Eigenvalues of 4, 8, and 16 coupled anharmonic oscillators

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We present graphs of 48–96 eigenvalues of 4, 8, and 16 anharmonic-oscillator approximations to $g{:}\phi^4{:}_2$ field theories for $g \geq 0$. The graphs show how the energy levels develop critical behavior as the number of oscillators increases. The graphs also illustrate how the large-g behaviors of the spectrum for Dirichlet and periodic boundary conditions differ.

INTRODUCTION

We present graphs of 48–96 eigenvalues of coupled-anharmonic-oscillator operators of the form

$$H^{(N)}=H^{(N)}(q_1\cdots q_N;g,\epsilon)=\frac{1}{2}\left(\sum_{j=1}^{N}[-\partial^2/\partial q_j{}^2+gq_j{}^4+(3-wg)q_j{}^2]\right)-\sum_{j=1}^{N-1}q_jq_{j+1}-\epsilon q_1q_N \tag{1}$$

for $N=4$, 8, and 16, $\epsilon=0$ or 1, and $0\leq g\leq 30$. The constant $w=w(N,\epsilon)$ is chosen so that $dm^{(N)}(g)/dg\,|_{g=0}=1$, where $m^{(N)}(g)$ is the gap between the lowest two eigenvalues of $H^{(N)}$ (this amounts to Wick ordering $gq_j{}^4$).

The operators $H^{(N)}$ result from a quantization (and Wick ordering or mass renormalization) of finite difference approximations to the classical Hamiltonian

$$\frac{1}{2}\int_{-l/2}^{l/2}(\phi_t{}^2+\phi_x{}^2+\phi^2+g\phi^4)dx \ .$$

We have taken the lattice spacing to be 1. Dirichlet boundary conditions correspond to $\epsilon=0$ and periodic boundary conditions to $\epsilon=1$.

The operators $H^{(N)}$ frequently appear in the physics literature not only because of their intrinsic interest but also because they provide a testing ground for methods designed to approximate the critical behavior and particle structure of lattice approximations to local field theories.

Several recent papers devoted to moment recursion methods for computing the eigenvalues of $H^{(1)}$ are Refs. 1–3. A method for computing the eigenvalues and eigenfunctions of $H^{(1)}$ that is uniformly accurate for all $g\geq 0$ is described in Ref. 4 (this article contains proofs of convergence and graphs of several of the lowest eigenvalues of $H^{(1)}$ for $0\leq g\leq 30$).

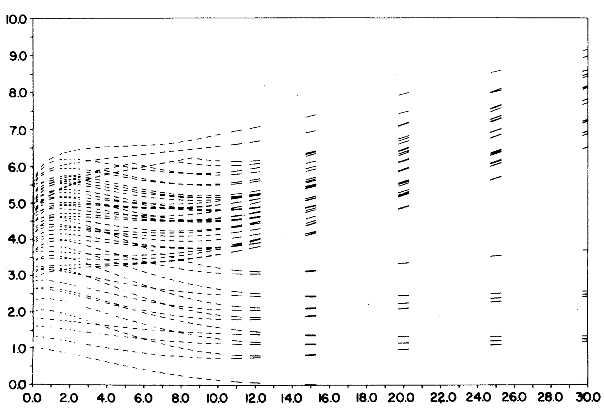

FIG. 1. Lowest 12 eigenvalues of $H^{(4)}(\epsilon=0)$ in $H_{i,j}$.

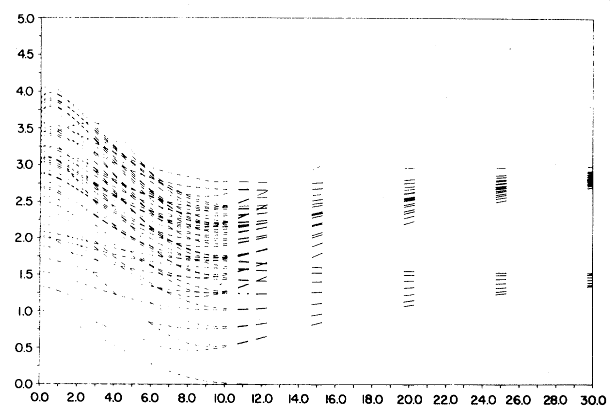

FIG. 2. Lowest 12 eigenvalues of $H^{(8)}(\epsilon=0)$ in $H_{i,j}$.

