New sets of asymptotically flat static and stationary solutions

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Asymptotically flat stationary solutions of the Einstein and Einstein-Maxwell equations, including the Kerr solution, are generated from solutions of Laplace's equation.

I. INTRODUCTION

Axially symmetric exact solutions with physical relevance are powerful guides for better understanding the implications of general relativity. A lot of work is, therefore, done in this field which falls into the category of axially symmetric stationary solutions. Nevertheless, only a few such solutions appear to be physically interesting. The situation for electrovac fields is not much better.

In a paper by Herlt,[1] several transformation techniques have been applied to the Van Stockum solution, and in particular the Kerr solution was reproduced. In the vacuum case, the central problem of maintaining a real metric with proper signature remains unmanageable, and no further progress has thus far been made. However, Herlt's technique beautifully applies to electrovac cases, where fewer difficulties are encountered. As regards the generation technique for stationary Einstein-Maxwell fields given by Israel and Wilson[2] and by Perjes,[3] the situation is not much better. Very few physically realistic axially symmetric stationary solutions of the Einstein-Maxwell equations have been worked out their way.

In this paper we show that physically realistic axially symmetric vacuum stationary solutions are obtainable from solutions of Laplace's equation by the same combinations of transformations given by Herlt.[1] A new stationary axially symmetric vacuum solution of the Einstein equations is presented here starting from a solution of Laplace's equation. The same solution of Laplace's equation is used as an input to the Israel-Wilson and Perjes technique, and another new axially symmetric stationary solution of Einstein-Maxwell equations is derived. Thirdly, a new set of static axially symmetric electrovac solutions is also presented.

In Sec. II, Herlt's procedure is briefly reviewed and it is shown how Laplace's equation may be considered to play a vital role instead of Eq. (2.2). Section III gives the new stationary solution of Einstein equation and also the electrovac solution derived from the harmonic function used in the former case. In Sec. IV, we present the new stationary solution of the Einstein-Maxwell equations with a brief review of the techniques of Israel and Wilson and Perjes. Our conclusions follow.

II. HERLT'S PROCEDURE

The Van Stockum metric can be written as

$$ds^2=\rho^{-1/2}(d\rho^2+dz^2)+2\rho\, d\varphi\, dt-\chi_\rho dt^2\ , \tag{2.1}$$

and all the metric coefficients are derivable from the solution of the single equation[1]

$$\chi_{\rho\rho}+\chi_{zz}-\frac{1}{\rho}\chi_\rho=0\ . \tag{2.2}$$

Any axially symmetric stationary metric on the other hand can be expressed in the form

$$ds^2=f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2 d\varphi^2]-f(dt+\omega\, d\varphi)^2\ . \tag{2.3}$$

Ernst[4] simplified the Einstein field equations and showed that the metric (2.3) can be obtained from the solution of the single equation

$$(\xi\xi^*-1)\nabla^2\xi=2\xi^*\nabla\xi\cdot\nabla\xi\ , \tag{2.4}$$

where $\mathscr{E}=(\xi-1)/(\xi+1)=f+i\phi$. ϕ is known as the twist potential.

Herlt has shown that if one obtains the solution of Eq. (2.2), one can get the solution of Eq. (2.4) and consequently f and ϕ can be determined. After performing a series of transformations for obtaining realistic solutions, Herlt finally has given f and ϕ in terms of the solution of Eq. (2.2), viz.

$$f=\beta_1\beta_2\left[\frac{1}{\chi}+\frac{1}{\rho(\chi_\rho)^{-1}({\chi_\rho}^2+{\chi_z}^2)-\chi}\right]\ , \tag{2.5}$$

$$\phi=i\beta_1\beta_2\left[\frac{1}{\chi}-\frac{1}{\rho(\chi_\rho)^{-1}(\chi_\rho{}^2+\chi_z{}^2)-\chi}+D\right] . \tag{2.6}$$

A subscript after χ indicates partial differentiation. χ is a complex solution of Eq. (2.2). Unfortunately, all the complex solutions of Eq. (2.2), it is seen, do not give f and ϕ real simultaneously, which is essential for obtaining realistic axially symmetric stationary vacuum solutions of Einstein's equations. Herlt was able to identify only one choice of χ, which ultimately leaves f and ϕ real. This choice is as follows:

$$\chi=A(2+e^{i\lambda}r_1+e^{-i\lambda}r_2) , \tag{2.7}$$

where

$$r_1+r_2=2x, \quad r_1-r_2=2y . \tag{2.8}$$

x and y are the prolate spheroidal coordinates and λ and A are two arbitrary real constants. Herlt actually obtained two real solutions of Eq. (2.2), and made a complex one using $e^{\pm i\lambda}$.

