# Local realism and measured correlations in the spin-$s$ Einstein-Podolsky-Rosen experiment

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Two aspects of the Clauser-Horne conditions for the compatibility with local realism of measured spin-$\frac{1}{2}$ Einstein-Podolsky-Rosen correlations are investigated in the spin-$s$ case. (1) A new set of necessary conditions is given for compatibility with local realism. These conditions are violated for a large range of geometries. The range does not diminish with increasing $s$, if the observed correlations are sufficiently near to the quantum-theoretic predictions. (2) A simple counterexample is given to the spin-1 generalization of a recent conjecture that the conditions tested by the Clauser-Horne spin-$\frac{1}{2}$ inequalities are sufficient as well as necessary for compatibility of the data with local realism.

## I. INTRODUCTION

In the spin-$s$ Einstein-Podolsky-Rosen experiment,[1–4] the spin components $m$ and $m'$ of two spin-$s$ particles in a spin singlet state $\phi$ are measured along directions $\hat{a}$ and $\hat{b}$, respectively. The data from many runs determine the joint distribution $p_{\hat{a},\hat{b}}(m,m')$.

Suppose we are given such distributions for several different pairs of axes, among which there are $N$ different directions $\hat{a}$ and $N'$ different $\hat{b}$. We shall say that such a set of distributions is compatible with local realism if it is possible to represent them in the form[5]

$$p_{\hat{a},\hat{b}}(m,m')=\langle f_{\hat{a}}(m,z)g_{\hat{b}}(m',z)\rangle \ . \tag{1.1}$$

Here, for each of the $N$ axes $\hat{a}$, the function $f_{\hat{a}}(m,z)$ is non-negative and viewed as the probability of a spin measurement along $\hat{a}$ yielding the value $m$, given that some set of hidden-variable parameters has the value represented by $z$. Similarly, for each of the $N'$ axes $\hat{b}$, the function $g_{\hat{b}}(m',z)$ is non-negative and viewed as the probability of the other spin measurement along $\hat{b}$ yielding the value $m'$, given the same value $z$ for the hidden-variable parameters. The angular brackets indicate an average over the distribution of hidden-variable parameters $z$, which is required to be independent of the particular pair of directions $\hat{a}\hat{b}$ along which the measurements were performed. Being probabilities, the $f_{\hat{a}}$ and $g_{\hat{b}}$ obey the normalization conditions

$$\sum_m f_{\hat{a}}(m,z)=\sum_{m'} g_{\hat{b}}(m',z)=1 \ . \tag{1.2}$$

If a set of distributions is compatible with local realism, then for any subsets $\hat{a}_1 \cdots \hat{a}_n$ and $\hat{b}_1 \cdots \hat{b}_{n'}$ of the $N$ directions $\hat{a}$ and $N'$ directions $\hat{b}$ we can define a non-negative function,

$$F_{\hat{a}_1 \cdots \hat{a}_n,\hat{b}_1 \cdots \hat{b}_{n'}}(m_1 \cdots m_n,m_1' \cdots m_{n'}')=\left\langle \prod_{i=1}^{n} f_{\hat{a}_i}(m_i,z) \prod_{j=1}^{n'} g_{\hat{b}_j}(m_j',z)\right\rangle \tag{1.3}$$

with the following properties.

(a) The function $F$ yields the given distributions as marginals:

$$\sum_{\substack{\text{all } m \text{ and } m' \\ \text{except } m_i,m_j'}} F_{\hat{a}_1 \cdots \hat{a}_n,\hat{b}_1 \cdots \hat{b}_{n'}}(m_1 \cdots m_n,m_1' \cdots m_{n'}')=p_{\hat{a}_i,\hat{b}_j}(m_i,m_j') \ . \tag{1.4}$$

(b) The sum of $F$ over any one of its arguments does not depend on the choice made for the corresponding axis:

$$\begin{aligned}&\sum_{m_i} F_{\hat{a}_1 \cdots \hat{a}_n,\hat{b}_1 \cdots \hat{b}_{n'}}(m_1 \cdots m_n,m_1' \cdots m_{n'}') \text{ independent of } \hat{a}_i \ , \\ &\sum_{m_j'} F_{\hat{a}_1 \cdots \hat{a}_n,\hat{b}_1 \cdots \hat{b}_{n'}}(m_1 \cdots m_n,m_1' \cdots m_{n'}') \text{ independent of } \hat{b}_j \ .\end{aligned} \tag{1.5}$$

 

Note that the existence of such functions follows directly from the conditions (1.1) and (1.2) of local realism, quite independently of whether the functions $F$ themselves have any meaning as higher-order distributions.

It is thus possible to show that a set of distributions is *not* compatible with local realism, by showing that the existence of any particular non-negative function or group of functions satisfying (1.4) and (1.5) is incompatible with the actual values of those distributions.

The simplest such test requires two directions $\hat{a}$, and two directions $\hat{b}$.[6] If the distributions are given for all four axis pairs $\hat{a}_i\hat{b}_j$, $i,j=1,2$, then in the spin-$\frac{1}{2}$ case the necessary and sufficient conditions for there to be non-negative three-axis functions $F_{\hat{a}_1\hat{a}_2,\hat{b}_i}(m_1m_2,m_i')$, $i=1,2$, satisfying conditions (1.4) and (1.5), are just the correlation inequalities of Clauser and Horne.[7,8] When the spin-$\frac{1}{2}$ distributions $p_{\hat{a}_i,\hat{b}_j}(m,m')$, $i,j=1,2$, are given by their quantum-theoretic forms, these conditions are found to be violated for a substantial range of geometries.

We shall examine below two questions suggested by this general point of view.

(1) Suppose we allow the spin to have a general value $s$, and continue to examine conditions for the existence of non-negative $F_{\hat{a}_1\hat{a}_2,\hat{b}_i}$ compatible with the four distributions $p_{\hat{a}_i,\hat{b}_j}$, $i,j=1,2$. Can we extract any characteristic behavior of the sets of axis pairs that violate local realism, as we approach the classical (large-$s$) limit?

