$$

The cancellation of singularities in higher loop orders and for other n-point functions is the subject of Sec. IV.

IV. HIGHER LOOP ORDERS AND OTHER N-POINT FUNCTIONS

A. Three-Loop Fermion Self-energy

To cancel the infrared singularities of the fermion self-energy at three loops, one has to add diagrams containing B photons. The diagrams that contribute to $\Sigma^{(3)}$ are shown in Fig. 5 together with their counterterms. As an example, we evaluate the diagrams in Figs. 5(b) and 5(b′),

$$\Sigma^{5(\mathrm{b})}+\Sigma^{5(\mathrm{b}')}=-i\mu^{1+\eta}\int\frac{d^Dk}{(2\pi)^D}\gamma^\mu\frac{1}{\not p+\not k}\Sigma_R^{2(\mathrm{a})}(p+k)\frac{1}{\not p+\not k}\gamma^\nu\frac{g_{\mu\nu}-(1-\alpha)k_\mu k_\nu/k^2}{k^2}\ , \tag{4.1}$$

where

$$\Sigma_R^{2(\mathrm{a})}(p+k)=\Sigma^{2(\mathrm{a})}(p+k)+\frac{\not p+\not k}{96\pi^2}\lambda^2(p+k)\frac{1}{\eta} \tag{4.2}$$

and $\Sigma^{2(\mathrm{a})}(p)$ is given by (2.8b). Evaluation of loop and parametric integrals yields

$$\Sigma^{5(\mathrm{b})}+\Sigma^{5(\mathrm{b}')}=-\frac{\alpha\not p}{16\times 96\pi^2}\lambda^{3+\eta}(p)\left[\lambda^{2\eta}(p)\Gamma(-\eta)+\frac{1}{\eta}\right] \tag{4.3a}$$

$$=\frac{\alpha\not p}{16\times 48\pi^2}\lambda^3(p)\left[\ln\lambda(p)+C^{(2)}\right]+O(\eta)\ . \tag{4.3b}$$

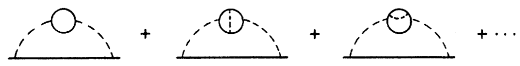

FIG. 4. Vanishing diagrams in the dimensional regularization method.

Hence $\Sigma^{5(b)}+\Sigma^{5(b')}$ is finite and $C^{(2)}$ is the same as in (2.9). The remaining three diagrams, Figs. 5(c), 5(d), and 5(e), have similar terms; thus, up to three loops the fermion self-energy is finite.

Total contribution to $\Sigma^{(3)}$ has the form

$$\Sigma^{(3)}=\alpha \not{p} \lambda^3(p)\left[A^{(3)}\left[\ln\lambda(p)+C^{(2)}\right]+F^{(3)}\right], \tag{4.4}$$

where $A^{(3)}$ and $F^{(3)}$ are finite constants, $F^{(3)}$ stands for finite diagrams.

B. Vacuum Polarization

Vacuum polarization at two loops is finite and at three loops there are two IR divergent diagrams (Fig. 6), but the divergences cancel in the sum[4]; so the vacuum polarization is finite up to this order. This cancellation is a consequence of gauge invariance and can be understood as follows. The IR divergences in both diagrams of Fig. 6 are produced by zero-momentum flow through the photon line with the bubble insertion, and these divergences can be canceled by B photons individually. In the sum the B photons must cancel by gauge invariance; hence the divergences must cancel.

Since logarithms of momentum associated with these singularities (that cancel) must contain a gauge-dependent scale, it follows that these logarithms do not appear in gauge-invariant amplitudes like the vacuum polarization.

This is in agreement with the second paper of Ref. 8, where it is shown that contributions of the leading-logarithm diagrams to $\Pi_{\mu\nu}$ vanish when summed to all orders.

C. Other N-Point Functions

Arguments based on gauge invariance strongly suggest that if one has been able to make finite the fermion propagator up to three loops, then up to the same loop order all other n-point functions should be finite in this gauge.

