New analytical stellar model in general relativity

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A new analytical solution has been obtained for stellar models by solving Einstein's field equation for the spherically symmetric and static case. The variation of density is smooth and gradual. The density remains positive under all conditions. For all finite pressures the configurations are stable under radial perturbations. For $dP/d\rho \leq 1$, the maximum mass of a neutron-star model is $4.56M_{\odot}$, and the surface and the central redshifts are 0.787 and 2.673, respectively. For an infinite central pressure the surface redshift is 1.575 which is greater than that for any other analytical solution with varying density.

I. INTRODUCTION

The study of fluid spheres in general relativity requires the solution of Einstein's field equations. Due to the nonlinearity of these equations it becomes difficult to obtain analytic solutions. Oppenheimer and Volkoff[1] were the first to give a computational solution of these equations for a degenerate neutron gas, while in the same issue of Physical Review, Tolman[2] showed the importance of analytic solutions by putting forth eight different exact solutions of Einstein's field equations. Five of them, namely III, IV, V, VI, and VII, were pertaining to the fluid spheres, and they have found applications to physical problems. Tolman's solutions have been discussed in detail by Durgapal and Pande.[3] The III solution corresponds to a sphere of constant density,[4] but for it we have $dP/d\rho = -\infty$. The V and VI solutions correspond to infinite central density. In the IV solution the value of P/ρ is not maximum at the center for u (= mass/radius) > 0.25, and hence it has a limited range of applicability. The VII solution has been applied to the neutron-star problem,[5] but it has been found to be unstable under radial perturbations.[6]

Though a large number of numerically computed solutions are available in the literature, the simplicity and elegance of the exact solutions to give a clear understanding of the internal structure of a relativistic sphere cannot be denied. In view of the shortcomings of Tolman's solutions mentioned above, a new exact solution is always welcome. In this paper, we have proposed a new analytic solution of Einstein's field equations which is free from any of the shortcomings of Tolman's solutions.

The general assumptions made for solving Einstein's field equations are the following. (a) The system is spherically symmetric and static. (b) The space-time is everywhere regular. The origin is taken as the center of spherical symmetry. (c) At the center of the system $e^{\lambda(0)} = 1$. (d) The space is empty outside a finite region of radius a. (e) At $r = a$, the internal solutions are continuous with the exterior Schwarzschild solutions. That is, at $r = a$ we have $e^{-\lambda(a)} = e^{\nu(a)} = 1 - 2u$ and $P(r = a) = 0$, where $u = m/a$ = mass/radius of the fluid sphere. (f) The pressure is positive and finite at all points within the structure. (g) The density is positive and finite at all points within the structure and $d\rho/dr < 0$. For obtaining the analytic solution conforming to all the above assumptions, it is assumed that the density is given by

$$\rho = (3C/16\pi)(3 + Cr^2)/(1 + Cr^2)^2 \ .$$

(h) The pressure and the density follow certain restrictions depending upon the constraints put upon them, viz.

(i) The trace of the energy-momentum tensor is not negative,

$$P \leq \tfrac{1}{3}\rho c^2 \ .$$

(ii) The signal cannot propagate with a speed greater than that of light,[7] $P \leq \rho c^2$ and $dP/d\rho \leq c^2$. In some cases the speed of the signal is not given by $\sqrt{dP/d\rho}$ (Ref. 8) and there will be no violation of causality if $dP/d\rho$ or P/ρ exceed c^2 in these cases.

(iii) The pressure at the center should be finite, $P<\infty$.

