Nonlinear static model

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The Abelian nonlinear Bose field interacting with a static source is treated by using the Tomonaga separation of the field into internal, corresponding to the classical field, and external parts. Two possible choices of the internal mode function are discussed. Quantum effects arise from quantization of the coordinate of the internal mode function; the relation to the result of the path-integral technique is shown. The simplest case, which occurs when the single-meson kinetic-energy term in the Hamiltonian is neglected, is used to show how the separation works.

I. INTRODUCTION

Consider the interaction of a scalar meson field Φ with a static source $\rho(\vec{x})$; the Hamiltonian density for the system is taken to be

$$\begin{aligned}\hat{H}&=\tfrac{1}{2}[\Pi^2+(\nabla\Phi)^2+m^2\Phi^2]\\&\quad-\eta\alpha m\Phi^3+\frac{\alpha^2}{4}\Phi^4-g\rho\Phi\\&=\tfrac{1}{2}[\Pi^2+(\nabla\Phi)^2]+U(\Phi)-g\rho\Phi\ ,\\U(\Phi)&=\frac{m^2}{2}\Phi^2-\eta\alpha m\Phi^3+\frac{\alpha^2}{4}\Phi^4\ .\end{aligned}\tag{1}$$

It is no restriction to require that $\eta\geq 0$ and $\alpha\geq 0$. For $\alpha>0$ this system will be called the Abelian nonlinear static model (ANSM). When $\alpha=0$ the Abelian linear static model (ALSM) is obtained; the solution of the ALSM is well known.[1]

One reason for considering systems described by the Hamiltonian density of Eq. (1) is that the covariant quantum field theory of a Dirac field interacting with a scalar field produces Φ^3 and Φ^4 counterterms as part of the perturbative renormalization procedure; the study of the system described by Eq. (1) may give information about interesting effects arising from the counterterms.

In this paper, the aim is rather to study the effects of the nonlinear term in systems related to the popular heuristic pictures that are used to describe how quantum chromodynamics may work.[2–5] In particular, the idea that there are two distinct states of some octet Bose field Φ_a is attractive. One of the states is stable in the vicinity of (color-singlet) systems of quarks, the other is stable away from quarks and corresponds to the ordinary vacuum. One of the states corresponds to $\Phi_a=0$ and the other to $\Phi_a\neq 0$ in some sense. The corresponding non-Abelian color nonlinear static model (CNSM) has the Hamiltonian density

$$\begin{aligned}\hat{H}&=\tfrac{1}{2}[\Pi^2+(\nabla\Phi)^2+m^2\Phi^2]\\&\quad-\eta\alpha m\sum_{abc}d_{abc}\Phi_a\Phi_b\Phi_c+\frac{\alpha^2}{4}\Phi^4-g\rho\lambda\cdot\Phi\ ,\end{aligned}\tag{2}$$

where

$$\Phi^2\equiv\sum_a\Phi_a\Phi_a,\ \text{etc.}\ ,\tag{3}$$

the color-changing octet λ_a acts on the source color variable or variables, and d_{abc} is the symmetric structure tensor of color SU_3. The corresponding non-Abelian linear static model (NALSM) has been extensively treated.[6] The present study of the ANSM of Eq. (1) is a preliminary to the study of the CNSM of Eq. (2). Clearly, the ANSM lacks the interesting group properties of the CNSM, particularly features that distinguish color singlets. On the other hand, it may provide some understanding of the hypothesis of two states of the Bose field. In particular, the heuristic picture relates the two states to the two minima of the quartic function $U(\Phi)-g\rho\Phi$, where the parameters α and η are chosen so that for ρ less than some critical density ρ_c the stable minimum occurs at $\Phi=0$, while for ρ exceeding the critical density ρ_c the stable minimum occurs at large values of Φ. This type of behavior was originally suggested by Lee and Wick[7] as a means of generating abnormal nuclear matter.

In previous studies[2–5] related to the present work, the density ρ in Eq. (1) has been due to fermion fields interacting with the scalar field; there the combined fermion-boson system has been solved self-consistently. The present work treats ρ as external and fixed with the aim of studying effects due specifically to particular features of the nonlinear Bose field Φ.

Section II gives a discussion of the case of constant uniform density merely as a simple introduction to the nonlinear field. In Sec. III various special cases of the ANSM Hamiltonian are listed. Section IV is concerned with the simplest of these, the case in which the single-meson kinetic energy can be neglected and the meson energy is $\omega(k)=m$. Up to this point, only classical field effects have been treated.

In Sec. V quantum field effects are calculated. Here the basic tool is the separation of the quantum field into internal and external parts; in the NALSM this separation was suggested by Tomonaga.[8] The internal mode function describes the part of the field that is like the classical field. The criterion for choosing the internal mode function and quantum effects arising from quantization of the coordinate Q of the internal mode function in the resolution of the field operator are discussed in Sec V. An interesting point is that the internal mode function can be chosen either to minimize the ground-state energy functional of the system, or so as to eliminate purely mesonic corrections to the ground state. In the ALSM the two choices are equivalent; the latter choice has been shown to be appropriate in the NALSM (Ref. 6) and both there and in the ANSM it leads to a simpler practical procedure for solving for the internal mode function. The effective Schrödinger equation for the quantized coordinate Q of the internal mode function is an improvement on the quadratic approximation used in path-integral calculations of quantum corrections. Section VI describes how these methods work in the simplest case.