For example, the Ward identity shows that the vertex function with zero external photon momentum can also be made finite with the same gauge transformation.

The cancellation of infrared divergences by a gauge transformation can be seen in all other n-point functions and everything works as in the case of the fermion propagator.

In the case of the vertex function, up to two loops the only divergent diagram is shown in Fig. 7(a).

From this diagram we can extract its infrared-singular part easily by realizing that the infrared behavior is not changed by making the replacement $1/(\not{p}'+\not{k})\to 1/\not{p}'$ [or, if one chooses, the replacement $1/(\not{p}+\not{k})\to 1/\not{p}$]. This just follows closely the treatment for the case of the fermion propagator, and in this case one can see that the diagram that cancels the infrared divergence of the graph in Fig. 7(a) is the one shown in Fig. 7(a').

The cancellation of the singularities up to three loops can be shown using similar techniques. For example, in Fig. 8(a) we can replace the part enclosed in a square, which is equal to

$$\frac{1}{\not{p}'+\not{k}}\Gamma_\mu^{(1)}(p'+k,p+k)\frac{1}{\not{p}+\not{k}}$$

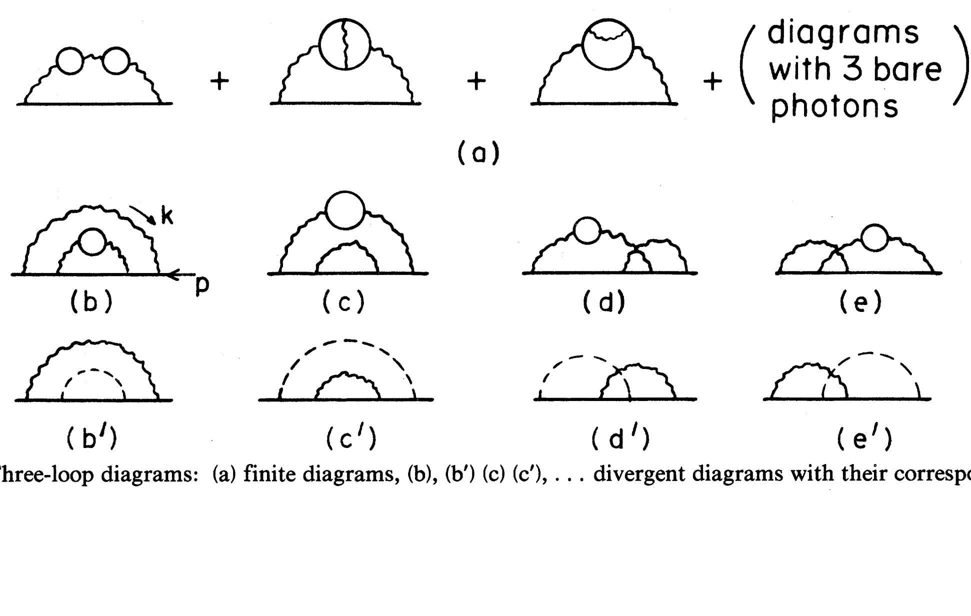

FIG. 5. Three-loop diagrams: (a) finite diagrams, (b), (b′) (c) (c′), . . . divergent diagrams with their corresponding counterterms.

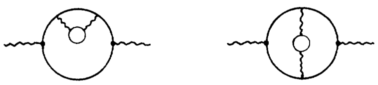

FIG. 6. Three-loop contributions to $\Pi_{\mu\nu}$ with mutually canceling infrared divergences.

by an expression that keeps the same infrared behavior for Fig. 8(a)—for example,

$$\frac{1}{\not p'+\not k}\Gamma_\mu^{(1)}(p',p)\frac{1}{\not p} .$$

After extracting the singular part of Fig. 8(a), it is easy to see that it is canceled by Fig. 8(a'). Other diagrams are treated in a similar way; for example, one can check that Fig. 8(b') cancels the singularity of Fig. 8(b).