(2) Suppose one is given the distributions $p_{\hat{a}_i,\hat{b}_j}$ for all axis pairs $\hat{a}_i\hat{b}_j$, $i=1\cdots N$, $j=1\cdots N'$, and suppose all the non-negative functions $F_{\hat{a}_i\hat{a}_j,\hat{b}_k}$ and $F_{\hat{a}_k,\hat{b}_i\hat{b}_j}$ exist and satisfy (1.4) and (1.5). The existence of all these $F$'s is a necessary condition for the compatibility of the given distributions with local realism, but could it also be sufficient? If so, the Clauser-Horne inequalities would be more than just a tool, reflecting the efficacy of a particular strategy which was sometimes successful in ruling out local realism; they would become, instead, the fundamental characterization of those sets of correlation experiments that could (or could not) be interpreted as compatible with local realism.

We discuss the first question in Sec. II. The issue was first raised by Mermin and Schwarz,[4] who outlined how the necessary and sufficient conditions for the existence of three-axis functions $F$ could be found for any spin $s$, and treated in detail the case of spin-1 inversion-symmetric distributions. The necessary and sufficient spin-1 conditions for the existence of $F_{\hat{a}_1\hat{a}_2,\hat{b}_i}$, $i=1,2$, are rather intricate, making it difficult to assess the behavior of the geometries incompatible with local realism even as $s$ goes from $\frac{1}{2}$ to 1. It would be a formidable task to pursue the question of necessary and sufficient conditions to still higher $s$.

In Sec. II we therefore revert to the less systematic but considerably simpler procedure of seeking only necessary conditions for the existence of the three-axis functions. We further only consider sets of experiments in which the geometry of the four axes $\hat{a}_i,\hat{b}_j$, $i,j=1,2$, is subject to two restrictions.

(i) We take the four axes to be coplanar. (This is, in fact, the case one generally has in mind, the plane being perpendicular to the line of flight of the two separating particles.) Since a spin component $m$ along a direction $\hat{n}$ is the same as a spin component $-m$ along $-\hat{n}$, we can adopt whatever sign convention is convenient for the axes. We choose these signs so that the four coplanar axes lie in a single half-plane.

(ii) We require the $\hat{a}_i$ and $\hat{b}_j$ to be four distinct axes, arranged within their half-plane so that precisely one $\hat{b}$ lies between the two $\hat{a}$'s, and therefore precisely one $\hat{a}$ lies between the two $\hat{b}$'s (Fig. 1).

For these geometries we give a very simple derivation of a new family of conditions necessary for the compatibility of the four distributions $p_{\hat{a}_i,\hat{b}_j}$ with local realism.[9] If the distributions agree with their quantum-theoretic forms, we show that these conditions are violated in *all* of the geometries considered, and for *all* values of the spin $s$. Thus for at least a third of all the possible coplanar geometries, the quantum-theoretic predictions for the four pairs of experiments are as peculiar for arbitrary spin $s$ as they are for spin $\frac{1}{2}$.

We discuss the second question in Sec. III. The issue was raised for spin $\frac{1}{2}$ by Fine,[8] who pointed

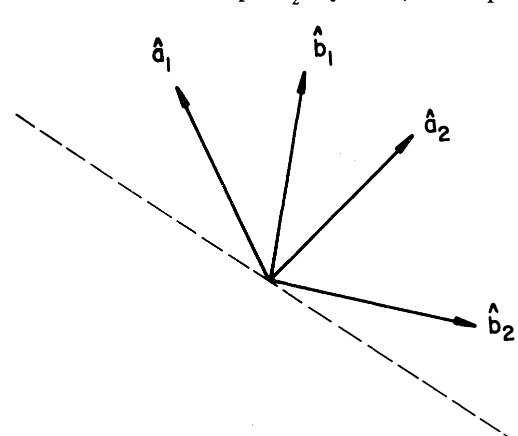


FIG. 1. The four coplanar axes are represented by unit vectors $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, and $\hat{b}_2$, with signs chosen so that the four lie in the same half-plane. The argument of Sec. II applies in the case shown here, in which the $\hat{a}$'s are separated by a $\hat{b}$, and vice versa.

out that when $N=N'=2$, the existence of all four three-axis functions ensures the existence of a non-negative four-axis function $F_{\hat{a}_1\hat{a}_2,\hat{b}_1\hat{b}_2}$ satisfying (1.4) and (1.5). If there are more than four pairs of axes, however, Fine's construction does not in general give a four-axis function satsifying (1.5).[10] He has conjectured that this limitation can be overcome[10] so that the answer to our question (2) may well be in the affirmative.

When $s=\frac{1}{2}$ we can indeed prove for *general* $N$ and $N'$ that if all non-negative three-axis functions $F_{\hat{a}_i\hat{a}_j,\hat{b}_k}$ and $F_{\hat{a}_k,\hat{b}_i\hat{b}_j}$ exist and satisfy (1.4) and (1.5) for all the distributions $p_{\hat{a}_i,\hat{b}_j}$, $i=1\cdots N$, $j=1\cdots N'$, then it is possible to construct all non-negative four-axis functions $F_{\hat{a}_i\hat{a}_j,\hat{b}_k\hat{b}_l}$ satisfying (1.4) *and* (1.5), *provided* the distributions $p_{\hat{a}_i,\hat{b}_j}$ do not violate inversion symmetry (Appendix A). Thus a spin-$\frac{1}{2}$ counterexample to Fine's conjecture must either use inversion-asymmetric distributions, or deal with $n$-axis functions with $n\geq 5$.[11]

For these reasons we consider here the spin-1 generalization of Fine's conjecture. This seems worth testing, since in the case $(N=N'=2)$ where it is known to be valid, Fine's construction can easily be shown to hold for arbitrary spin $s$. In Sec. III we give an explicit set of nine joint distributions $p_{\hat{a}_i,\hat{b}_j}(m,m')$ corresponding to nine geometries $\hat{a}_i\hat{b}_j$, $i,j=1,2,3$, with $m$ and $m'$ each taking on the values 1, 0, $-1$. We also give explicit non-negative forms for all the required three-axis functions $F_{\hat{a}_i\hat{a}_j,\hat{b}_k}$ and $F_{\hat{a}_k,\hat{b}_i\hat{b}_j}$ satisfying (1.4) and (1.5). Finally, however, we prove that there is no set of non-negative functions $F_{\hat{a}_i\hat{a}_j,\hat{b}_k\hat{b}_l}$ satisfying (1.4) and (1.5). The distributions we use in the counterexample are actually the quantum-theoretic ones for a particular set of spin-1 geometries (and therefore are inversion symmetric).