The idea behind resolving the field operator is that the strong-coupling phenomena should all be associated with the internal mode, while the external modes can be treated perturbatively. Section VII describes the one-external-meson sector of the ANSM; this sector can be meaningful if the aim of the internal-external resolution is ever shown to be realized. Section VIII summarizes this work.

II. CONSTANT UNIFORM DENSITY

As a first orientation for the ANSM, consider the case of constant uniform density ρ. Then the field that minimizes the Hamiltonian is the constant uniform field that minimizes $U(\Phi)-g\rho\Phi$. In order to have distinct stable Bose field states, the parameters α and η must be chosen so that there is a ρ_c such that for $\rho<\rho_c$ the stable uniform field is at the minimum of $U-g\rho\Phi$ that lies closest to $\Phi=0$, while for $\rho>\rho_c$ the minimum of $U-g\rho\Phi$ that is farthest from $\Phi=0$ becomes the lower one. These conditions are most simply expressed by noting that $U-g\rho\Phi$ can be written in the form

$$\begin{aligned} U(\Phi)-g\rho\Phi&=\frac{m^4}{\alpha^2}V_\sigma(X)\ ,\\ V_\sigma(X)&=\tfrac{1}{4}(X-\eta)^4-\tfrac{1}{2}(3\eta^2-1)(X-\eta)^2\\ &\quad+(\sigma_c-\sigma)(X-\eta)+\eta(\tfrac{1}{2}\eta-\tfrac{3}{4}\eta^3-\sigma)\ ,\\ X&=\frac{\alpha}{m}\Phi\ ,\\ \sigma&=\frac{\alpha}{m^3}g\rho\ ,\\ \sigma_c&=\eta(1-2\eta^2)\ . \end{aligned} \tag{4}$$

Only for $\eta^2>\frac{1}{3}$ can there be two minima of $V_\sigma(X)$, and when $\sigma=0$ the two minima are equal when σ_c is also zero, that is, $\eta^2=\frac{1}{2}$; for $\eta^2>\frac{1}{2}$ the second minimum is lower than the first one when $\sigma=0$. Thus the limits on η^2 are

$$\tfrac{1}{3}<\eta^2\le\tfrac{1}{2}\ . \tag{5}$$

The condition that there actually be two minima of $V_\sigma(X)$ is that σ lie between the values $\sigma_c-2(\eta^2-\frac{1}{3})^{3/2}$ and $\sigma_c+2(\eta^2-\frac{1}{3})^{3/2}$; let the corresponding values of X be $X_1(\sigma)$ and $X_2(\sigma)$ with $X_1(\sigma)<X_2(\sigma)$. For $\sigma<\sigma_c$ it is clear that $V_\sigma(X_1(\sigma))<V_\sigma(X_2(\sigma))$ and *vice versa*, so that σ_c is the critical value of σ; then $\rho_c=\sigma_c m^3/\alpha g$ is the corresponding critical value of ρ. For σ less than $\sigma_c-2(\eta^2-\frac{1}{3})^{3/2}$ there is a single minimum of $V_\sigma(X)$ at $X_1(\sigma)$, and for σ greater than $\sigma_c+2(\eta^2-\frac{1}{3})^{3/2}$ there is a single minimum of $V_0(X)$ at $X_2(\sigma)$. For fixed σ let the lower of the two minima be called $V_m(\sigma)$ and the value of X for which it occurs be $X_m(\sigma)$; then clearly

$$\begin{aligned} X_m(\sigma)&=\begin{cases} X_1(\sigma), & \sigma<\sigma_c\ ,\\ X_2(\sigma), & \sigma>\sigma_c\ ,\end{cases}\\ V_m(\sigma)&=\begin{cases} V_\sigma(X_1(\sigma)), & \sigma<\sigma_c\ ,\\ V_\sigma(X_2(\sigma)), & \sigma>\sigma_c\ .\end{cases} \end{aligned} \tag{6}$$

Note that $\eta^2=\frac{1}{2}$ is the "symmetric" case, in that then the two minima of $V_{\sigma=0}(X)$ are at the same value of $V_{\sigma=0}(X)$. In this sense, it is possible to regard the deviation of η^2 from the value $\frac{1}{2}$ to be a measure of symmetry breaking in the system.

III. HAMILTONIANS

With the notation of Eq. (4), the Hamiltonian density of Eq. (1) is

$$\hat{H}=m_\alpha{}^4\hat{H}' ,$$
$$\hat{H}'=V_\sigma(X)+\frac{1}{2m^2}(\nabla X)^2+\frac{m^2}{2m_\alpha{}^8}\Pi_X{}^2 , \quad (7)$$
$$m_\alpha{}^4=m^4/\alpha^2 .$$

In this form the function of the various parameters is clearly separated. The parameter η determines the shape of $V_\sigma(X)$, which is independent of m and m_α. The mass m of the meson enters $\hat{H}'$ in the single-meson kinetic-energy term proportional to $(\nabla X)^2$. The meson field kinetic-energy term involves both m and m_α, and m_α is also involved in the energy scale that relates $\hat{H}$ and $\hat{H}'$. The length scale is determined by the length scale of ρ or, equivalently, σ. From Eqs. (7) it is clear that $\alpha \rightarrow 0$ is a difficult limit.