D. Singularities at higher loop orders

1. The leading and the first subleading diagrams of Σ

We now show how the cancellation of the leading and the first subleading singularities works to all orders for the fermion self-energy Σ. Similar analysis can be carried out for other n-point functions.

The diagrams with the highest power of the logarithm at a given loop order we call the leading diagrams; similarly, the first subleading (1SL) diagrams have one loop more than the leading counterparts for the same power of the logarithm. The leading diagrams contain leading singularities, before renormalization, and each singularity contributes a logarithm. Previously considered diagrams [Fig. 2(a) and Figs. 5(b), 5(c), 5(d), and 5(e)] are the lowest-order examples of the leading and the 1SL diagrams, respectively.

Now we extend the analysis to arbitrary loop orders where one can easily prove, by induction, the following classification of singularities. The leading diagrams of Σ at $2n$ loop order can be of two types.

(i) N-photon lines, with the bubble insertion, attached to the bare fermion line; e.g., for $n=2$ such diagrams are shown in Figs. 9(a) and 9(b). Each photon line contributes a singularity, hence logarithm, from the soft-photon region.

(ii) Bare photon attached to a fermion line where

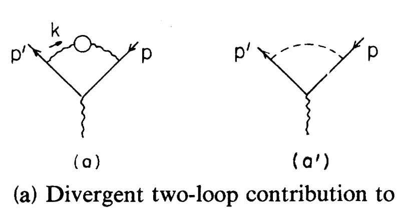

FIG. 7. (a) Divergent two-loop contribution to the vertex function. (a') The corresponding counterterm.

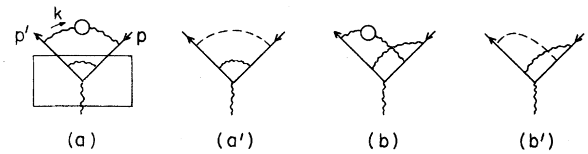

FIG. 8. (a), (b) Divergent three-loop contributions to the vertex function. (a'), (b') Their corresponding counterterms.

the only dressing occurs in the internal fermion line with a $(2n-1)$-loop 1SL diagram; e.g., for $n=2$, Fig. 9(c). One singularity is produced by the soft momentum flow through the fermion line and the remaining $(n-1)$ singularities come from the subdiagram. Diagrams of this type are proportional to gauge parameter α.

The 1SL diagrams of Σ appear at odd loop orders and they are related to the leading diagrams at preceding, even, orders. All the 1SL diagrams at $(2n+1)$ order are obtained by attaching a bare photon to the fermion line of the leading $2n$-loop diagram; they are proportional to gauge parameter α.

Now we choose the $\alpha=0$ gauge in which both 1SL and the type (ii) leading diagrams vanish. Therefore, all the leading diagrams of Σ are of the type (i) and they can be renormalized by B photons order by order [see the four-loop example in Figs. 9(a), 9(a'), 9(a''), and 9(a''')] and the finite part of the $2n$ loop order becomes

$$\Sigma_L^{(2n)}=A_L^{(2n)}\not p\lambda^{2n}(p)[\ln\lambda(p)+C^{(2)}]^n , \qquad (4.5)$$

where $A_L^{(2n)}$ is a constant. Explicit evaluation of $A_L^{(2n)}$ [for $n=1$, see (2.9)] is in agreement with Ref. 8. However, in contrast to Refs. 4 and 8, we find that the logarithms do not have to contain the coupling constant because the scale of the logarithms is gauge dependent (as explained for $n=1$ in Sec. III B).