This counterexample demonstrates that the answer to question (2) is in the negative. For spin 1, necessary and sufficient conditions for the existence of all three-axis functions do not have the fundamental significance that Fine suggested they might have for spin $\frac{1}{2}$: Even if all non-negative three-axis functions exist and are compatible with (1.4) and (1.5), there will not, in general, be non-negative four-axis functions compatible with (1.4) and (1.5), and therefore the distributions will not, in general, be compatible with local realism.

## II. NECESSARY SPIN-$s$ CONDITIONS FOR LOCAL REALISM

We shall show that if the four axes $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, and $\hat{b}_2$ are coplanar and arranged as in Fig. 1, and if the observed distributions $p_{\hat{a}_i,\hat{b}_j}(m,m')$, $i,j=1,2$, agree sufficiently well with the quantum-theoretic predictions, then there can be no non-negative functions $F_{\hat{a}_1\hat{a}_2,\hat{b}_1}$ and $F_{\hat{a}_1\hat{a}_2,\hat{b}_2}$ satisfying (1.4) and (1.5). The general structure of the argument is this: Given that both non-negative three-axis functions exist and satisfy (1.4) and (1.5) for a given set of four distributions $p_{\hat{a}_i,\hat{b}_j}$, we use those three-axis functions to construct a quantity $K(\hat{a}_1\hat{a}_2,\hat{b}_1\hat{b}_2)$ with the following properties: (1) $K$ can be expressed entirely in terms of the observed distributions $p_{\hat{a}_i,\hat{b}_j}$, and vanishes if they have the quantum-theoretic form; (2) $K$ can be given a lower bound in terms of the observed distributions, which exceeds zero if they have the quantum-theoretic form.

These two steps can, of course, be combined to give a single rather clumsy inequality that the observed distributions must satisfy if the three-axis functions are to exist, and which the quantum-theoretic distributions fail to satisfy. However we find it clearer to present the argument in the form just described.

The essential feature of the coplanar geometries specified in (ii) of Sec. I, is that the axes $\hat{a}$ and $\hat{b}$ can be numbered so that (Fig. 2)

$$\hat{b}_1=C_1\hat{a}_1+C_2\hat{a}_2,\quad C_1,C_2>0\,, \tag{2.1}$$

$$\hat{b}_2=-D_1\hat{a}_1+D_2\hat{a}_2,\quad D_1,D_2>0\,. \tag{2.2}$$

If non-negative three-axis functions $F_{\hat{a}_1\hat{a}_2,\hat{b}_1}$ and $F_{\hat{a}_1\hat{a}_2,\hat{b}_2}$ exist and satisfy (1.4) and (1.5), then we define

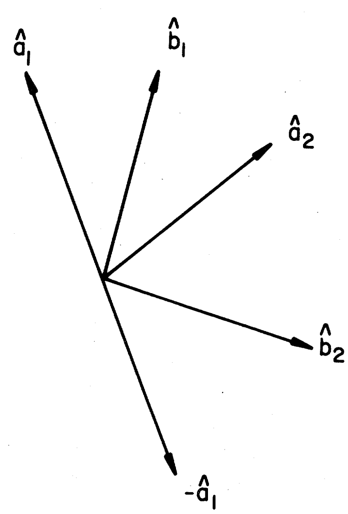


FIG. 2. The four unit vectors of Fig. 1 and a fifth unit vector, $-\hat{a}_1$. It is evident from the figure that $\hat{b}_1$ is a linear combination of $\hat{a}_1$ and $\hat{a}_2$ with positive coefficients, and $\hat{b}_2$ is a linear combination of $-\hat{a}_1$ and $\hat{a}_2$ with positive coefficients, as asserted in Eqs. (2.1) and (2.2).

$$G(\hat{a}_1\hat{a}_2,\hat{b}_1)=D_1D_2\sum_{m_1m_2m}(C_1m_1+C_2m_2+m)^2 \times F_{\hat{a}_1\hat{a}_2,\hat{b}_1}(m_1m_2,m)\,, \tag{2.3}$$

$$H(\hat{a}_1\hat{a}_2,\hat{b}_2)=C_1C_2\sum_{m_1m_2m}(-D_1m_1+D_2m_2+m)^2 \times F_{\hat{a}_1\hat{a}_2,\hat{b}_2}(m_1m_2,m)\,, \tag{2.4}$$

and

$$K(\hat{a}_1\hat{a}_2,\hat{b}_1\hat{b}_2)=G(\hat{a}_1\hat{a}_2,\hat{b}_1)+H(\hat{a}_1\hat{a}_2,\hat{b}_2)\,. \tag{2.5}$$

If the squared trinomials in (2.3) and (2.4) are explicitly multiplied out, then in either case each of the nine terms is independent of at least one of the summation variables. It follows from (1.4) that every term in each sum can be expressed in terms of a measured distribution $p_{\hat{a}_i,\hat{b}_j}$ except for the terms in $m_1m_2$, which give terms containing the unknown function

$$\sum_m F_{\hat{a}_1\hat{a}_2,\hat{b}_i}(m_1m_2,m)\,, \tag{2.6}$$

with $i=1$ in (2.3) and $i=2$ in (2.4). It follows from (1.5), however, that the sum in (2.6) is independent of $i$. We have taken advantage of this fact to choose the coefficients multiplying the sums in (2.3) and (2.4) so that the term proportional to the unknown function in $G$ is precisely canceled by the corresponding term in $H$, when the two are added together to give $K$.

We can therefore express $K$ entirely in terms of measured distributions:

$$K(\hat{a}_1\hat{a}_2,\hat{b}_1\hat{b}_2)=D_1D_2(C_1{}^2\langle m^2\rangle_{\hat{a}_1}+C_2{}^2\langle m^2\rangle_{\hat{a}_2}+\langle m^2\rangle_{\hat{b}_1}+2C_1\langle mm'\rangle_{\hat{a}_1\hat{b}_1}+2C_2\langle mm'\rangle_{\hat{a}_2\hat{b}_1}) +C_1C_2(D_1{}^2\langle m^2\rangle_{\hat{a}_1}+D_2{}^2\langle m^2\rangle_{\hat{a}_2}+\langle m^2\rangle_{\hat{b}_2}-2D_1\langle mm'\rangle_{\hat{a}_1\hat{b}_2}+2D_2\langle mm'\rangle_{\hat{a}_2\hat{b}_2})\,. \tag{2.7}$$

Here

$$\langle mm'\rangle_{\hat{a}\hat{b}}=\sum_{m,m'}mm'p_{\hat{a},\hat{b}}(m,m') \tag{2.8}$$

and

$$\langle m^2\rangle_{\hat{a}_i}=\sum_m m^2p_{\hat{a}_i}(m)\,,\qquad \langle m^2\rangle_{\hat{b}_i}=\sum_m m^2p_{\hat{b}_i}(m)\,. \tag{2.9}$$

[The one-axis distributions in (2.9) are defined by

$$p_{\hat{a}_i}(m)=\sum_{m'}p_{\hat{a}_i,\hat{b}_j}(m,m')\,,\qquad p_{\hat{b}_i}(m)=\sum_{m'}p_{\hat{a}_j,\hat{b}_i}(m',m)\,, \tag{2.10}$$

where the sums in (2.10) must be[12] independent of $j$.]