In all static models σ is taken to be constant. In a classical static model, X is zero and therefore Π_X is zero, so the classical static-model Hamiltonian density is

$$\hat{H}'_{\rm cl}=V_\sigma(X)+\frac{(\nabla X)^2}{2m^2} . \quad (8)$$

The $\Pi_X{}^2$ term in $\hat{H}'$ is responsible for quantum corrections to the ground state of $\hat{H}'_{\rm cl}$ of Eq. (8); these are discussed in Sec. V. The static field $X_{\rm cl}$ that minimizes the classical Hamiltonian $H'_{\rm cl}=\int \hat{H}'_{\rm cl} d\vec{x}$ satisfies

$$-\frac{1}{m^2}\nabla^2 X+\frac{\partial V_\sigma(X)}{\partial X}=0 , \quad (9)$$

and is the solution of Eq. (9) that minimizes $H'_{\rm cl}$. In general, Eq. (9) is a rather complicated nonlinear partial differential or (in the case of radial dependence only) nonlinear ordinary differential equation to be solved for the classical static field $X(\vec{x})$. A simpler case occurs when the single-meson kinetic energy is neglected. In this "massive-meson" ANSM the Hamiltonian density is

$$\hat{H}'_{\rm mm}=V_\sigma(X)+\frac{m^2}{2m_\alpha{}^8}\Pi_X{}^2$$
$$=m_\alpha{}^{-4}[\tfrac{1}{2}(\Pi^2+m^2\Phi^2)+U(\Phi)-g\rho\Phi] , \quad (10)$$

with the corresponding classical Hamiltonian density given by

$$\hat{H}'_{\rm mm,cl}=V_\sigma(X) . \quad (11)$$

Now the classical static field is the solution of the algebraic equation for minimizing $V_\sigma(X)$, rather than the differential equation of Eq. (9).

IV. SIMPLEST NONLINEAR STATIC MODEL

The Hamiltonian density $\hat{H}'_{\rm mm}$ of Eq. (10) describes what is undoubtedly the simplest nonlinear static model. Its classical counterpart $\hat{H}'_{\rm mm,cl}$ is relatively easily solved; quantum corrections are discussed in Secs. VI and VII.

Since $\hat{H}'_{\rm mm,cl}$ is just $V_\sigma(X)$, it follows that for any $\vec{x}$ the field $X(\vec{x})$ is just that value that minimizes $V_{\sigma(\vec{x})}(X)$; that is, $X(\vec{x})$ is determined by the density $\rho(\vec{x})$ at $\vec{x}$, and the "local-density approximation" is exact in this case. In the terminology of Sec. II, the Hamiltonian is minimized when $X(\vec{x})$ is equal to $X_m(\sigma(\vec{x}))$; the energy density at $\vec{x}$ is $V_m(\sigma(\vec{x}))$.

In this simplest model, consider uniform spherical densities $\rho(\vec{x})$:

$$\rho(\vec{x})=\begin{cases}\rho_0, & |\vec{x}|<R \\ 0, & |\vec{x}|>R ,\end{cases}$$
$$g\rho_0=\frac{3G}{4\pi R^3} , \quad (12)$$

where G is the integrated coupling of the source. The corresponding $\sigma(\vec{x})$ is

$$\sigma(\vec{x})=\begin{cases}\sigma_0, & |\vec{x}|<R , \\ 0, & |\vec{x}|>R ,\end{cases}$$
$$\sigma_0=\frac{3\alpha G}{4\pi(mR)^3} . \quad (13)$$

Classically the total energy of the source and field is

$$H_{\rm mm,cl}=\int \hat{H}_{\rm mm,cl} d\vec{x}$$
$$=\frac{4\pi R^3}{3}m_\alpha{}^4 V_m(\sigma_0)=\frac{m}{\alpha}G\frac{V_m(\sigma_0)}{\sigma_0} . \quad (14)$$

Figures 1–4 show the functions $V_m(\sigma)/\sigma$ and $X_m(\sigma)$ for various values of η.

If the source has an internal energy proportional to R^{-2} or R^{-1}, as is the case if the internal energy is fermion kinetic energy, for example, then it is evident that the sum of source and field energies will in most cases have a minimum near $\sigma=\sigma_c$, that is, for R near R_c, where

$$R_c=\left[\frac{3\alpha G}{4\pi m^3\eta(1-2\eta^2)}\right]^{1/3} . \quad (15)$$

The case of nonuniform monotonic density is easy to visualize. Of course it is assumed that the density goes to zero as $|\vec{x}|$ goes to ∞. Then, if the density at the origin is less than ρ_c, the field $X(\vec{x})$ is just $X_1(\sigma(\vec{x}))$ everywhere. If the density at the origin is greater than ρ_c, the field $X(\vec{x})$ is equal to $X_2(\sigma(\vec{x}))$ until $\vec{x}$ reaches a value at which $\sigma(\vec{x})=\sigma_c$; there the

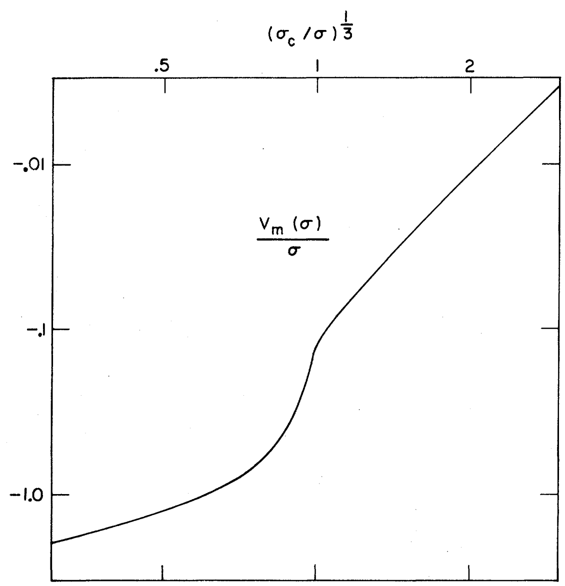

FIG. 1. $V_m(\sigma)/\sigma$ for $\eta^2=0.35$.