We show in the following that solutions of the more familiar Laplace equation can be used as generating functions to obtain axially symmetric stationary vacuum fields instead of solutions of Eq. (2.2).

(i) If we put

$$\chi_\rho=\rho\psi , \tag{2.9}$$

Eq. (2.2) implies

$$\psi_{\rho\rho}+\psi_{zz}+\psi_\rho/\rho=0 , \tag{2.10}$$

i.e., ψ is a harmonic function which is independent of azimuthal angle φ in cylindrical coordinates (ρ,z,φ).

(ii) Equation (2.2) implies $(\rho^{-1}\chi_\rho)_\rho+(\rho^{-1}\chi_z)_z=0$. Hence there exists a potential ζ such that $\chi_\rho=\rho\zeta_z$ and $\chi_z=-\rho\zeta_\rho$. The potential ζ is found to satisfy Laplace's equation.

(iii) Again, if ζ is a harmonic function, then so is $\psi=\zeta_z$. In this paper we shall display ψ rather than ζ.

In prolate spheroidal coordinates (x,y), two simple solutions of Eq. (2.10) turn out to be

$$\psi=\frac{1}{x+y}=\frac{1}{r_1} \text{ and } \frac{1}{x-y}=\frac{1}{r_2} , \tag{2.11}$$

where

$$\rho=(x^2-1)^{1/2}(1-y^2)^{1/2} , \quad z=xy . \tag{2.12}$$

From Eqs. (2.9) and (2.11) one reproduces Herlt's result (2.7), i.e.,

$$\chi=A[2+e^{i\lambda}(x+y)+e^{-i\lambda}(x-y)] . \tag{2.13}$$

Here A and $e^{\pm i\lambda}$ are two arbitrary constants, the latter is used only to make χ complex. It is seen that the χ we obtained using the above procedure actually satisfies Eq. (2.2), while in principle it need have satisfied only

$$\frac{\partial}{\partial\rho}\left[\chi_{\rho\rho}+\chi_{zz}-\frac{1}{\rho}\chi_\rho\right]=0 .$$

From χ, given by (2.13), when used in Eqs. (2.5) and (2.6), we obtain f and ϕ. The solution so derived is of course the Kerr solution corresponding to

$$f=-\frac{\beta_1\beta_2}{2A}\,\frac{x^2\cos^2\lambda+y^2\sin^2\lambda-1}{(x\cos\lambda+1)^2+y^2\sin^2\lambda} , \tag{2.14}$$

and

$$\phi=-\frac{i\beta_1\beta_2}{2A}\,\frac{-2iy\sin\lambda}{(x\cos\lambda+1)^2+y^2\sin^2\lambda} . \tag{2.15}$$

Thus the harmonic functions (2.11) have given rise to the Kerr metric. All solutions of Laplace's equation, however, do not give f and ϕ real simultaneously, and therefore the central problem remains where it was.

We have discovered another set of harmonic functions which give f and ϕ real. The new stationary vacuum solution generated from this set is given in Sec. III. It is interesting to speculate that this set in combination with other known sets might generate the Tomimatsu and Sato metric for $\partial=2$. But this awaits further investigation.

III. STATIONARY GRAVITATIONAL AND STATIC ELECTROVAC FIELDS

In Sec. II it was shown how a complex harmonic function can be used to generate a stationary gravitational field. The central problem lies in identifying a complex solution of Laplace's equation which ultimately leaves f and ϕ real. We shall see that an arbitrary real harmonic function can be used to construct a realistic static electrovac field.

The second harmonic function in prolate spheroidal coordinates which we have identified for generating a stationary gravitational field is of the form

$$\psi=\frac{xy+1}{(x+y)^3} . \tag{3.1}$$

This solution in combination with the previous one gives, using Eq. (2.9),

$$\chi=A(x+y)+iB\frac{xy+1}{x+y}+C . \tag{3.2}$$

With the use of Eqs. (2.5) and (2.6) and after straightforward but lengthy calculations, we obtain f and ϕ; namely

$$f=\beta\frac{B^2(x^2-1)(1-y^2)-A^2(x+y)^4}{A^2C(x+y)^4+2AC^2(x+y)^3+C^3(x+y)^2+C^3(xy+1)^2} , \tag{3.3}$$