If the distributions $p_{\hat{a}_i,\hat{b}_j}$ are taken to be the quantum-theoretic ones for the singlet state, then[13]

$$\langle m^2\rangle_{\hat{a}_i}=\langle m^2\rangle_{\hat{b}_i}=s(s+1)/3 \tag{2.11}$$

and

$$\langle mm'\rangle_{\hat{a}_i\hat{b}_j}=-(\hat{a}_i\cdot\hat{b}_j)s(s+1)/3\,. \tag{2.12}$$

If these forms are substituted into (2.7) the resulting expression can be cast in the form

$$K(\hat{a}_1\hat{a}_2,\hat{b}_1\hat{b}_2)=\tfrac{1}{3}s(s+1)D_1D_2(\hat{b}_1-C_1\hat{a}_1-C_2\hat{a}_2)^2+\tfrac{1}{3}s(s+1)C_1C_2(\hat{b}_2+D_1\hat{a}_1-D_2\hat{a}_2)^2 \tag{2.13}$$

(since the "extra" terms in $\hat{a}_1\cdot\hat{a}_2$ cancel identically). But the right-hand side of (2.13) vanishes, in view of (2.1) and (2.2). Thus $K$ vanishes when the observed distributions agree with the quantum-theoretic ones.

On the other hand, returning to the definitions (2.3)–(2.5), we can construct lower bounds on $K$ as follows.

Define $g_1(m,m')$ to be the minimum over all $m_2$ in integral steps from $-s$ to $s$ of the quantity $(C_1m+C_2m_2+m')^2$,

$$g_1(m,m')=\min_{m_2}(C_1m+C_2m_2+m')^2\,, \tag{2.14}$$

and similarly define

$$g_2(m,m')=\min_{m_1}(C_1m_1+C_2m+m')^2\,, \tag{2.15}$$

$$h_1(m,m')=\min_{m_2}(-D_1m+D_2m_2+m')^2\,, \tag{2.16}$$

$$h_2(m,m')=\min_{m_1}(-D_1m_1+D_2m+m')^2\,. \tag{2.17}$$

Since the quantities $D_1D_2$ and $C_1C_2$ are positive [see (2.1) and (2.2)], and since by hypothesis the three-axis functions $F$ are non-negative, the quantity $K$ has four simple lower bounds:

$$K(\hat{a}_1\hat{a}_2,\hat{b}_1\hat{b}_2)\geq D_1D_2\sum_{m,m'} g_i(m,m')p_{\hat{a}_i,\hat{b}_1}(m,m') +C_1C_2\sum_{m,m'} h_j(m,m')p_{\hat{a}_j,\hat{b}_2}(m,m') . \quad (2.18)$$

If the distributions $p_{\hat{a}_i,\hat{b}_j}$ agree with the quantum-theoretic ones, then not all of these lower bounds can vanish. For if $C_1 \geq C_2$,[14] then

$$g_1(s,s)=s^2(1+C_1-C_2)^2>0 . \quad (2.19)$$

Combining this with the quantum-theoretic value,[15]

$$p_{\hat{a}_1,\hat{b}_1}(s,s)=(2s+1)^{-1}(\sin\tfrac{1}{2}\theta_{\hat{a}_1\hat{b}_1})^{4s} ,$$

we have the nonzero lower bound

$$K\geq D_1D_2 s^2(1+C_1-C_2)^2(2s+1)^{-1} \times(\sin\tfrac{1}{2}\theta_{\hat{a}_1\hat{b}_1})^{4s}>0 . \quad (2.20)$$

This establishes our main result.

If one wishes to know by how much the quantum-theoretic distributions violate local realism or, equivalently, how accurately the observed distributions must agree with the quantum-theoretic forms to establish a violation, then it is important to realize that (2.20) will generally grossly underestimate the actual values of the bounds (2.18) (which can, of course, be explicitly computed for any specified geometry and set of measured distributions). For most geometries, for example, $g_i$ and $h_j$ will not vanish for any values of their arguments, owing to the discreteness of the $m$'s. The question of how better to estimate these bounds is an intriguing one, but we shall not pursue it further here.

## III. A FAILURE OF LOCAL REALISM WHEN ALL THREE-AXIS FUNCTIONS EXIST

We give here a simple spin-1 counterexample to the conjecture that the existence of all three-axis functions is *sufficient* as well as necessary for local realism. The counterexample is based on distributions $p_{\hat{a}_i,\hat{b}_j}(m,m')$, hereafter abbreviated to $p_{ij}(m,m')$, associated with nine axis pairs $\hat{a}_i\hat{b}_j$, $i,j=1,2,3$, and with variables $m$ and $m'$ assuming the values 1, 0, and $-1$. We take these distributions to be given by

$$p_{ij}(m,m')=\tfrac{1}{3},\ m=m', \quad =0,\ m\neq m', \qquad ij=11,22 \quad (3.1)$$

$$p_{ij}(m,m')=f(m,m')=\tfrac{1}{27},\ m=m', \quad =\tfrac{4}{27},\ m\neq m', \qquad i\neq j \text{ or } ij=33 . \quad (3.2)$$

We note in passing that this counterexample is a natural one, in that the distributions defined by (3.1) and (3.2) are precisely the quantum-theoretic spin-1 distributions for the sets of axes[16]:

$$\hat{a}_1=3^{-1/2}(1,1,1),\ \hat{a}_2=3^{-1/2}(-1,-1,1),\ \hat{a}_3=3^{-1/2}(1,-1,-1) ,$$
$$\hat{b}_1=3^{-1/2}(-1,-1,-1),\ \hat{b}_2=3^{-1/2}(1,1,-1),\ \hat{b}_3=3^{-1/2}(1,-1,1) . \quad (3.3)$$

The set of distributions given in (3.1) and (3.2) is compatible with the existence of non-negative three-axis functions $F_{ij,k}$ and $F_{k,ij}$ with all the properties required by (1.4) and (1.5). One such set, for example, is given explicitly by the following.