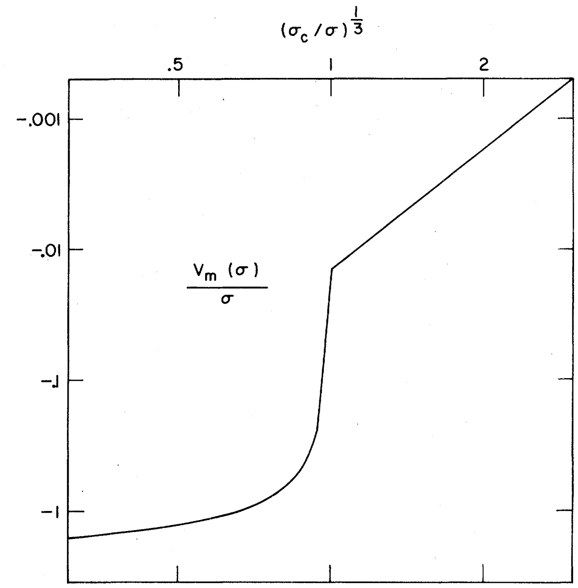

FIG. 3. $V_m(\sigma)/\sigma$ for $\eta^2=0.48$.

field X jumps down from $X_2(\sigma_c)$ to $X_1(\sigma_c)$ and follows $X_1(\sigma(\vec{x}))$ as $\vec{x}$ moves away from the origin. The total energy of source and field is just $m_\alpha{}^4 \int V_m(\sigma(\vec{x}))d\vec{x}$.

V. QUANTUM FIELD EFFECTS

In order to write the quantum field Φ as the sum of two parts, an internal part that corresponds to the classical field and an external part that is the orthogonal complement of the internal part of the quantum field, it is first necessary to go to the representation of the quantum field in which orthogonality is meaningful, namely, the representation in terms of creation and annihilation operators $a^\dagger(\vec{k})$ and $a(\vec{k})$:

$$\Phi(\vec{x})=\int \frac{e^{i\vec{k}\cdot\vec{x}}}{[16\pi^3\omega(k)]^{1/2}}[a(\vec{k})+a^\dagger(-\vec{k})]d\vec{k}\ , \qquad (16)$$

$$\Pi(\vec{x})=-i\int \left[\frac{\omega(k)}{16\pi^3}\right]^{1/2} e^{i\vec{k}\cdot\vec{x}}[a(\vec{k})-a^\dagger(-\vec{k})]d\vec{k}\ ,$$

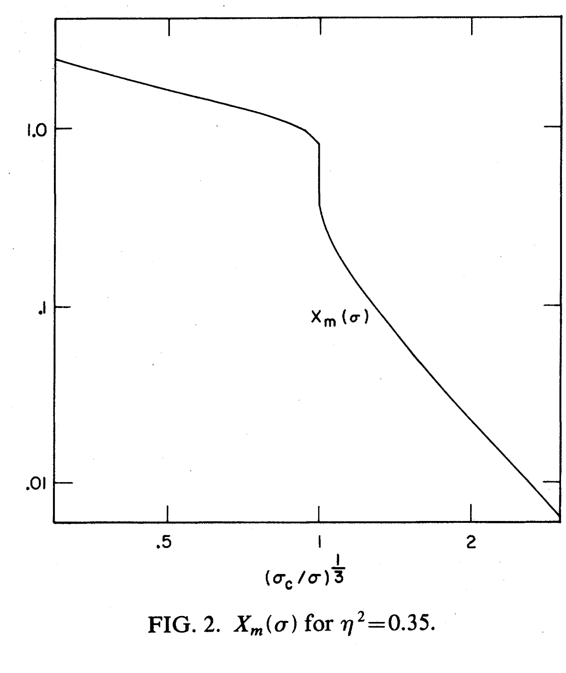

FIG. 2. $X_m(\sigma)$ for $\eta^2=0.35$.

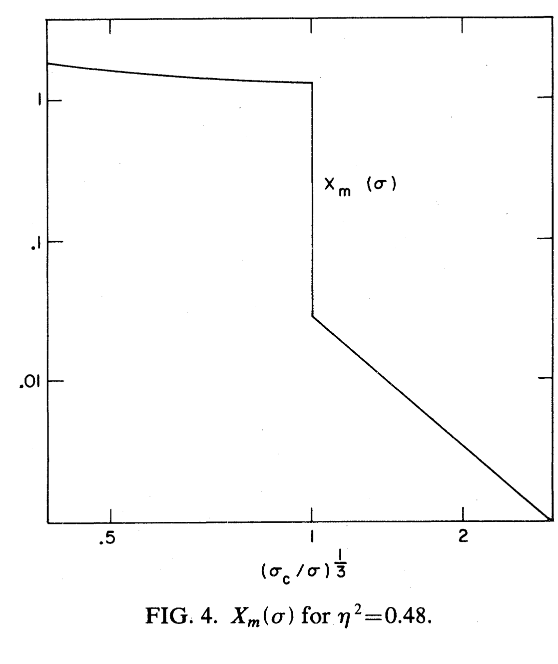

FIG. 4. $X_m(\sigma)$ for $\eta^2=0.48$.