$$\phi=2\beta C\frac{(xy+1)(x+y)}{(x+y)^2(Ax+By+C)^2+C^2(xy+1)^2} . \tag{3.4}$$

It is to be noted that in the above, neither f nor ϕ is complex although χ in (3.2) is complex. Ernst[4] has already shown that the above two potentials f and ϕ are sufficient to determine the metric (2.3) uniquely. β is an arbitrary constant which is connected with A and B intimately in making the metric coefficients of (2.3) flat at spatial infinity with proper signature. The constants B and C in Eq. (3.2) and D in Eq. (2.6) are suitably adjusted to make f and ϕ real, otherwise no realistic stationary gravitational field is obtainable:

$$B=C=K \text{ and } D=-1/K . \tag{3.5}$$

The above asymptotically flat stationary gravitational solution is new. f and ϕ when expanded asymptotically take the following forms:

$$f=-\beta\left[1+2/Ax-\frac{8A^2y(2y+A)+(6y+1)}{A^2x^2}+\cdots\right] , \tag{3.6}$$

$$\phi=2\beta\{y/x^2+(1/A^2x^3)[(1+y^2)A^2-y(4y+2A)]+\cdots\} , \tag{3.7}$$

$$K=1 .$$

These expansions show that $\beta=-1$ so that f approaches unity at spatial infinity ($x\to\infty$). ϕ contains a dipole term as well as higher multipole terms and f contains a mass monopole term as well as other higher multipole terms.

Thus we obtain asymptotically flat stationary gravitational field with mass term present from a harmonic generating function. In addition, we know that static vacuum metrics are also determined by solutions of Laplace's equation. Therefore, the technique discussed is a technique which may be used to map static gravitational fields into stationary fields, and except in special cases there will be no functional relationship between f and ϕ.

In the following we discuss briefly the technique of obtaining electrovac fields with an illustration using the same solution (3.1) of Laplace's equation.

A static axially symmetric metric may be written as

$$ds^2=e^{-2u}[e^{2\gamma}(dp^2+dz^2)+\rho^2d\varphi^2] -e^{2u}dt^2 . \tag{3.8}$$

Bonnor[5] and Kramer and Neugebauer[6] discussed a theorem which maps stationary vacuum fields into static Einstein-Maxwell ones. It connects the twist potential ϕ and f [Eq. (2.3)] by

$$f\to e^u,\ \ \phi\to i(k_0/2)^{1/2}\psi \tag{3.9}$$

with electric potential ψ and metric coefficient e^{2u} of the static Einstein-Maxwell fields [Eq. (3.8)].

Equation (3.9) shows that if ϕ is originally real, then the electric potential ψ becomes imaginary. Even then elimination of the imaginary term can be achieved by redefining certain constant parameters present in f and ϕ. For a full discussion of the technique one may refer to a paper by Das and Banerji.[7]

In the following example we choose χ to be a real function and therefore get, with the use of Eqs. (2.5) and (2.6), f real but ϕ imaginary. f and ϕ so derived for a stationary metric are transformed by (3.9) into e^u and real ψ for static electrovac metric (3.8). Thus any real solution of Eq. (2.2) or, alternatively Eq. (2.10), gives rise to an electrovac metric. Here also in general no functional relationship exists between e^{2u} and ψ.

We select for illustrative purposes

$$\chi=A(x+y)+B(xy+1)/(x+y)+C \tag{3.10}$$

as a generating function which is equivalent to a solution of Laplace's equation, and we construct e^{2u} and ψ using Eqs. (2.5), (2.6), and (3.9). Thus, we find

$$e^u=-\beta\frac{A^2(x+y)^4+B^2(x^2-1)(1-y^2)}{A^2C(x+y)^4+A(c^2-B^2)(x+y)^3-B^2C[(x^2-1)(1-y^2)+2(xy+1)^2]}\,, \tag{3.11}$$

$$\psi=2\beta(2/k_0)^{1/2}\frac{BC(x+y)(xy+1)}{A^2C(x+y)^4+A(C^2-B^2)(x+y)^3-BC[(x^2-1)(1-y^2)+2(xy+1)^2]}\,. \tag{3.12}$$