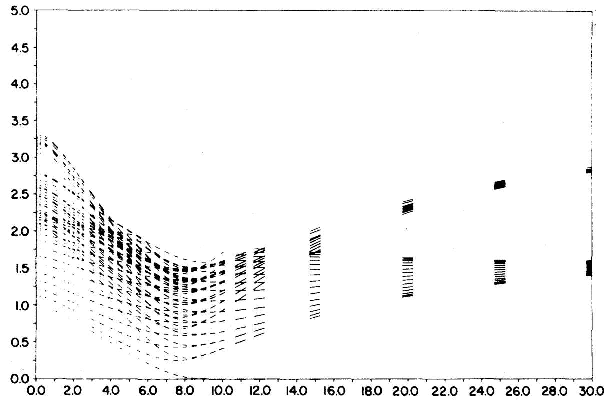

FIG. 3. Lowest 12 eigenvalues of $H^{(16)}(\epsilon=0)$ in $H_{i,j}$.

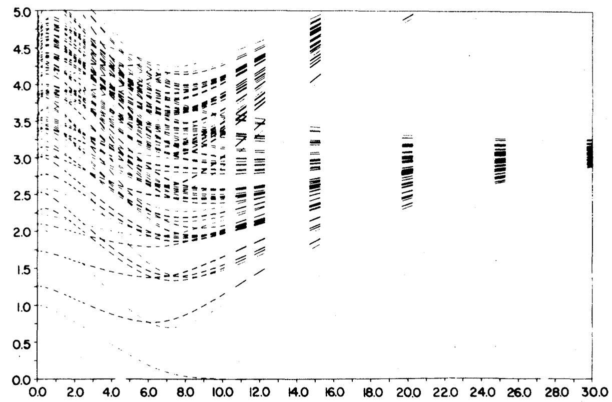

FIG. 5. Lowest 12 eigenvalues of $H^{(8)}(\epsilon=1)$ in $H_{i,j,k}$.

The moment recursion method was generalized and applied to operators of the form $H^{(2)}$ in Ref. 2 by Blankenbecler, DeGrand, and Sugar. $H^{(2)}$ is also studied in Ref. 5, where a method that is uniform in $g \geqslant 0$ is proven to converge rapidly and graphs of several of the lowest eigenvalues as functions of g are given.

Variational trial functions for the ground state of $H^{(N)}$ are constructed in Richardson and Blankenbecler,[6] Bronzan and Sugar,[7] and Pacheco.[8]

These trial functions were used to study the $N \rightarrow \infty$ limit of the vacuum energy per particle and the critical behavior of the two-point function.

In this paper we describe briefly a method (which is convergent and, for each N, uniform in $g \geqslant 0$) for computing eigenvalues and eigenfunctions of $H^{(N)}$ when $N=2^n$, $n=2,3,4$. This algorithm may be used to study the limit of $H^{(N)}$ as N tends to infinity.[9]

The main purpose of this paper is to present graphs of the energy levels of $H^{(N)}$ when $N=4$, 8, and 16. These graphs are interesting because with just 16 oscillators many features of the infinite oscillator limit can be seen. Some of the features that can be seen in Figs. 1–6 are (i) the approaching critical behavior of the mass and degeneracy of the vacuum for g large; (ii) the formation and approaching critical behavior of the two-particle threshold; (iii) the periodic energy spectrum for g large is missing the elementary particle present in the Dirichlet spectrum.

METHOD

We describe briefly the method used to approximate the energy levels of $H^{(N)}$ for $N=2^n$, $n=2,3,4,\ldots$. Details of the method are given in Ref. 10. For simplicity we describe the Dirichlet ($\epsilon=0$) case.