Case (i), $i=j$.

$$F_{ij,k}(m_1m_2,m_3)=F_{k,ij}(m_3,m_1m_2) =\delta_{m_1m_2}p_{ik}(m_1,m_3) . \quad (3.4)$$

Case (ii), $i\neq j$ and either $k=3$ or $i\neq k\neq j$.

$$F_{ij,k}(m_1m_2,m_3)=F_{k,ij}(m_3,m_1m_2) =\tfrac{1}{27},\ m_1=m_2=m_3 , \quad =\tfrac{4}{27},\ m_1\neq m_2\neq m_3\neq m_1 , \quad =0, \text{ all other cases} . \quad (3.5)$$

Case (iii), $i\neq j$ and $k\neq 3$ and either $k=i$ or $k=j$.

$$F_{ij,k}(m_1m_2,m_3)=F_{k,ij}(m_3,m_1m_2) =3p_{ik}(m_1,m_3)p_{jk}(m_2,m_3) . \quad (3.6)$$

As defined these functions are explicitly non-negative. The reader can verify that they give the distributions (3.1) and (3.2) as marginals, i.e., that they satisfy (1.4), which in the present case amounts to the conditions

$$\sum_{m_2} F_{ij,k}(m_1m_2,m_3)=\sum_{m_2} F_{ji,k}(m_2m_1,m_3) =\sum_{m_2} F_{i,jk}(m_1,m_2m_3) =\sum_{m_2} F_{i,kj}(m_1,m_3m_2) =p_{ik}(m_1,m_3) \quad (3.7)$$

for all 27 values of $ijk$. It can also be verified by explicit computation that the functions satisfy (1.5), i.e., that

$$\sum_{m_2} F_{ij,k}(m_1m_3,m_2) \text{ and } \sum_{m_2} F_{k,ij}(m_2,m_1m_3) \tag{3.8}$$

are independent of $k$ for all nine values of $ij$.

We show next, that in spite of the existence of all the required three-axis functions, there can be no set of four-axis functions $F_{ij,kl}(m_1m_2,m_3m_4)$ that are non-negative and satisfy (1.4) and (1.5). The requirement (1.4) that the four-axis function returns the given distributions as marginals requires, in conjunction with (3.2), that

$$\sum_{m_am_b} F_{13,23}(m_1m_2,m_3m_4)=f(m_c,m_d) \tag{3.9}$$

for $ab,cd$ equal to 13,24; 14,23; 23,14; or 24,13. We first show that (3.9) must also hold for the remaining choices 12,34 and 34,12.

Condition (1.5) requires that

$$\sum_{m_3m_4} F_{13,23}(m_1m_2,m_3m_4) = \sum_{m_3m_4} F_{13,13}(m_1m_2,m_3m_4) \tag{3.10}$$

since the sum on $m_3$ must be independent of the corresponding axis. Now according to (3.1), $p_{11}(m_1,m_3)=0$, $m_1\neq m_3$. Writing $p_{11}$ as the marginal of $F_{13,13}$ we then have

$$0=\sum_{m_2m_4} F_{13,13}(m_1m_2,m_3m_4),\quad m_1\neq m_3 . \tag{3.11}$$

But (3.11) can hold for non-negative $F_{13,13}$ only if every term in the sum vanishes:

$$F_{13,13}(m_1m_2,m_3m_4)=0,\quad m_1\neq m_3 . \tag{3.12}$$

Using (3.12) to rewrite the right-hand side of (3.10), we have

$$\sum_{m_3m_4} F_{13,23}(m_1m_2,m_3m_4) = \sum_{m_4} F_{13,13}(m_1m_2,m_1m_4) . \tag{3.13}$$

But we can again use (3.12) to rewrite the right-hand side of (3.13), casting it in the form

$$\sum_{m_3m_4} F_{13,23}(m_1m_2,m_3m_4) = \sum_{m_3m_4} F_{13,13}(m_3m_2,m_1m_4) . \tag{3.14}$$

But the right-hand side of (3.14) is just the marginal of the four-axis function $F_{13,13}$ that is equal to the distribution $p_{31}(m_2,m_1)$. According to (3.2) this distribution is just $f(m_1,m_2)$ (note that $f$ is symmetric in its two variables) and therefore (3.14) reduces to (3.9) in the case $ab,cd=34,12$:

$$\sum_{m_3m_4} F_{13,23}(m_1m_2,m_3m_4)=f(m_1,m_2) . \tag{3.15}$$

The other case, $ab,cd=12,34$, is similarly dealt with:

$$\begin{aligned}\sum_{m_1m_2} F_{13,23}(m_1m_2,m_3m_4) &= \sum_{m_1m_2} F_{23,23}(m_1m_2,m_3m_4)\\ &= \sum_{m_2} F_{23,23}(m_3m_2,m_3m_4)\\ &= \sum_{m_1m_2} F_{23,23}(m_3m_2,m_1m_4)\\ &= p_{23}(m_3,m_4)\\ &= f(m_3,m_4) .\end{aligned} \tag{3.16}$$

We have therefore established that (3.9) holds for all six possible choices of the pair of variables summed over. We next use this fact to evaluate the non-negative quantity:

$$W=\sum_{\substack{m_1m_2\\ m_3m_4}} (m_1+m_2+m_3+m_4)^2 \times F_{13,23}(m_1m_2,m_3m_4) . \tag{3.17}$$

If we multiply out the square of $m_1+m_2+m_3+m_4$ then in each of the 16 terms that result the sum of $F$ on two of the indices will be given by (3.9). The 4 diagonal ($m_i^2$) and 12 off-diagonal ($m_im_j$, $i\neq j$) terms combine to give the simple result

$$W=\sum_{mm'} (4m^2+12mm')f(m,m') . \tag{3.18}$$

But the explicit form (3.2) for $f(m,m')$ gives

$$\sum_{mm'} m^2 f(m,m')=\tfrac{2}{3},\quad \sum_{mm'} mm'f(m,m')=-\tfrac{2}{9} , \tag{3.19}$$

and these, together with (3.18), require that $W=0$.

