c-number time-energy uncertainty relation in the quark model

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The time-energy uncertainty relation is studied for the time- and energy-separation variables between two constituent quarks inside a hadron bounded by a harmonic-oscillator force. The O(3,1) and O(3)-like internal space-time symmetries discussed in a previous paper are formulated in terms of covariant commutators. It is shown that the resulting commutation relations constitute a covariant realization of Dirac's c-number time-energy uncertainty relation.

The time-energy uncertainty relation in the form of $(\Delta t)(\Delta E)\simeq 1$ was known to exist even before the present form of quantum mechanics was formulated.[1] This problem can be studied either with or without reference to the Schrödinger equation.[2–7] Indeed, the time-energy uncertainty relation is universal and is expected to hold even in systems which cannot be described by the Schrödinger equation.

In studying this uncertainty relation, we encounter the following problems:

While there exists the time-energy uncertainty relation in the real world, possibly with the form $[t,H]=-i$,[8] this commutator is zero in the case of Schrödinger quantum mechanics. As was noted by Dirac in 1927,[2] the time variable is a c number. Then, is this form of the c-number time-energy uncertainty relation universal, or true only in nonrelativistic quantum mechanics?

If the time variable is a c number and the position variables are q numbers, then the coordinate variables in a different Lorentz frame are mixtures of c and q numbers. This cannot be consistent with special relativity, as was also pointed out by Dirac.[2] These were the fundamental problems in 1927, and today's situation does not appear to be different.

The reason for this slow rate of progress is very simple. While the time-energy uncertainty relation is to be formulated from experimental observations,[9] there are not many experimental phenomena which can be regarded as direct manifestations of this relation. In fact, the connection between the lifetime and the energy width of unstable states is the only well-known direct application of this important relation.[1,4]

The purpose of the present note is to point out that the relativistic quark model is one physical example in which this point can be studied in detail. The time-separation variable between the quarks confined within a hadron plays an important role here. While this variable is not contained in the Schrödinger equation, it is an interesting proposition to see whether the uncertainty relation applicable to the time-separation variable is the same as the currently accepted form of the time-energy uncertainty relation largely based on nonrelativistic quantum mechanics.[10]

When we say that the time variable is a c number or write

$$[t,H]=0 \ , \tag{1}$$

we are implying that the Robertson procedure[11] applicable to Heisenberg's position-momentum uncertainty relation does not work here. Classically, this corresponds to the fact that t and H are not canonically conjugate variables.[10] In quantum mechanics, the above commutator means that there is no Hilbert space in which t and $i\partial/\partial t$ act as operators.[12] However, it is important to note that there still exists a "Fourier" relation between time and energy,[1,10] which limits the precision to $(\Delta t)(\Delta E)\simeq 1$.

Using the language of wave functions in the covariant harmonic-oscillator model which was formulated to explain basic high-energy features of quark-model hadrons,[6,13–15] we have pointed out in our previous paper[16] that the time-separation variable between the quarks in the hadronic rest frame exhibits the above-mentioned form of time-energy uncertainty relation. It was noted in earlier papers[6] that the parton phenomenon is a manifestation of the relation $(\Delta t')(\Delta E')\simeq 1$, where t' and E' are, respectively, the time and energy separations in the hadronic rest frame. It was emphasized that, in spite of this relation, there is no experimental evidence to indicate the existence of excitations along the time-separation axis.[13]

In addition, it was pointed out repeatedly in the literature that this form of time-energy uncertainty relation can be covariantly combined with the uncertainty relations applicable to three spatial variables.[17] This form of space-time asymmetry has been shown to be perfectly consistent with the concept of the O(3)-like little group of the Poincaré group for massive particles.[16,18]

In this paper, we would like to summarize all these into a commutator form. From a mathematical standpoint, this is not a new story. We are, in fact, quite familiar with the commutator form for vector-meson quantization which deals covariantly with vector mesons with three spatial components.[19] The question then is whether we can use a similar form to represent the above-mentioned time-energy uncertainty relation combined covariantly with position-momentum uncertainty.

Let us consider for simplicity a hadron consisting of two quarks, and let x_1 and x_2 denote the space-time coordinates for these quarks. Then the standard procedure is to define the new variables

$$X=(x_1+x_2)/2 \ , \qquad x=(x_1-x_2)/2\sqrt{2} \ , \tag{2}$$

where X and x correspond, respectively, to the overall hadronic coordinate and space-time separation between the quarks. While the hadronic space-time coordinate is specified by X, its structure is determined by the internal space-time separation between the quarks.