Taking the velocity of light $c=1$ and the gravitational constant $G=1$, the relations between the density ρ, the pressure P, and the energy-momentum tensor of a perfect fluid are given by

$$\rho=T_0^0,\quad P=-T_1^1=-T_2^2=-T_3^3\ . \tag{1}$$

II. FIELD EQUATIONS AND THEIR SOLUTIONS

Field equations

The line element is given by

$$ds^2=g_{00}dt^2+g_{ij}dx^i dx^j\ , \tag{2}$$

where $(i,j=1,2,3)$

$$g_{00}=e^{\nu(r)},\quad g_{11}=-e^{\lambda(r)},\quad g_{22}=-r^2\ ,$$

$$g_{33}=-r^2\sin^2\theta,\quad g_{ij}=0 \text{ for } i\neq j\ .$$

Here ν and λ are functions of r alone. The resulting field equations are

$$-8\pi T_1^1=8\pi P=e^{-\lambda}[(d\nu/dr)r^{-1}+r^{-2}]-r^{-2}\ , \tag{3}$$

$$\begin{aligned}-8\pi T_2^2=-8\pi T_3^3&=8\pi P\\ &=e^{-\lambda}[\tfrac{1}{2}(d^2\nu/dr^2)+\tfrac{1}{4}(d\nu/dr)^2\\ &\quad-\tfrac{1}{4}(d\nu/dr)(d\lambda/dr)\\ &\quad+\tfrac{1}{2}(d\nu/dr-d\lambda/dr)r^{-1}]\ ,\end{aligned} \tag{4}$$

$$8\pi T_0^0=8\pi\rho=e^{-\lambda}[(d\lambda/dr)r^{-1}-r^{-2}]+r^{-2}\ . \tag{5}$$

Let us assume the density as

$$\rho=(3C/16\pi)(3+Cr^2)/(1+Cr^2)^2\ . \tag{6}$$

From Eqs. (5) and (6) we get

$$\begin{aligned}e^{-\lambda}&=1-\frac{1}{r}\int 8\pi\rho r^2 dr+\frac{K}{r}\\ &=1-(3Cr^2)/2(1+Cr^2)+K/r\ ,\end{aligned} \tag{7}$$

where K is a constant. For making the solution regular at the origin we must take K equal to zero. Thus

$$e^{-\lambda}=(2-Cr^2)/2(1+Cr^2)\ . \tag{8}$$

Let us now assume

$$Cr^2=x,\quad e^{-\lambda}=Z,\ \text{and}\ e^{\nu}=Ay^2\ , \tag{9}$$

where A is a constant. Equations (3) and (4) can now be expressed as

$$8\pi P/C=4Z(y'/y)-(1-Z)/x\ , \tag{10}$$

$$(4Zx^2)y''+(2Z'x^2)y'+(Z'x-Z+1)y=0\ , \tag{11}$$

where the primes denote differentiation with respect to x. Substituting

$$Z=(2-x)/2(1+x) \tag{12}$$

into Eq. (11), we obtain a particular solution of this equation as

$$y=y_i=(1+x)^{3/2}\ . \tag{13}$$

A more general solution of the Eq. (11) is obtained as

$$y=y_i\left[1+B\int dx/{y_i}^2 Z^{1/2}\right] \tag{14}$$

or

$$y=(1+x)^{3/2}+B(2-x)^{1/2}(5+2x)\ , \tag{15}$$

where B is a constant. From Eqs. (10), (12), and (15) the expression for the pressure can be obtained as

$$\frac{8\pi P}{C}=\frac{9(1-x)(1+x)^{1/2}-B(17+14x)(2-x)^{1/2}}{2(1+x)[(1+x)^{3/2}+B(5+2x)(2-x)^{1/2}]}\ . \tag{16}$$

The constants A, B, and C appearing in the solutions can be evaluated from the boundary conditions given in Sec. I, assumption (e). Thus

$$\begin{aligned}&Ca^2=X=4u/(3-4u)\ ,\\ &B=9(1-X)(1+X)^{1/2}/(17+14X)(2-X)^{1/2}\ ,\\ &A=(1-2u)[(1+X)^{3/2}\\ &\qquad+B(5+2X)(2-X)^{1/2}]^{-2}\ .\end{aligned} \tag{17}$$

III. APPLICATIONS

Some useful quantities in relativistic objects which can be obtained by simple calculations are the surface red-shift, the central red-shift, and the size a_N and mass m_N of a neutron-star model.