Returning to the definition (3.17) of $W$ and noting that $F_{13,23}$ must be non-negative, we conclude that $W$ can be zero only if $F_{13,23}(m_1m_2,m_3m_4)$ vanishes wherever $m_1+m_2+m_3+m_4$ does not:

$$F_{13,23}(m_1m_2,m_3m_4)=0$$
$$\text{unless } m_1+m_2+m_3+m_4=0\ . \qquad (3.20)$$

But consider now the relations (3.9) when $m_c=1$, $m_d=0$. Because of condition (3.20) there can be only two non-vanishing terms in the sum, given when the pair $m_am_b$ is either $0\bar{1}$ or $\bar{1}0$ (we define $\bar{m}=-m$). Since (3.2) gives $f(1,0)=\frac{4}{27}$, we then have for the choices $cd=12$, 13, 14, respectively,

$$\begin{aligned}\tfrac{4}{27}&=F_{13,23}(10,0\bar{1})+F_{13,23}(10,\bar{1}0)\ ,\\ \tfrac{4}{27}&=F_{13,23}(10,0\bar{1})+F_{13,23}(1\bar{1},00)\ ,\\ \tfrac{4}{27}&=F_{13,23}(10,\bar{1}0)+F_{13,23}(1\bar{1},00)\ .\end{aligned} \qquad (3.21)$$

Equations (3.21) are three linear equations in three unknowns with the unique solution

$$\begin{aligned}F_{13,23}(10,0\bar{1})&=F_{13,23}(10,\bar{1}0)\\ &=F_{13,23}(1\bar{1},00)=\tfrac{2}{27}\ .\end{aligned} \qquad (3.22)$$

Consider, finally, the relation giving $p_{32}(0,0)$ as a marginal of $F_{13,23}$:

$$p_{32}(0,0)=\sum_{mm'}F_{13,23}(m0,0m')\ . \qquad (3.23)$$

This sum includes the term $F_{13,23}(10,0\bar{1})$, and since every other term in the sum is non-negative, we have

$$p_{32}(0,0)\geq F_{13,23}(10,0\bar{1})=\tfrac{2}{27}\ . \qquad (3.24)$$

But this is incompatible with the actual value $p_{32}(0,0)=\frac{1}{27}$, given in (3.2). There can therefore be no set of non-negative four-axis functions satisfying (1.4) and (1.5).

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## APPENDIX A: EXISTENCE OF FOUR-AXIS FUNCTIONS IN THE SPIN-$\frac{1}{2}$ INVERSION SYMMETRIC CASE

We specialize here to the spin-$\frac{1}{2}$ case, proving a result we quoted without proof in Ref. 10, which shows why a spin-$\frac{1}{2}$ counterexample to Fine's conjecture must either use asymmetric distributions, or deal with $n$-axis functions with $n\geq 5$.

Suppose we are given distributions $p_{ik}(m,m')$ for all axis pairs $\hat{a}_i\hat{b}_k$, $i=1\cdots N$, $k=1\cdots N'$, and suppose all non-negative three-axis functions $F_{ij,k}(m_1m_2,m_3)$ and $F_{i,kl}(m_1,m_2m_3)$ exist and satisfy (1.4) and (1.5):

$$\begin{aligned}\sum_{m_2}F_{ij,k}(m_1m_2,m_3)&=\sum_{m_2}F_{ji,k}(m_2m_1,m_3)\\ &=\sum_{m_2}F_{i,lk}(m_1,m_2m_3)\\ &=\sum_{m_2}F_{i,kl}(m_1,m_3m_2)\\ &=p_{ik}(m_1,m_3)\ ,\end{aligned} \qquad (A1)$$

and

$$\sum_{m_2}F_{ij,k}(m_1m_3,m_2)=r_{ij}(m_1m_3)\ ,\quad \text{independent of } k\ ,$$
$$\sum_{m_2}F_{i,kl}(m_2,m_1m_3)=s_{kl}(m_1m_3)\ ,\quad \text{independent of } i\ . \qquad (A2)$$

We show that a full set of non-negative four-axis functions $F_{ij,kl}(m_1m_2,m_3m_4)$ can be found satisfying (1.4) and (1.5) *provided* the given distributions $p_{ik}(m,m')$ all have inversion symmetry[17]:

$$p_{ik}(\bar{m},\bar{m}')=p_{ik}(m,m')\quad (\bar{m}=-m,\ \text{etc.})\ . \qquad (A3)$$

To establish the existence of the required four-axis functions we first define a set of inversion-symmetrized three-axis functions:

$$\begin{aligned}G_{ij,k}(m_1m_2,m_3)&=\tfrac{1}{2}F_{ij,k}(m_1m_2,m_3)+\tfrac{1}{2}F_{ij,k}(\bar{m}_1\bar{m}_2,\bar{m}_3)\ ,\\ G_{i,kl}(m_1,m_2m_3)&=\tfrac{1}{2}F_{i,kl}(m_1,m_2m_3)+\tfrac{1}{2}F_{i,kl}(\bar{m}_1,\bar{m}_2\bar{m}_3)\ .\end{aligned} \qquad (A4)$$

Because the $F$'s are non-negative, so are the $G$'s. Because the given distributions $p$ satisfy (A3), it follows that the $G$'s satisfy the same condition (A1) as the $F$'s:

$$\begin{aligned}\sum_{m_2}G_{ij,k}(m_1m_2,m_3)&=\sum_{m_2}G_{ji,k}(m_2m_1,m_3)\\ &=\sum_{m_2}G_{i,lk}(m_1,m_2m_3)\\ &=\sum_{m_2}G_{i,kl}(m_1,m_3m_2)\\ &=p_{ik}(m_1,m_3)\ .\end{aligned} \qquad (A5)$$

Finally, the $G$'s satisfy

$$\sum_{m_2} G_{ij,k}(m_1m_3,m_2)=R_{ij}(m_1m_3)\ , \quad \text{independent of } k\ ,$$

$$\sum_{m_2} G_{i,kl}(m_2,m_1m_3)=S_{kl}(m_1m_3)\ , \quad \text{independent of } i\ , \tag{A6}$$

where $R_{ij}$ and $S_{kl}$ are the inversion-symmetrized forms of the $r_{ij}$ and $s_{kl}$ appearing in (A2):

$$R_{ij}(m_1m_3)=\tfrac{1}{2}r_{ij}(m_1m_3)+\tfrac{1}{2}r_{ij}(\bar{m}_1\bar{m}_3)\ ,$$
$$S_{kl}(m_1m_3)=\tfrac{1}{2}s_{kl}(m_1m_3)+\tfrac{1}{2}s_{kl}(\bar{m}_1\bar{m}_3)\ . \tag{A7}$$