Let $E_j^{(N)}$ and $\Psi_j^{(N)}(q_1\cdots q_N)$ for $j=1,2,\ldots$ denote the eigenvalues and eigenfunctions of $H^{(N)}(q_1\cdots q_N)$. Here $E_1^{(N)} < E_2^{(N)} \leqslant E_3^{(N)} \leqslant \cdots$, and $H^{(N)}\Psi_j^{(N)} = E_j^{(N)}\Psi_j^{(N)}$. Assume we know the eigenvalues $E_j^{(N)}$ and matrix elements

$$X_{j,l}^{(N,\alpha)} = \int \Psi_j^{(N)}(q_1\cdots q_N)q_\alpha \times \Psi_l^{(N)}(q_1\cdots q_N)dq_1\cdots dq_N$$

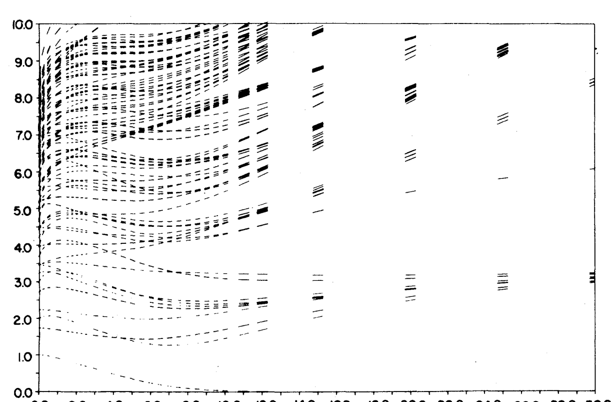

FIG. 4. Lowest 12 eigenvalues of $H^{(4)}(\epsilon=1)$ in $H_{i,j,k}$.

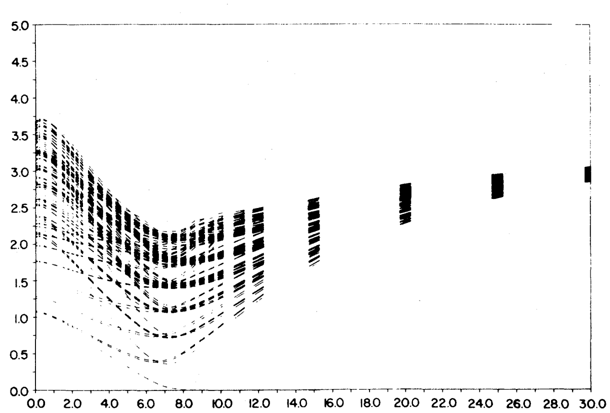

FIG. 6. Lowest 12 eigenvalues of $H^{(16)}(\epsilon=1)$ in $H_{i,j,k}$.

for $\alpha=1,N$ (computation of these for $N=2$ is explained in Ref. 5). We may compute approximations to $E_j^{(2N)}$ and $X_{j,l}^{(2N,\alpha)}$ as follows. Expand $\Psi_j^{(2N)}(q_1 \cdots q_{2N})$ in the basis

$$\{\Psi_i^{(N)}(q_1 \cdots q_N)\Psi_k^{(N)}(q_{N+1} \cdots q_{2N})\}$$

so that

$$\Psi_j^{(2N)} = \sum_{i,k=1}^{\infty} C_{i,k}^j \Psi_i^{(N)}(q_1 \cdots q_N)\Psi_k^{(N)}(q_{N+1} \cdots q_{2N}) \ .$$

Then the equation $H^{(2N)}\Psi_j^{(2N)} = E_j^{(2N)}\Psi_j^{(2N)}$ becomes equivalent to the infinite-matrix eigenvalue problem

$$\sum_{i,k=1}^{\infty} [(E_i^{(N)} + E_k^{(N)})\delta_{ii'}\delta_{kk'} - X_{i'i}^{(N,N)} X_{k'k}^{(N,1)}] C_{ik}^j = E_j^{(2N)} C_{i'k'}^j \tag{2}$$

for $i',k'=1,2,\ldots$. To approximate $E_j^{(2N)}$ just replace the infinite sum (2) by a finite one and numerically compute the lowest eigenvalues and eigenvectors of the resulting finite-dimensional eigenvalue problem.[10] Having computed approximations to $E_j^{(2N)}$ we may compute approximations to $E_j^{(4N)}$ in an analogous manner once we have approximations to $X_{j,l}^{(2N,\alpha)}$. These may be obtained by truncating the infinite sums in the identities

$$X_{j,l}^{(2N,l)} = \sum_{i,k}^{\infty} \sum_{i',k'}^{\infty} C_{i,k}^j C_{i',k}^l X_{i,i'}^{(N,l)} \delta_{kk'} \ ,$$

$$X_{j,l}^{(2N,N)} = \sum_{i,k}^{\infty} \sum_{i',k'}^{\infty} C_{i,k}^j C_{i',k}^l X_{i,i'}^{(N,N)} \delta_{kk'} \ .$$