The above static electrovac solution with metric (3.8) is asymptotically flat, and mass and charge as well as higher multipoles are present. In the stationary case no monopole term is present in the expression for ϕ. There, ϕ varies as $\sim x^{-2}$ at spatial infinity, which is essential for vanishing of w at infinity. This property is exposed from the study of Tomimatsu and Sato metrics.[7] Moreover, in the solution given by Eqs. (3.11) and (3.12), no restriction on constants B and C is necessary, which on the other hand, is necessary in the stationary axially symmetric case (3.5) to make f and ϕ real. As it is easy to tackle the static electrovac case, several new electrovac solutions have been given by Herlt. But as regards stationary solutions, we get only one such solution from his work, which is already known as the Kerr solution. In this paper we have presented a new stationary solution of Eqs. (2.3) and (2.4) and observe that its counterpart can similarly be constructed with the following solution of Laplace's equation instead of (3.1):

$$\psi=A+B/(x-y)+C(xy-1)/(x-y)^3\,. \tag{3.13}$$

There exists a hopeful possibility of getting a series of new stationary or electrovac fields from a combination of the series of harmonic functions presented above and other solutions already known. A third simple harmonic function in oblate spheroidal coordinates exists unnoticed, namely,

$$\psi=(3x^2y^2-x^2-4ixy+y^2-3)(x-iy)^{-5}\,. \tag{3.14}$$

It is hoped that all these solutions of Laplace's equation in suitable combination may lead to very interesting and yet unexplored stationary gravitational fields.

IV. STATIONARY EINSTEIN-MAXWELL FIELDS

Israel and Wilson[2] and Perjes[3] independently published a method of generating a solution of the stationary Einstein-Maxwell equations from a complex harmonic function. They have shown that the generated solution can be interpreted as the exterior field of a static or steadily moving distribution of charged dust having numerically equal charge and mass densities. Their method is summarized below.

The stationary axially symmetric line element

$$ds^2=f^{-1}\{e^{2\gamma}[(d\rho^2+dz^2)+\rho^2d\varphi^2]\}-f(dt-\omega\,d\varphi)^2 \tag{4.1}$$

may be constructed if we know Ernst's potentials $\mathscr{E}=f-\Phi\Phi^*-i\phi$ and Φ in detail. f, Φ, and ϕ are considered for this investigation as functions of ρ and z only. Φ and ϕ are known as complex electromagnetic scalar potential and twist potential, respectively.

Israel and Wilson and Perjes have shown that stationary electrovac solutions can be obtained directly from complex solutions of Laplace's equation when mass and charge densities become numerically equal. In this case $\mathscr{E}=2/\psi-1$, where

$$\nabla^2\psi=0 \tag{4.2}$$

and ∇^2 is the Laplacian operator. Expressing ψ in the form

$$\psi=1+L+iM\,, \tag{4.3}$$

where L and M are real, we have

$$\mathscr{E}=(1-L^2-M^2-2iM)/[(1+L)^2+M^2]\,, \tag{4.4}$$

$$\Phi=e^{i\alpha}(L+iM)/(1+L+iM)\,, \tag{4.5}$$

$$\phi=-2M/[(1+L)^2+M^2]\,, \tag{4.6}$$

and

$$f=[(1+L)^2+M^2]^{-1}\,. \tag{4.7}$$

where α is a real constant.

After a lengthy but straightforward calculation, ω may be obtained by solving the following two equations in prolate spheroidal coordinates:

$$\rho=(x^2-1)^{1/2}(1-y^2)^{1/2},\quad z=xy\,, \tag{4.8}$$

$$\omega_x=-2(1-y^2)[ML_y-(L+1)M_y]\,, \tag{4.9}$$

$$\omega_y=2(x^2-1)[ML_x-(L+1)M_x]\,, \tag{4.10}$$

where subscripts x and y indicate partial differentiations with regard to x and y, respectively.

The central problem therefore reduces to selecting a complex harmonic function. The various metric coefficients of (4.1) can be obtained then from Eqs. (4.5)–(4.7) and (4.9) and (4.10). As an illustration we shall select the harmonic function

$$\psi=1+A(x+y)^{-1}+iB(xy+1)(x+y)^{-3}\,, \tag{4.11}$$

where x and y are prolate spheroidal coordinates and A,B are arbitrary real constants. A lengthy and laborious calculation gives the metric coefficients as follows:

$$f=(x+y)^6[(x+y+A)^2+B^2(xy+1)^2]^{-1}\,, \tag{4.12}$$

$$\omega=-2(1-y^2)\left[-\frac{AB(1-y^2)}{2(x+y)^4}-\frac{3ABy+3B(1-y^2)}{3(x+y)^3}-\frac{4By-AB}{2(x+y)^2}+\frac{B}{x+y}\right]+\text{const}\,. \tag{4.13}$$

An asymptotic expansion of f takes the form

$$f=1-\frac{2A}{x}+\frac{2Ay}{x^2}-\frac{B^2}{(x+y)^2}-\frac{2xy}{(x+y)^2}-\frac{x^2y^2}{(x+y)^2}+\cdots \tag{4.14}$$

This shows that $f\to 1$ when $x\to\infty$, i.e., at spatial infinity f goes to asymptotically flat form. At infinity, it is observed that ω and Φ also vanish and therefore the derived stationary Einstein-Maxwell field is well behaved at infinity.