Suppose we were able to find non-negative functions $F_{ij,kl}(m_1m_2,m_3m_4)$ that returned the $G$'s as marginals; i.e., the $F$'s obeyed

$$\sum_{m_4} F_{ij,kl}(m_1m_2,m_3m_4)=G_{ij,k}(m_1m_2,m_3) \tag{A8}$$

and the corresponding three equations in which $m_3$, $m_2$, and $m_1$ were summed on. The very structure of these equations would guarantee that the $F$'s obeyed condition (1.5), and property (A5) of the $G$'s would ensure that the $F$'s returned the given distributions as marginals, as required by (1.4). The problem therefore reduces to showing that such a set of $F$'s can be constructed when the given distributions satisfy condition (A3).

If we ignore the requirement of non-negativity it is easy to specify a family of four-axis functions returning the $G$'s as marginals. Define

$$\begin{aligned}F^0_{ij,kl}(m_1m_2,m_3m_4)=&\tfrac{1}{2}G_{ij,k}(m_1m_2,m_3)+\tfrac{1}{2}G_{ij,l}(m_1m_2,m_4)+\tfrac{1}{2}G_{i,kl}(m_1,m_3m_4)+\tfrac{1}{2}G_{j,kl}(m_2,m_3m_4)\\&-\tfrac{1}{4}p_{ik}(m_1,m_3)-\tfrac{1}{4}p_{il}(m_1,m_4)-\tfrac{1}{4}p_{jk}(m_2,m_3)-\tfrac{1}{4}p_{jl}(m_2,m_4)-\tfrac{1}{4}R_{ij}(m_1m_2)\\&-\tfrac{1}{4}S_{kl}(m_3m_4)+\tfrac{3}{16}\ .\end{aligned} \tag{A9}$$

It follows from (A5), (A6), and the inversion symmetry of the two-axis functions[18] $p$, $R$, and $S$, that the $F^0$'s do indeed return the $G$'s as marginals. Furthermore, so do the functions

$$F_{ij,kl}(m_1m_2,m_3m_4)=F^0_{ij,kl}(m_1m_2,m_3m_4)+m_1m_2m_3m_4W_{ij,kl} \tag{A10}$$

for any set of numbers $W$, since the sum over any $m$ of the product $m_1m_2m_3m_4$ vanishes. We complete the proof by showing that numbers $W_{ij,kl}$ can always be found to make the functions $F_{ij,kl}$ defined in (A10) non-negative. We can do this separately for each set of indices $ij,kl$, and therefore drop these subscripts for the rest of the argument.

We require a number $W$ that will make the function

$$F(m_1m_2,m_3m_4)=F^0(m_1m_2,m_3m_4)+m_1m_2m_3m_4W \tag{A11}$$

non-negative for all 16 ways of assigning the values $\pm\frac{1}{2}$ to the variables $m_1\cdots m_4$. Since the term in $W$ assumes only the values $\pm W/16$, we require

$$F^0(m_1m_2,m_3m_4)+W/16\geq 0\ , \quad m_1m_2m_3m_4 \text{ positive}\ , \tag{A12}$$

$$F^0(m_1m_2,m_3m_4)-W/16\geq 0\ , \quad m_1m_2m_3m_4 \text{ negative}\ . \tag{A13}$$

Such a $W$ can be found if and only if every upper bound on $W/16$ given by (A13) exceeds every lower bound given by (A12)

$$F^0(m_1^0m_2^0,m_3^0m_4^0)\geq -F^0(m_1^1m_2^1,m_3^1m_4^1)\ , \tag{A14}$$

where the $m_i^0$ are any set of $m_i$ with negative product and the $m_i^1$, any set with positive product. This condition can be put more symmetrically in the completely equivalent form,

$$F^0(m_1m_2,m_3m_4)+F^0(m_1'm_2',m_3'm_4')\geq 0\ , \tag{A15}$$

where the $m_i$ are unrestricted, and the $m_i'$ are obtained from the $m_i$ by changing an odd number of signs.

We now make our only essential use of the assumption of inversion symmetry. Because all the functions on the right-hand side of (A9) are inversion symmetric, it follows that $F^0$ has inversion symmetry:

$$F^0(m_1m_2,m_3m_4)=F^0(\bar{m}_1\bar{m}_2,\bar{m}_3\bar{m}_4)\ (\bar{m}_i=-m_i)\ . \tag{A16}$$

It is therefore enough to consider the case in which the $m_i'$ in (A15) are obtained from the $m_i$ by a single change of sign (since in the only other case—three sign changes—we could, in addition, reverse the signs of *all* the $m_i'$ without altering the value of $F^0$).

Suppose it is $m_4'$ that differs in sign from $m_4$. Then (A15) requires

$$F^0(m_1m_2,m_3m_4)+F^0(m_1m_2,m_3\overline{m}_4)\geq 0 \ . \tag{A17}$$

But since the $m_i$ only can assume the values $\pm\frac{1}{2}$, the left-hand side of (A17) is just

$$\sum_{m_4} F^0(m_1m_2,m_3m_4) \tag{A18}$$

which is precisely the three-axis function $G(m_1m_2,m_3)$. Condition (A17) is thus guaranteed by the non-negativity of the three-axis functions. Evidently the same conclusion holds if the sign reversal is made for any other value of $i$, and the result is therefore established.

This result should be contrasted with an apparently similar conclusion of Fine.[8] Fine constructs his four-axis function in a very different way, which does not require the given distributions to be inversion symmetric. However his construction fails to satisfy condition (1.5) unless $N$ or $N'$ is less than 3, whereas ours has all the required properties for any $N$ and $N'$.