(a) The surface red-shift is given by

$$1+z_s=(1-2u)^{-1/2}\ .$$

(b) The central red-shift is given by

$$1+z_c=A^{-1/2}(1+5B\sqrt{2})^{-1} .$$

(c) The neutron-star model can be constructed by considering the surface density $\rho_s=2\times10^{14}$ g/cc (Ref. 9) or by using the P vs ρ curve of a standard equation of state for nuclear matter and setting the scale of central density.[10] Here we have made use of the former method for our calculations. From Eq. (6) we obtain

$$16\pi\rho_s a^2=3X(3+X)/(1+X)^2$$

or

$$a_N=13.375u^{1/2}(9-8u)^{1/2}\text{ km}$$

and

$$m_N=9.074u^{3/2}(9-8u)^{1/2}M_\odot .$$

The limiting values of z_s, z_c, a_N, and m_N will be determined from the restrictions on the pressure and density inside the configuration.

Limitations on pressure and density

From Eqs. (6) and (16) we see that the ratio of pressure and density at the center σ, is given by

$$\sigma=P_0/\rho_0=(9-17B\sqrt{2})/9(1+5B\sqrt{2}) .$$

For $\sigma=\frac{1}{3}$, we have $B=3\sqrt{2}/32$, which corresponds to a value of $X=0.1712$ and $u=0.29609$. The surface red-shift and the central red-shift are 0.566 and 1.474, respectively. The radius and the mass of the neutron-star model are 18.74 km and 3.76 $M_\odot$, respectively. For $\sigma=1$, we have

$$B=0,\quad X=1,\quad u=3.375 ,$$

$$z_s=1.0,\text{ and } z_c=4.657 .$$

For $\sigma=\infty$, we have

$$B=\sqrt{2}/10,\quad X=1.305,\quad u=0.4246 ,$$

$$z_s=1.575,\text{ and } z_c=\infty .$$

The value of $dP/d\rho$ can be calculated very easily. For $u\leq0.3434$ the value of $dP/d\rho\leq1$ throughout the configuration. For $u>0.3434$ the value of $dP/d\rho$ exceeds 1 inside the configuration.

IV. STABILITY

The static character of any solution is in itself only sufficient to assure us that the solution describes a possible state of equilibrium, but it is not sufficient to tell us whether or not that state of equilibrium would be stable towards disturbances. Now, this spherically symmetric and static configuration can have a relativistic gravitational collapse before it attains a large value of central red-shift. Hence, it is necessary to study the stability of these configurations. The criterion to study the pulsational stability of a model is judged by studying its behavior by introducing a radial perturbation. Since the variation of density inside the configuration is slow and smooth, we have used a variational method[11,12] to ascertain the pulsational stability. It is seen that the solutions with finite central pressure are stable under radial perturbation.

TABLE I. Some important parameters.

u	σ	z_s	z_0	a_N (km)	$m(M_\odot)$
0.05	0.022	0.054	0.095	8.77	0.0297
0.10	0.050	0.118	0.216	12.11	0.822
0.20	0.138	0.291	0.606	16.27	2.208
0.2961	0.333	0.566	1.438	18.74	3.764
0.3434	0.584	0.787	2.673	19.60	4.566
0.375	1.000	1.000	4.657	20.06	5.104
0.4246	Infinite	1.575	Infinite	21.90	5.940

V. DISCUSSION

The new analytic solution given here has a finite and positive density with a very smooth variation. The values of z_s, z_c, a_N, and m_N are shown for various values of u in Table I. In the extreme limiting condition $\sigma=\infty$, the values of u and z_s are found to be greater than the corresponding values for other analytic solutions with varying density ($d\rho/dr<0$). The maximum mass of a neutron star for $dP/d\rho\leq1$ is 4.56 $M_\odot$ which is just short of the maximum mass obtained under similar conditions by computations.[9] The stability of the structures under radial perturbations for extremely large central red-shifts makes this solution important for applications to physical problems.

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