with the associated commutation relation

$$[a(\vec{k}),a^\dagger(\vec{k}\,')]=\delta(\vec{k}-\vec{k}\,') . \tag{17}$$

The expansion of the operator $a(\vec{k})$ in terms of a finite orthonormal set $\tilde{\phi}_i(\vec{k})$ is easy:

$$\begin{aligned} a(\vec{k}) &= \sum_{i=1}^{N} A_i\tilde{\phi}_i(\vec{k})+a_\perp(\vec{k}) , \\ [A_i,A_j^\dagger] &= \delta_{i,j} , \\ [A_i,a_\perp^\dagger(\vec{k})] &= 0 , \\ [a_\perp(\vec{k}),a_\perp^\dagger(\vec{k}\,')] &= \delta(\vec{k}-\vec{k}\,') - \sum_{i=1}^{N} \tilde{\phi}_i(\vec{k})\tilde{\phi}_i^*(\vec{k}\,') , \\ \int \tilde{\phi}_i^*(\vec{k})a_\perp(\vec{k})d\vec{k} &= 0, \quad i=1,2,\ldots,N . \end{aligned} \tag{18}$$

In the present case, a single mode $\tilde{\phi}(\vec{k})$ will represent the internal quantum field that is the quantum analog of the classical field, so that the expression is

$$\begin{aligned} a(\vec{k}) &= A\tilde{\phi}(\vec{k})+a_\perp(\vec{k}) , \\ \tilde{\phi}(\vec{k}) &= \tilde{\phi}^*(-\vec{k}) , \\ \int |\tilde{\phi}(\vec{k})|^2 d\vec{k} &= 1 . \end{aligned} \tag{19}$$

The relation between $\phi(\vec{x})$ and $\tilde{\phi}(\vec{k})$ is taken to be the usual boson Fourier transform with an additional factor α/m:

$$\phi(\vec{x})=\frac{\alpha}{m}\int \frac{e^{i\vec{k}\cdot\vec{x}}}{[8\pi^3\omega(k)]^{1/2}}\tilde{\phi}(\vec{k})d\vec{k} , \tag{20}$$

so that

$$\phi^*(\vec{x})=\phi(\vec{x}) ,$$

$$\int \phi^2(\vec{x})d\vec{x}=\left[\frac{\alpha}{m}\right]^2\int \frac{|\tilde{\phi}(\vec{k})|^2}{\omega(k)}d\vec{k} = \left[\frac{\alpha}{m}\right]^2\langle\omega^{-1}\rangle . \tag{21}$$

Then the resolution of the field operator $\Phi(\vec{x})$ corresponding to Eq. (19) is

$$\begin{aligned} \Phi(\vec{x}) &= Q\frac{m}{\alpha}\phi(\vec{x})+\Phi_\perp(\vec{x}) , \\ Q &= \frac{A+A^\dagger}{\sqrt{2}} , \\ \Phi_\perp(\vec{x}) &= \int \frac{e^{i\vec{k}\cdot\vec{x}}}{[16\pi^3\omega(k)]^{1/2}} \times[a_\perp(\vec{k})+a_\perp^\dagger(-\vec{k})]d\vec{k} , \end{aligned} \tag{22}$$

and similarly for $\Pi(\vec{x})$,

$$\begin{aligned} \Pi(\vec{x}) &= P\frac{m}{\alpha}\psi(\vec{x})+\Pi_\perp(\vec{x}) , \\ P &= \frac{A-A^\dagger}{i\sqrt{2}} , \\ \psi(\vec{x}) &= \frac{\alpha}{m}\int\left[\frac{\omega(k)}{8\pi^3}\right]^{1/2} e^{i\vec{k}\cdot\vec{x}}\tilde{\phi}(\vec{k})d\vec{k} , \\ \Pi_\perp(\vec{x}) &= -i\int\left[\frac{\omega(k)}{16\pi^3}\right]^{1/2} e^{i\vec{k}\cdot\vec{x}} \times[a_\perp(\vec{k})-a_\perp^\dagger(-\vec{k})]d\vec{k} , \end{aligned} \tag{23}$$

where P and Q satisfy the commutation relation

$$[P,Q]=\frac{1}{i} . \tag{24}$$

From the above equations, it is seen that $\phi(\vec{x})$ is the special mode of the field, normalized in a somewhat unusual way; the quantum coordinate of the special mode is Q. The form of the internal mode function $\phi(\vec{x})$ will be chosen later; it should resemble the classical field of the previous sections multiplied by a normalization factor. The field $\Phi_\perp(\vec{x})$ is the external part of the quantum field $\Phi(\vec{x})$.