Without loss of generality, we assume that the hadron has definite four-momentum P_μ and moves along the z direction with velocity parameter β. In terms of the above coordinate variables, we can write down the ten generators of the Poincaré group and their Casimir operators corresponding to $(\text{mass})^2$ and $(\text{intrinsic spin})^2$ of the hadron.[17] It has been shown that the covariant harmonic-oscillator formalism, while being consistent with observed high-energy hadronic features,[6,13–15] provides relativistic wave functions which are diagonal in the Casimir operators.[16,17] While the exact form for the hadronic wave function is somewhat complicated, the essential element of the wave function takes the form

$$\psi_{nk}(x)=(1/\pi 2^{n+k} n!k!)^{1/2} H_n(z')H_k(t') \times \exp[-(z'^2+t'^2)] \ , \tag{3}$$

where

$$z'=(z-\beta t)/(1-\beta^2)^{1/2} \ ,$$
$$t'=(t-\beta z)/(1-\beta^2)^{1/2} \ .$$

In the above expression, we have suppressed all the factors which are not affected by the Lorentz transformation along the z axis. This is possible because the oscillator wave functions are separable in both the Cartesian and spherical coordinate systems. z' and t' are the longitudinal and timelike coordinate variables, respectively, in the hadronic rest frame.

In terms of the standard step-up and step-down operators,

$$a_\mu=(1/\sqrt{2})(x_\mu-\partial/\partial x^\mu) \ , \qquad a_\mu^\dagger=(1/\sqrt{2})(x_\mu+\partial/\partial x^\mu) \ , \tag{4}$$

the oscillator wave function of Eq. (3) satisfies the differential equation[17]

$$a_\mu^\dagger a^\mu \psi(x)=(\lambda+1)\psi(x) \ , \tag{5}$$

where the eigenvalue λ, together with transverse excitations, determines the $(\text{mass})^2$ of the hadron.[17]

The operators given in Eq. (4) satisfy the algebraic relation

$$[a_\mu, a_\nu^\dagger]=-g_{\mu\nu} \ . \tag{6}$$

This commutation relation is Lorentz invariant.[20] The timelike component of the above commutator is -1 in every Lorentz frame. This allows timelike excitations. Indeed, in his recent paper,[21] Rotbart discussed the covariant Hilbert space of harmonic-oscillator wave functions in which time-like excitations are allowed in all Lorentz frames.

On the other hand, there is no evidence to indicate the existence of such timelike excitations in the real world.[13] This is perfectly consistent with the fact that the basic spacetime symmetry of confined quarks is that of the O(3)-like little group of the Poincaré group.[16,18] We can suppress timelike excitations in the hadronic rest frame by imposing the subsidiary condition[16,17,22,23]

$$P^\mu a_\mu^\dagger \psi_{nk}(x)=0 \ , \tag{7}$$

where P_μ is the hadronic four-momentum. Then only the solutions with $k=0$ are allowed, and the commutator given in Eq. (6) is not consistent with the above subsidiary condition.

How can we then construct a covariant commutator consistent with Eq. (7)? In order to attack this problem, let us divide the four-dimensional Minkowskian space-time into the one-dimensional timelike space parallel to the hadronic four-momentum and to the three-dimensional space-like hyperplane perpendicular to the four-momentum.[24] This hyperplane accommodates the internal space-time symmetry dictated by the O(3)-like little group.[16,18] This leads us to consider the operator

$$b_\mu=a_\mu-(P_\mu P^\nu/M^2)a_\nu \ . \tag{8}$$

Then b_μ satisfies the constraint condition

$$P^\mu b_\mu=P^\mu b_\mu^\dagger=0 \ , \tag{9}$$

and has only three independent components. Thus, for b_μ and $b_\mu^\dagger$, we can write the covariant commutation relation[25]

$$[b_\mu, b_\nu^\dagger] = -g_{\mu\nu} + P_\mu P_\nu / M^2 \ . \tag{10}$$

The right-hand side of the above expression is symmetric in μ and ν, and satisfies the relation

$$P^\mu(-g_{\mu\nu} + P_\mu P_\nu / M^2) = 0 \ . \tag{11}$$

Therefore the covariant commutation relation given in Eq. (10) is consistent with the subsidiary condition of Eq. (7).

The covariant form of Eq. (10) represents the usual Heisenberg uncertainty relations on the three-dimensional spacelike hypersurface perpendicular to the hadronic four-momentum. This form enables us to treat separately the uncertainty relation applicable to the timelike direction, without destroying covariance. The existence of the t' distribution due to the ground-state wave function in Eq. (3) restricted by Eq. (7) allows us to write the time-energy uncertainty relation in the form

$$(\Delta t')(\Delta E') \simeq 1 \ , \tag{12}$$

without postulating the commutation relation. E' in this case is the energy separation between the quarks in the Lorentz frame in which the hadron is at rest. The parton phenomenon,[6] together with other high-energy features in the quark model,[13–15] indicates clearly the existence of this uncertainty relation.

As has been expected, the uncertainty relation exists between the time- and energy-separation variables in the quark model. The remarkable fact is that this uncertainty relation is just like the one expected in all other physical phenomena.[10]

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