2. Higher subleading and gauge-invariant singularities

Singularities of Σ discussed so far are gauge effects, but this need not be the case for higher subleading singularities. The second subleading (2SL) diagrams at $2n$-loop order have one power of the logarithm less than the leading diagrams at the same order, thus give the term proportional to

$$\lambda^{2n}(p)[\ln\lambda(p)+\text{const}]^{n-1} ;$$

for an example of 2SL, see Appendix B. The second and the higher subleading diagrams of Σ may involve a new type of singularities not removable by a gauge transformation, e.g., if gauge-invariant vacuum polarization develops a singularity it will affect Σ.

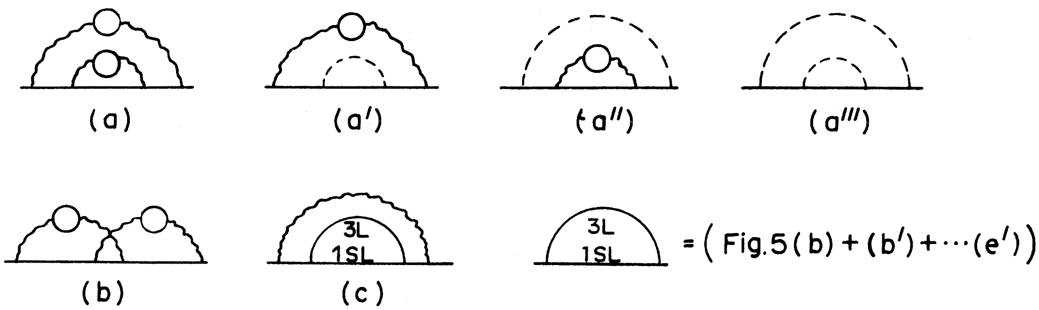

FIG. 9. Leading-logarithm diagrams at four loops. 3L: three-loop contributions, 1SL: first subleading contributions. (a′), (a″), (a‴) are counterterms of (a).

Singularities in the vacuum polarization may appear at four or higher loop orders and gauge-invariant counterterms would be necessary. In this case one would be able to calculate only the leading logarithms (leading in the vacuum polarization), at each loop order, leaving remaining terms uncalculable.

V. CONCLUSIONS

We have been able to eliminate all the leading and the first subleading singularities to all orders and all singularities up to three loops in the loop expansion; to four loops for the fermion self-energy (Appendix B).

In the present approach, after these singularities have been removed by a gauge transformation, we are left with logarithms of the kinematical variables divided by a gauge-dependent scale. This gauge-dependent scale appears, rather than nonanalytic terms in the coupling constant, up to three loops and for all the leading and the first subleading logarithms if the scale is not chosen to be e^2. Therefore, both nonanalytic and uncalculable terms found in Refs. 4 and 8 are a consequence of working in an α gauge with standard boundary conditions, and do not have gauge-invariant meaning.

ACKNOWLEDGMENTS

We thank A. Guth, R. Jackiw, and S. Templeton for many useful discussions and criticisms. This work was supported in part through funds provided by the U.S. Department of Energy under Contract No. DE-AC02-76ER03069.

APPENDIX A: BOUNDARY CONDITIONS

The following procedure, due to Professor A. Guth, shows more carefully the origin of the additional conditions defined by Eqs. (3.1) and (3.5). The argument will be shown for three dimensions; however, it is immediately generalizable to any integer number of dimensions and by analytic continuation to any number of dimensions.

Let us assume that our field variables satisfy periodic boundary conditions, so that they only need to be specified in a box defined by

$$-\frac{L}{2}\le x_\mu \le \frac{L}{2}\ . \tag{A1}$$

The generating functional in an α gauge is given by

$$Z=\int \mathscr{D}A_\mu \mathscr{D}\psi\, \mathscr{D}\bar{\psi}\, e^{i\int d^3x \mathscr{L}_{\rm eff}} \tag{A2a}$$

$$=\int \mathscr{D}A_\mu \mathscr{D}\psi\, \mathscr{D}\bar{\psi} \exp\left[i\int d^3x \left[-\tfrac{1}{4}F_{\mu\nu}{}^2 + i\bar{\psi}\partial\!\!\!/\psi - e\bar{\psi}A\!\!\!/\psi - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \right]\right] \tag{A2b}$$

In this generating functional many gauge copies of the same configuration are considered, because after introducing the gauge-fixing term we still have a residual gauge freedom (A3) that respects the periodic boundary conditions

$$\psi \to e^{-i2\pi k\cdot x/L}\psi\ ,\qquad A_\mu \to A_\mu + \frac{2\pi}{e}\frac{k_\mu}{L}\ , \tag{A3}$$

where k_μ are integers.