The quantum Hamiltonian density is

$$\hat{H}=:\{\tfrac{1}{2}[\Pi^2+(\nabla\Phi)^2]+U(\Phi)-g\rho\Phi\}: , \tag{25}$$

where $U(\Phi)$ is given in Eq. (1), and the normal ordering indicated by the colons is defined in terms of the creation and annihilation operators $a^\dagger(\vec{k})$ and $a(\vec{k})$; note that definition of the normal ordering is not changed by the separation of the special mode. Now substitution of the forms for Φ and Π into the Hamiltonian $\int \hat{H}$ gives

$$\begin{aligned} H=\int\hat{H} &= :\left\{\int \omega(k)a^\dagger(\vec{k})a(\vec{k})d\vec{k}+\int\left[-\eta\alpha m\Phi^3(\vec{x})+\frac{\alpha^2}{4}\Phi^4(\vec{x})-g\rho\Phi(\vec{x})\right]d\vec{x}\right\}: \\ &= \frac{m^4}{\alpha^2}H_A+\frac{m^3}{\alpha}\int(\hat{H}_1+\hat{H}_1^\dagger)d\vec{x}+\int\hat{H}_2 d\vec{x} , \end{aligned} \tag{26}$$

where the Hamiltonian H_A is independent of $a_\perp$ and $a_\perp^\dagger$, the Hamiltonian density $\hat{H}_1$ is linear in $a_\perp^\dagger$, and the

Hamiltonian density $\hat{H}_2$ has terms of degree greater than 1 in $a_\perp$ and $a_\perp^\dagger$:

$$H_A = :\left[\frac{w}{2}(P^2+Q^2) - Q\int\sigma\phi - \eta Q^3\int\phi^3 + \frac{Q^4}{4}\int\phi^4\right]: \quad w = \frac{\alpha^2}{m^4}\int\omega(k)\,|\tilde{\phi}(\vec{k})|^2 d\vec{k} = \frac{\alpha^2}{m^4}\langle\omega\rangle = \int\phi(\vec{x})\left[1-\frac{\nabla^2}{m^2}\right]\phi(\vec{x})d\vec{x} ;$$

$$\hat{H}_1 = \int\frac{e^{-i\vec{k}\cdot\vec{x}}}{[16\pi^3\omega(k)]^{1/2}}a_\perp^\dagger(\vec{k})d\vec{k}\,\hat{J}(\vec{x}), \quad \hat{J} = (Q+iP)\left[1-\frac{\nabla^2}{m^2}\right]\phi - \sigma + :(Q^3\phi^3 - 3\eta Q^2\phi^2): ; \tag{27}$$

$$\hat{H}_2 = :\{\tfrac{1}{2}[\Pi_\perp{}^2 + (\nabla\Phi_\perp)^2] + U(\Phi_\perp) + m^2(\tfrac{3}{2}Q^2\phi^2 - 3\eta Q\phi)\Phi_\perp{}^2 + \alpha m Q\phi\Phi_\perp{}^3\}: .$$

In the classical case where all commutators of P and Q are zero and normal ordering has no effect, and for static classical fields where $P=0$, it is evident that

$$\hat{J}_{sc}(\vec{x}) = \frac{\delta H_{A,sc}}{\delta(Q\phi(\vec{x}))} , \tag{28}$$

so that the classical field $Q\phi$ that minimizes $H_{A,sc}$ also makes the source $\hat{J}_{sc}$ for $\perp$ meson fields vanish.

Quantum effects are now simply computed by taking into account that P and Q in H_A satisfy the commutation relation of Eq. (24). In the quantum case, the normal ordering is important. Without it, there would be an uncontrolled distribution of zero-point energy between the internal and external modes. By using the forms of Q and P in terms of A and $A^\dagger$ given in Eqs. (22) and (23), it is easy to see that

$$\begin{aligned} :Q: &= Q , \\ :Q^2: &= Q^2 - \tfrac{1}{2} , \\ :P^2: &= P^2 - \tfrac{1}{2} , \\ :Q^3: &= Q^3 - \tfrac{3}{2}Q , \\ :Q^4: &= Q^4 - 3Q^2 + \tfrac{3}{4} , \end{aligned} \tag{29}$$

so that

$$H_A = \frac{w}{2}P^2 + V_A(Q) , \tag{30}$$

where

$$V_A(Q) = \frac{Q^4}{4}\int\phi^4 - \eta Q^3\int\phi^3 + \left[\frac{w}{2} - \tfrac{3}{4}\int\phi^4\right]Q^2 + \left[\tfrac{3}{2}\eta\int\phi^3 - \int\sigma\phi\right]Q + \tfrac{3}{16}\int\phi^4 - \frac{w}{2} , \tag{31}$$

and

$$\hat{J} = (Q+iP)\left[1-\frac{\nabla^2}{m^2}\right]\phi - \sigma + \tfrac{3}{2}\eta\phi^2 - \tfrac{3}{2}Q\phi^3 - 3\eta Q^2\phi^2 + Q^3\phi^3 . \tag{32}$$

Once a choice of $\phi(\vec{x})$ has been made, H_A is completely determined. Its eigenvalues and eigenstates are the eigenvalues and eigenfunctions of the operator

$$\hat{h} = -\frac{w}{2}\frac{d^2}{dQ^2} + V_A(Q) , \tag{33}$$

and can be determined as accurately as may be required by using standard numerical techniques. The quadratic approximation to the lowest eigenvalue is

$$V_A(Q_0) + \tfrac{1}{2}[wV_A''(Q_0)]^{1/2} , \tag{34}$$

where the minimum of $V_A(Q)$ occurs at $Q=Q_0$. This quadratic approximation is exact in the case of the ALSM and is the sort of formula that is derived in path-integral treatments of similar problems. In the present case, there is no need to use the quadratic approximation, which, for many situations of interest, may not be accurate.