## APPENDIX B: WHY THE CLAUSER-HORNE TEST REQUIRES FOUR AXES

Suppose we are given the distributions characterizing the Einstein-Podolsky-Rosen data in two sets of runs characterized by axis pairs $\hat{a}_1\hat{b}$ and $\hat{a}_2\hat{b}$, and suppose those distributions satisfy the condition

$$\sum_{m_1} p_{\hat{a}_1,\hat{b}}(m_1,m)=\sum_{m_2} p_{\hat{a}_2,\hat{b}}(m_2,m)=p_{\hat{b}}(m) \tag{B1}$$

required by the constraints (1.1) and (1.2) of local realism. One can then always find a non-negative three-axis function with the required marginals by a simple construction similar to one used by Fine in Ref. 8:

$$\begin{aligned} &F_{\hat{a}_1\hat{a}_2,\hat{b}}(m_1m_2,m)=0, \ \text{if } p_{\hat{b}}(m)=0 \ , \\ &F_{\hat{a}_1\hat{a}_2,\hat{b}}(m_1m_2,m)=p_{\hat{a}_1,\hat{b}}(m_1,m)p_{\hat{a}_2,\hat{b}}(m_2,m)/p_{\hat{b}}(m), \ \text{if } p_{\hat{b}}(m)\neq 0 \ . \end{aligned} \tag{B2}$$

One easily verifies that (1.4) holds, and (1.5) is without content, since there is not enough data available to test it. It is thus necessary to have data from at least two orientations of each[19] detector to test local realism.

Note that if one tries to use the construction (B2) in the four-axis case, defining $F_{\hat{a}_1\hat{a}_2,\hat{b}_i}$ by (B2) with $\hat{b}$ set equal to $\hat{b}_i$ for $i=1,2$, then condition (1.4) continues to be satisfied. However condition (1.5) now has content and is in general not satisfied, since there is nothing in the construction to ensure that

$$\sum_m F_{\hat{a}_1\hat{a}_2,\hat{b}_1}(m_1m_2,m)=\sum_m F_{\hat{a}_1\hat{a}_2,\hat{b}_2}(m_1m_2,m) \ . \tag{B3}$$

Indeed, we know that in general it is impossible to ensure condition (B3), since we know that in general the Clauser-Horne conditions are violated. This limitation to the usefulness of (B2) in constructing three-axis functions from two-axis distributions is entirely analogous to our criticism[10] of Fine's method for constructing four-axis functions from three-axis functions.

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[1]A. Einstein *et al.*, Phys. Rev. 47, 777 (1935).

[2]D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, New Jersey, 1951), pp. 614—619.

[3]N. D. Mermin, Phys. Rev. D 22, 356 (1980).

[4]N. D. Mermin and Gina M. Schwarz, Found. Phys. 12, 101 (1982).

[5]An elegant discussion of the representation (1.1) can be found in J. S. Bell, J. Phys. (Paris) 42, C2-41 (1981).

[6]If there are two directions $\hat{a}$ and only one direction $\hat{b}$, then it is easy to find a non-negative $F_{\hat{a}_1\hat{a}_2,\hat{b}}$ satisfying (1.4) and (1.5) (see Appendix B).

[7]J. F. Clauser and M. A. Horne, Phys. Rev. D 10, 526 (1974). For a review of these inequalities and their relation to the earlier inequalities of Clauser, Horne, Shimony, and Holt, or of J. S. Bell, see J. F. Clauser and A. Shimony, Rep. Prog. Phys. 41, 1881 (1978). It was shown in Ref. 4 that for general spin $s$ the necessary and sufficient conditions for the existence of non-negative three-axis functions satisfying (1.4) and (1.5) continue to be a set of inequalities that the pair distributions must satisfy. We shall refer to such a set of necessary and sufficient conditions for any spin as "Clauser-Horne inequalities"; the spin-$s$ Clauser-Horne inequalities are, of course, considerably more intricate

than the spin-$\frac{1}{2}$ set.

[8]Sufficiency has been proved by Arthur Fine, Phys. Rev. Lett. 48, 291 (1982). An alternative proof, valid when the distributions $p$ are inversion symmetric, can be found in Ref. 4.

[9]We have used a similar approach to derive spin-$s$ Bell inequalities. See Anupam Garg and N. D. Mermin, Phys. Rev. Lett. 49, 901 (1982). Our present argument is somewhat more intricate and our conclusions are different, because we must now deal with the fact that some of the marginals of the three-axis functions are not given by observable distributions.

[10]This is pointed out in a comment on Ref. 8, Anupam Garg and N. D. Mermin, Phys. Rev. Lett. 49, 242 (1982). See also Arthur Fine, *ibid.* 49, 243 (1982).

[11]Since writing this we have found counterexamples of both types. See Anupam Garg and N. D. Mermin, Phys. Rev. Lett. 49, 1220 (1982).

[12]If the sums did depend on $j$ this would violate (1.1) and (1.2) and local realism could be ruled out without resorting to the study of possible three-axis functions. (Note also that the quantum-theoretic distributions meet this requirement.)

[13]For an elementary proof see, for example, N. D. Mermin, Phys. Rev. D 22, 356 (1980).

[14]The case $C_2 > C_1$ can be dealt with in the same way.

[15]See, for example, Ref. 4, Eq. (2.1) and L. D. Landau and E. M. Lifshitz, *Quantum Mechanics,* 3rd edition (Pergamon, New York, 1977), Eq. (58.26).

[16]This observation has no bearing on the argument that follows, but it does indicate that there is nothing pathological or contrived about our counterexample. We show in Appendix A that a similarly "natural" counterexample cannot be constructed for spin-$\frac{1}{2}$ at the level of four-axis functions.

[17]Inversion symmetry requires that $p_{\hat{a},\hat{b}}(m,m') = p_{-\hat{a},-\hat{b}}(m,m')$. In addition, $p_{\hat{a},\hat{b}}(m,m') = p_{-\hat{a},-\hat{b}}(\overline{m},\overline{m}')$, since a spin component of $-m$ along an axis $-\hat{a}$ is identical to a spin component $m$ along $\hat{a}$. Thus inversion symmetry also requires that $p_{\hat{a},\hat{b}}(\overline{m},\overline{m}') = p_{\hat{a},\hat{b}}(m,m')$.

[18]If the two-axis functions are inversion symmetric so are their one-axis marginals. But $p(m) = p(\overline{m})$ requires $p(m) = p(\overline{m}) = \frac{1}{2}$, in the spin-$\frac{1}{2}$ case. This permits the final term in (A19) to be simply $\frac{3}{16}$, rather than the sum of one-axis functions it would have to be in the general case.

[19]If there is only one axis $\hat{b}$, then three-axis functions of the required form are given for any number of axes $\hat{a}$, by definitions (B2) for all possible pairs $\hat{a}_i\hat{a}_j$.