We can use the gauge freedom (A3) to ensure that (A4) is satisfied,

$$-\frac{\pi}{eL}\le \bar{A}_\mu \le \frac{\pi}{eL}\ \text{for each } \mu\ ,\qquad \bar{A}_\mu = \frac{1}{V}\int d^3x A_\mu(x)\ . \tag{A4}$$

An arbitrary vector potential in the α gauge can be written as

$$A_\mu(x) = \frac{2\pi}{e}\frac{k_\mu}{L} + A'_\mu(x)\ ,$$

where $\bar{A}'_\mu$ is inside the box defined in (A4) which we will call C_L. Then we have that (A2) can be written as follows:

$$Z=\int \mathscr{D}A_\mu \mathscr{D}\psi\, \mathscr{D}\bar{\psi} \exp\left[i\int d^3x\, \mathscr{L}_{\text{eff}}(A_\mu,\psi,\bar{\psi})\right]$$
$$=\sum_{k_\mu}\int_{\bar{A}'_\mu\in C_L} \mathscr{D}A'_\mu \mathscr{D}\psi' \mathscr{D}\bar{\psi}' \exp\left[i\int d^3x\, \mathscr{L}_{\text{eff}}(A'_\mu,\psi',\bar{\psi}')\right]$$
$$=\Big(\sum_{k_\mu} 1\Big)\int_{\bar{A}'_\mu\in C_L} \mathscr{D}A'_\mu \mathscr{D}\psi' \mathscr{D}\bar{\psi}' \exp\left[i\int d^3x\, \mathscr{L}_{\text{eff}}(A'_\mu,\psi',\bar{\psi}')\right]. \tag{A5}$$

The factor $\sum_{k_\mu} 1$ is a constant normalization and can be therefore dropped. The resulting form of the generating functional corresponds to the conditions satisfied by (3.1), since when $L\to\infty$, $\bar{A}_\mu\to 0$.

The form (3.5) can be obtained as follows. Instead of choosing the gauge (A4), choose A^1_μ to be

$$A^1_\mu=\frac{2\pi}{e}\frac{k_\mu}{L}+A_\mu\equiv\frac{g_0}{e}B_\mu+A_\mu\ ,$$

where $\bar{A}_\mu$ satisfies (A4). Now denote A^1_μ, ψ^1, $\bar{\psi}^1$ the dynamical variables in this gauge. Then up to an irrelevant normalization we have that

$$Z=\int_{\bar{A}^1-(g_0/e)B_\mu\in C_L} \mathscr{D}A^1_\mu \mathscr{D}\psi^1 \mathscr{D}\bar{\psi}^1 \exp\left[i\int d^3x\, \mathscr{L}_{\text{eff}}(A^1_\mu,\psi^1,\bar{\psi}^1)\right]$$
$$=\int_{\bar{A}_\mu\in C_L} \mathscr{D}A_\mu \mathscr{D}\psi^1 \mathscr{D}\bar{\psi}^1 \exp\left[i\int d^3x\left[-\frac{1}{4}F_{\mu\nu}{}^2+\bar{\psi}^1(i\partial\!\!\!/-eA\!\!\!/)\psi^1-g_0\bar{\psi}^1 B\!\!\!/\psi^1-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2\right]\right].$$