For a particular choice of ϕ, the ground state of H_A is given by

$$\begin{aligned} |\phi,0\rangle &= f_0\left[\frac{A+A^\dagger}{\sqrt{2}}\right]|\Omega\rangle , \\ a_\perp(\vec{k})|\Omega\rangle &= A|\Omega\rangle = 0 , \\ \hat{h}f_0(Q) &= \epsilon_0 f_0(Q) , \\ H_A|\phi,0\rangle &= \epsilon_0|\phi,0\rangle , \end{aligned} \tag{35}$$

where the operators H_A and $\hat{h}$ depend on the choice of ϕ.

Two different criteria for choosing the function ϕ seem worth considering. The first is variational, based on the observation that ϵ_0 of Eq. (35) is a functional of ϕ or $\tilde{\phi}$. The normalized ϕ that minimizes ϵ_0 is obtained by requiring that the functional derivative of $\epsilon_0 - \mu \int |\tilde{\phi}(\vec{k})|^2 d\vec{k}$ vanish. This criterion gives an integrodifferential equation for ϕ that depends on first solving for the ground state of $\hat{h}$ as a function of its parameters. The Lagrange multiplier μ is chosen so that ϕ is normalized according to Eq. (19).

A simpler alternative criterion for the meson mode function ϕ that has been shown to work well in the NALSM (Ref. 6) and in the meson-nucleon shell model[9] is to choose ϕ so that there are no purely mesonic corrections to the ground state of H_A. This is achieved by requiring that the expectation value of $\hat{J}(\vec{x})$ be proportional to $\psi(\vec{x})$:

$$\langle \hat{J} \rangle = -\sigma + \tfrac{3}{2}\eta\phi^2 + \langle Q \rangle \left[\left[1 - \frac{\nabla^2}{m^2} \right] \phi - \tfrac{3}{2}\phi^3 \right] - \langle Q^2 \rangle 3\eta\phi^2 + \langle Q^3 \rangle \phi^3 = \mu\psi \,, \tag{36}$$

where $\langle\ \rangle$ denotes expectation value in the ground state of H_A and $\langle P \rangle = 0$ since $V_A(Q)$ is real. With the condition (36), $\int \hat{H}_1$ of Eq. (27) has expectation value

$$\left\langle \int \hat{H}_1 \right\rangle = \mu \int \frac{e^{i\vec{k}\cdot\vec{x}}}{[16\pi^3\omega(k)]^{1/2}} \psi(\vec{x}) a_\perp^\dagger(\vec{k}) d\vec{x}\, d\vec{k} = \frac{\alpha\mu}{m\sqrt{2}} \int a_\perp^\dagger(\vec{k}) \tilde{\phi}(\vec{k}) d\vec{k} = 0 \,, \tag{37}$$

so that the ground state of H_A cannot emit mesons without changing to an excited state of H_A, and there are therefore no purely mesonic corrections to the ground state of H_A.

The function $\phi(\vec{x})$ is also required to be normalized according to Eq. (19), where $\tilde{\phi}(\vec{k})$ and $\phi(\vec{x})$ are related by Eq. (20). For arbitrary μ, the solution $\phi_\mu(\vec{x})$ of Eq. (36) has normalization given by

$$n(\mu) = \int |\tilde{\phi}_\mu(\vec{k})|^2 d\vec{k} = \left[\frac{m}{\alpha} \right]^2 \int \psi_\mu(\vec{x}) \phi_\mu(\vec{x}) d\vec{x} \,; \tag{38}$$

then μ must be chosen so that $n(\mu) = 1$. Clearly $\phi_\mu(\vec{x}) \to 0$ as $\mu \to -\infty$, since ψ is of the order of ϕ; thus $n(-\infty) = 0$. Presumably, the desired value of μ is the least μ for which $n(\mu) = 1$. Obviously the properties of solutions of Eq. (36) need further investigation. In the ALSM and NALSM it can be shown[6] that $n(0) = 1$, so that $\mu = 0$ in that case.

This second criterion for choosing ϕ has the advantage that only the ground-state expectation values of powers of Q enter into the equation for ϕ, so that it is a criterion that is easier to apply in practice.

The equations and conditions given above completely define the functions $\phi(\vec{x})$ and $f_0(Q)$ and the values of μ, w, $\langle Q \rangle$, $\langle Q^2 \rangle$, and $\langle Q^3 \rangle$. When these functions and values have been determined, the expectation value of H_A is given by ϵ_0, the lowest eigenvalue of $\hat{h}$. For the same function ϕ the no-meson (really no-⊥-meson) subspace has an orthonormal basis consisting of the states $f_i(Q)|\Omega\rangle$, where the f_i are the orthonormal eigenfunctions of $\hat{h}$.

The relative ease of solution of H_A in the ANSM is a reflection of the extreme simplicity of the corresponding ALSM for the neutral scalar field.