In practice we take advantage of the fact that $H^{(N)}$ commutes with the unitary operators R and S. Here

$$R\Psi(q_1 \cdots q_N) = \Psi(q_N \cdots q_1)$$

and

$$S\Psi(q_1 \cdots q_N) = \Psi(-q_1, \cdots -q_N) \ .$$

This allows us to compute the lowest few eigenvalues in each of the 4 invariant subspaces $\mathcal{H}_{ij}$ where for $i,j=\pm 1$

$$\mathcal{H}_{ij} = \{\Psi | R\Psi = i\Psi \text{ and } S\Psi = j\Psi\} \ .$$

For periodic boundary conditions ($\epsilon=1$) $H^{(N)}$ also commutes with T, where

$$T\Psi(q_1 \cdots q_N, q_{N+1} \cdots q_{2N}) = \Psi(q_{N+1} \cdots q_{2N}, q_1 \cdots q_N) \ .$$

Thus when $\epsilon=1$ we compute the lowest few eigenvalues in each of the 8 invariant spaces

$$\mathcal{H}_{i,j,k} \equiv \{\Psi | R\Psi = i\Psi, S\Psi = j\Psi, T\Psi = k\Psi\} \ .$$

GRAPHS

Figures 1–6 were made by computing approximations to the lowest 12 eigenvalues $E_j^{(N)}(g)$ and their derivatives $\partial E_j^{(N)}(g)/\partial g$ in each of the 4 (8) invariant subspaces $\mathcal{H}_{i,j}$ ($\mathcal{H}_{i,j,k}$) for a sequence of values of g between 0 and 30.

The eigenvalues and their derivatives were computed to graphical accuracy ($\approx 1\%$). For each value of g we formed the normalized values

$$\tilde{E}_j^{(N)}(g) \equiv [E_j^{(N)}(g) - E_1^{(N)}(g)]/M^{(N)}(0)$$

and plotted a short line segment whose center had the coordinates $(g, \tilde{E}_j^{(N)}(g))$ and whose slope was $\partial \tilde{E}_j^{(N)}(g)/\partial g$.

The subtraction of $E_1^{(N)}(g)$ is just the usual vacuum energy renormalization. The division by $M^{(N)}(0)$ has little effect because for periodic boundary conditions $M^{(N)}(0)=1$, and for Dirichlet boundary conditions $M^{(N)}(0)$ converges to 1 rapidly as N approaches infinity.

For $g=0$ the graphs display the normalized eigenvalues of N coupled harmonic oscillators which are

$$\frac{\sum_{j=1}^{N} n_j [1 + 4\sin^2(j\pi/2N+2)]^{1/2}}{[1+4\sin^2(\pi/2N+2)]^{1/2}}$$

for the Dirichlet case, and

$$\sum_{j=1}^{N/2-1} (n_j + l_j)[1 + 4\sin^2(j\pi/N)]^{1/2} + n_0 + n_{N/2}\sqrt{5}$$

for the periodic case (here n_j and $l_j = 0, 1, 2, \ldots$).

The graphs show the asymptotic degeneracy of all the energy levels as g increases. Although for finite N the vacuum is never degenerate, the vacuum becomes degenerate to graphical accuracy at a sequence of values of g that decrease as N increases.

The two-particle threshold is seen to come down and cross the one-particle states and become the bottom of the gap in the two-phase (approximately degenerate vacuum) region.

A striking feature of the graphs is the difference between the Dirichlet and periodic spectra for large g. The Dirichlet spectrum has the kink (or soliton) bound state but the periodic spectrum is missing this particle. An analogous phenomenon occurs in the two-dimensional Ising model as is nicely described by O'Carroll and Schor in Ref. 11.

CONCLUSION

We have described a method for approximating the low-energy spectrum of N anharmonic-oscillator approximations to $g{:}\phi^4{:}_2$ field theories. Graphs of the spectra when $N=4$, 8, and 16 were presented which illustrated asymptotic eigenvalue degeneracy for fixed

N as g goes to infinity. The graphs also illustrated the manner in which the vacuum becomes degenerate as N increases, and the fact that the particle structure of the spectrum in general depends on the boundary conditions.

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