Perjes has pointed out that if one obtains for large values of the radial coordinate r,

$$\psi=1+\frac{m}{r}+iJ\frac{\cos\theta}{r^2}+\eta\,, \tag{4.15}$$

where η stands for real terms of order r^{-2} and for imaginary ones of order r^{-3}, and θ is the polar angle, then the derived stationary Einstein-Maxwell field is well behaved in the sense that the derived metric is asymptotically flat at large distance and ω and Φ go to zero appropriately. Our ψ given by Eq. (4.11) fits the above requirements. A drawback of the Israel-Wilson-Perjes method is that we cannot simply switch off the electromagnetic field and obtain thereby a purely gravitational analog.

Rotating stars take the form of oblate spheroidal objects and therefore solutions in oblate spheroidal coordinates are of much interest. In this coordinate system two other simple solutions of Laplace's equation besides the one given in (3.14) may be cited, namely,

$$\psi=A/(x-iy)\,, \tag{4.16}$$

and

$$\psi=i(xy-i)(x-iy)^{-3}\,. \tag{4.17}$$

The stationary Einstein-Maxwell solution derived from the potential (4.16) by the Israel-Wilson-Perjes method corresponds to

$$\begin{aligned} f&=(x^2+y^2)[(x+1)^2+y^2]^{-1}\,,\\ \Phi&=e^{i\alpha}[(x+1)-iy]^{-1}\,,\\ \phi&=-2y[(x+1)^2+y^2]^{-1}\,,\\ \omega&=(1-y^2)(2x+1)(x^2+y^2)^{-1}\,. \end{aligned} \tag{4.18}$$

This solution does not have a purely gravitational analog, although it is well behaved in the sense that $f\to 1$, $\Phi\to 0$, $\omega\to 0$ at spatial infinity. Additional solutions in oblate spheroidal coordinates may be derived from linear combinations of the potentials given in (3.14), (4.16), and (4.17).

IV. CONCLUSION

No physically realistic axially symmetric and asymptotically flat stationary gravitational solution other than the Kerr and Tomimatsu-Sato[8] series was available in the literature till a few years back. Recently Kinnersley and Chitre,[9] Hoenselaers, Kinnersley, and Xanthopoulos,[10] Yamazaki[11] *et al.* have added a few to the list but their solutions are more complicated than the former and therefore it is difficult to analyze those properly. The stationary field given by Eqs. (3.3) and (3.4) is fairly simple and no doubt constitutes an addition to the list of stationary gravitational solutions. Moreover the harmonic function (3.1), when used as input to Herlt's transformation equations (2.5) and (2.6), results in only the second example of the successful use of Herlt's procedure; namely, it leaves f and ϕ as real functions. It is hoped that suitable finite combinations of the stated solutions of Laplace's equation may be selected and mingled with $e^{\pm i\lambda}$ to give rise to the Tomimatsu-Sato series. Further investigation along these lines is suggested. With a little effort, solutions similar to (3.3) and (3.4) may be constructed from the complex harmonic function

$$\psi = A(x-y)^{-1} + iB(xy+1)(x-y)^{-3} . \quad (5.1)$$

This will be the third example of a harmonic function which gives f and ϕ real. With a procedure similar to that discussed in the previous section the above solution (5.1), when used as input to the Israel-Wilson-Perjes method, will yield another interesting solution of the stationary Einstein-Maxwell equation similar to (4.12) and (4.13). Of course, other linear combinations of the cited harmonic functions will also produce electrovac fields. Thus Herlt's transformation equations (2.5) and (2.6) are very promising and they have opened up an avenue towards generation of new stationary gravitational fields: very little work has yet been done and a lot remains unexplored.

The electrovac solution given by Eqs. (3.11) and (3.12) has been derived directly from a real harmonic function by using the transformation Eqs. (2.5) and (2.6). Another class of electrovac fields may be obtained from solutions (3.3) and (3.4) by a procedure described in Ref. 7. This procedure may lead to new solutions other than (3.11) and (3.12).

In oblate spheroidal coordinates three sets of complex harmonic functions (3.14), (4.16), and (4.17) will also generate stationary Einstein-Maxwell fields when used separately or in combination. We have constructed just one of them, (4.18). In prolate coordinates we have already obtained stationary Einstein-Maxwell fields given by Eqs. (4.12) and (4.13) from two sets of harmonic functions. The remaining ones may be utilized too.

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