VI. QUANTUM CORRECTIONS IN THE SIMPLEST CASE

For the case of $H_{\rm mm}$ it follows that $\langle \omega \rangle = m$ and $w = \alpha^2/m^3$, so that for the uniform spherical source of Sec. IV, the function $\phi(\vec{x})$ is

$$\phi(\vec{x}) = \begin{cases} \phi_0 = \left[\dfrac{3\alpha^2}{4\pi(mR)^3} \right]^{1/2}, & |\vec{x}| < R \,, \\ 0, & |\vec{x}| > R \,, \end{cases} \tag{39}$$

and $\psi(\vec{x})$ is also uniform inside R; thus, Eq. (36) is automatically satisfied. In this case only the lowest eigenvalue of $\hat{h}$ requires computation. Here

$$\begin{aligned} &\int \phi^2 = \frac{\alpha^2}{m^3} = \frac{4\pi R^3}{3}\phi_0{}^2 = w \,, \\ &\int \phi^n = \frac{4\pi R^3}{3}\phi_0{}^n \,, \\ &\int \sigma\phi = \frac{4\pi R^3}{3}\sigma_0\phi_0 \,, \\ &\sigma_0 = \frac{G}{\alpha}\phi_0{}^2 \,, \end{aligned} \tag{40}$$

so that

$$V_A(Q) = \frac{4\pi R^3}{3} \left[\frac{(Q\phi_0)^4}{4} - \eta(Q\phi_0)^3 + (\tfrac{1}{2} - \tfrac{3}{4}\phi_0{}^2) \times (Q\phi_0)^2 + (\tfrac{3}{2}\eta\phi_0{}^2 - \sigma_0) Q\phi_0 + \tfrac{3}{16}\phi_0{}^4 - \tfrac{1}{2}\phi_0{}^2 \right]. \tag{41}$$

Now the canonical transformation

$$Q = (q + \eta)/\phi_0, \quad P = p\phi_0 \tag{42}$$

gives

$$\hat{h}=\frac{4\pi R^3}{3}[\tfrac{1}{2}\phi_0{}^4p^2+\tfrac{1}{4}q^4-\tfrac{1}{2}(3\eta^2+\tfrac{3}{2}\phi_0{}^2-1)q^2 +(\sigma_c-\sigma_0)q+\eta(\tfrac{1}{2}\eta-\tfrac{3}{4}\eta^3-\sigma_0) +\phi_0{}^2(\tfrac{3}{16}\phi_0{}^2+\tfrac{3}{4}\eta^2-\tfrac{1}{2})]\ , \quad (43)$$

where σ_c is defined in Eq. (4).

From the form of $V_A(Q)$ it is clear that if η^2 is in the range $\frac{1}{3}-\frac{1}{2}\phi_0{}^2$ to $\frac{1}{2}$, then $V_A(Q)$ has two minima. If $\sigma_0<\sigma_c$, the lowest eigenstate of $\hat{h}$ has Q mainly near zero, while if $\sigma_0>\sigma_c$, then Q is mainly in the second minimum of $V_A(Q)$. The transition around $\sigma_0=\sigma_c$ is associated with the near degeneracy at $\sigma_0=\sigma_c$ of two eigenstates of $\hat{h}$ with Q equally distributed between the two minima but differing in relative sign of the amplitudes in the minima. If $\eta^2<\frac{1}{3}$, the potential $V_\sigma(X)$ has only a single minimum. In such a case, the presence of a source with $\phi_0{}^2>2(\frac{1}{3}-\eta^2)$ gives $V_A(Q)$ with two minima; such a qualitative change is likely to be associated with a singularity in some function, in this case, of α, since $\phi_0{}^2$ is proportional to α^2.

VII. ONE-MESON SCATTERING HAMILTONIAN

Let the one-meson normal order $:\ :_1$ be defined to pick out just that part of $\Phi_\perp{}^2$ or $\Pi_\perp{}^2$ that has one $a_\perp^\dagger$ and one $a_\perp$ in normal order. Then the Hamiltonian in the one-meson sector is

$$H_A+\int\hat{H}_1+\int\hat{H}_1^\dagger+\int\hat{H}_{2,11}\ , \quad (44)$$

where

$$\hat{H}_{2,11}=:\{\tfrac{1}{2}[\Pi_\perp{}^2+(\nabla\Phi_\perp)^2] +m^2[\tfrac{1}{2}-3\eta Q\phi+\tfrac{3}{2}Q^2\phi^2]\Phi_\perp^2\}:_1\ , \quad (45)$$

and Q is an operator in the Tomonaga subspace generated by $A^\dagger$. The term $\hat{H}_{2,11}$ describes scattering of $\perp$ mesons by an operator (in the Tomonaga subspace) potential, while the $\hat{H}_1$ terms produce emission and absorption of $\perp$ mesons.

The one-meson sector has no superlinear terms and can be solved by standard techniques. Unlike the ALSM, the ANSM does have meson scattering in the one-meson sector.

VIII. SUMMARY

The classical and quantum behavior of a nonlinear neutral scalar field interacting with a static source has been shown to be relatively easily treated by separating the field into internal and external parts. The internal part is in a single mode, as proposed by Tomonaga[8]; hence, it is simple enough so that its Hamiltonian can be solved even when the internal field is strongly coupled. The ground-state energy is obtained by solving a Schrödinger equation; the quadratic approximation to this Schrödinger equation is what is obtained in the usual path-integral treatments of the quantum field problem. The internal mode function can be chosen either to minimize the internal-mode ground-state energy, or to eliminate purely mesonic corrections to this ground-state energy; the latter choice appears to be simpler to apply in practice. A simple "massive-meson" limiting case leads to particularly transparent equations. It serves as a useful guide to more general cases.

The treatment of the Bose field also applies when the source is due to fermion fields. Then, as is clear from the form of the $g\rho\Phi$ term, which becomes $g\Psi^\dagger\Psi\Phi$ or $g\bar{\Psi}\Psi\Phi$, the internal Bose field acts as a potential in which the fermions are confined in the usual self-consistent way.

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