Now one can drop the index 1 and perform an average with weight $e^{iB^2/2}$ over all possible B. Up to normalization

$$Z=\sum_{k_\mu} e^{iB^2/2}\int_{\bar{A}_\mu\in C_L} \mathscr{D}A_\mu \mathscr{D}\psi\, \mathscr{D}\bar{\psi} \exp\left[i\int d^3x\left[-\frac{1}{4}F_{\mu\nu}{}^2+\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/)\psi-g_0\bar{\psi}B\!\!\!/\psi-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2\right]\right],$$

when $L\to\infty$, $\bar{A}_\mu\to 0$, and $\sum_{k_\mu}\to\int d^3B$, and we get Eq. (3.5).

APPENDIX B: FOUR-LOOPS FERMION SELF-ENERGY

Here we consider four-loop contributions to the fermion propagator. Among the contributions to this loop order, let us consider first the leading diagrams, shown in Fig. 9, where the counterterms needed to make diagram 9(a) finite are shown [9(a′), 9(a″), 9(a‴)]. Diagrams 9(a) and 9(b) each have a second-order pole at $\eta=0$: each photon line with a

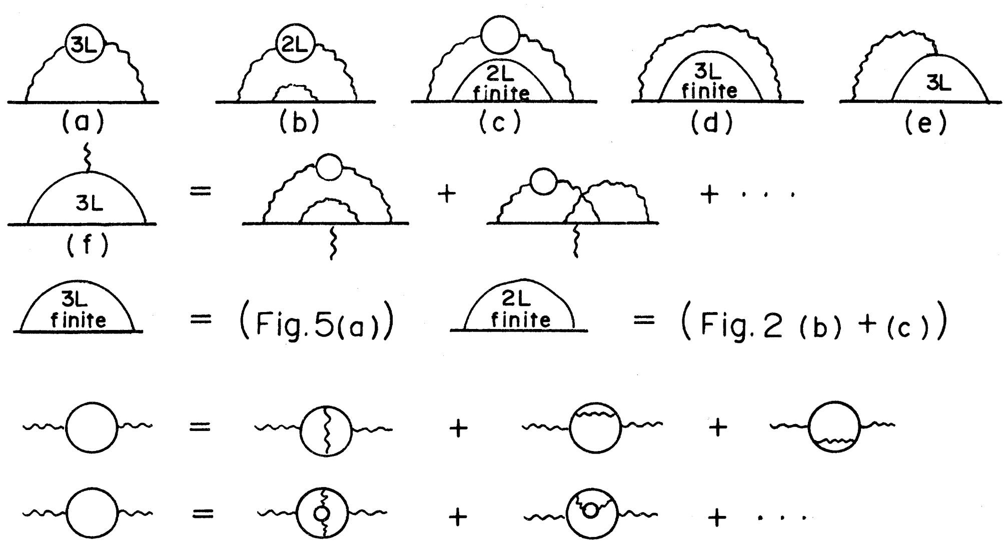

FIG. 10. Four-loop second subleading contributions to fermion self-energy.

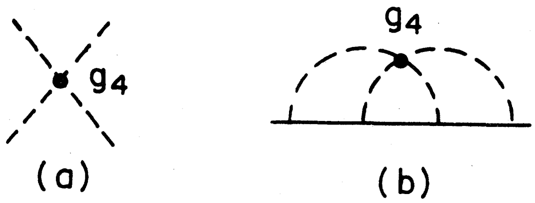

FIG. 11. (a) B^4 interaction vertex; (b) counterterm needed to cancel second subleading singularities of the fermion propagator at four loops.

bubble insertion contributes one order to the pole. The graphs represented by Fig. 9(c) have a new type of singularity, arising from soft momentum flow through the fermion line.

Diagram 9(c) cannot be renormalized by B photons, but it can be eliminated using the $\alpha=0$ gauge (Landau gauge if $g_0=0$), since it is proportional to α. Thus the leading diagrams of $\Sigma^{(4)}$ in the $\alpha=0$ gauge, renormalized by B-photon counterterms, give the finite contribution

$$\Sigma_L^{(4)}=A_L^{(4)}\not{p}\lambda^4(p)[\ln\lambda(p)+C^{(2)}]^2 , \tag{B1}$$

where $\lambda(p)$ is the expansion parameter defined in Eq. (2.7), $A_L^{(4)}$ is a constant, and $C^{(2)}$ is a gauge-dependent constant, as before.

There are no first subleading diagrams because they appear only at odd loop orders and in an α gauge they are proportional to α.

The second subleading diagrams at four loops contain a first-order pole at $\eta=0$; they contribute only terms proportional to $\ln\lambda(p)$ and powers of $\lambda(p)$ which redefine gauge-dependent terms of the leading diagrams.

We still have to show that the poles of these diagrams can be canceled. All four-loop diagrams are obtained by inserting three loops in both photon and electron propagators and in the vertex, (Fig. 10) (counterterms are suppressed). Diagrams 10(a) and 10(b) are singular because of soft momentum flow through the photon line; in diagram 10(d) the singularity comes from soft momentum through the fermion line, while in 10(e) the singularity is contained in subdiagram 10(f).

In the $\alpha=0$ gauge, diagrams 10(b) and 10(d) vanish, while diagrams 10(c) and 10(e) are renormalized by B-photon counterterms (see Figs. 3 and 8), and the only remaining pole is in diagram 10(a). This pole can be canceled by a generalization of the B-photon counterterms introducing a B^4 interaction, as will be explained below. Here we point out that all the singularities at this loop order are either proportional to α or removable by B-photon counterterms, hence of gauge origin. The total contribution of second subleading and finite diagrams (in $\alpha=0$ gauge) has the form

$$\Sigma_{2\mathrm{SL}}^{(4)}=A_{2\mathrm{SL}}^{(4)}\lambda^4(p)[\ln\lambda(p)+C^{(2)}+C^{(4)}+F^{(4)}] , \tag{B2}$$

where $A_{2\mathrm{SL}}^{(4)}$ is a constant, $C^{(4)}$ is a constant due to the B^4 interaction, and $F^{(4)}$ is a constant representing finite diagrams.

To cancel singularities of the diagram in Fig. 10(a), we introduce additional diagrams by modifying the definition of the gauge fixing from (3.5) to

$$Z[J_\mu,\xi,\bar{\xi},\alpha,g_0,g_4] = \int d^DBe^{iB^2/2+ig_4B^4}Z[J_\mu,\xi,\bar{\xi},\alpha,g_0B_\mu] .$$

The introduction of this additional term produces additional vertices of the form shown in Fig. 11(a). We are going to choose $g_4=K\eta$ with K a finite constant independent of $\eta(=3-D)$, so that $g_4\to 0$ as we approach three dimensions. Under these circumstances the only singular contribution where this interaction enters at four loops in the fermion propagator is shown in Fig. 11(b), where K is now chosen so as to cancel the singularities coming from diagrams of type (a) in Fig. 10.

[1]For a review, see D. Gross, R. Pisarski, and L. Yaffe, Rev. Mod. Phys. 53, 43 (1981).
[2]R. P. Feynman, Nucl. Phys. B188, 479 (1981).
[3]A. Linde, Phys. Lett. 96B, 289 (1980).
[4]R. Jackiw and S. Templeton, Phys. Rev. D 23, 2291 (1981).
[5]T. Appelquist and R. Pisarski, Phys. Rev. D 23, 2305 (1981).
[6]G. 't Hooft, in *Field Theory and Strong Interactions*, proceedings of the XIX Internationale Universitatswochen fur Kernphysik, Schladming (Acta Phys. Austriaca Suppl. 22), edited by P. Urban (Springer, Vienna, 1980), p. 531.
[7]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).
[8]S. Templeton, Phys. Lett. 103B, 134 (1981); Phys. Rev. D 24, 3134 (1981).