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Slowly rotating fluid spheres in general relativity

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We show that solutions of Einstein's field equations for static fluid spheres can be immediately generalized to slow differential rotation with a particular form for the rotation function, without solving any additional differential equations.

In a recent article[1] we introduced slow rotation to some of the solutions given by Vaidya, which correspond to radiating fluid spheres in general relativity. We gave several new analytic solutions, some of which correspond to uniform rotation and others to differentially rotating fluid spheres. We also studied the stationary field equations for slowly rotating and nonradiating fluids spheres and presented a new analytic solution. In this Brief Report we will present some additional information that will immediately allow one to obtain a solution to the field equations with slow but differential rotation, once a static solution is given.

The field equations for static fluid spheres with the Schwarzschild metric are given as[2]

$$ds^2=A^2(r)dt^2-B^2(r)dr^2-r^2d^2\Omega \quad , \tag{1}$$

$$8\pi P_r=\frac{1}{B^2}\left(\frac{2A'}{Ar}+\frac{1}{r^2}\right)-\frac{1}{r^2} \quad , \tag{2}$$

$$8\pi P_\perp=\frac{1}{B^2}\left[\frac{A''}{A}-\frac{A'B'}{AB}+\frac{1}{r}\left(\frac{A'}{A}-\frac{B'}{B}\right)\right] \quad , \tag{3}$$

$$8\pi\rho=\frac{1}{B^2}\left(\frac{2B'}{Br}-\frac{1}{r^2}\right)+\frac{1}{r^2} \quad , \tag{4}$$

where P_r and $P_\perp$ are the pressures along the radial and tangential directions, which in general could be different in anisotropic fluid spheres.[3] The above system of differential equations has five unknowns and three equations. Hence, in general we need two more relations to be supplied to complete the set. For isotropic fluids $P_r=P_\perp$; hence, equating (2) and (3) gives us a coupled differential equation in $A(r)$ and $B(r)$ as

$$\frac{d}{dr}\left(\frac{1-B^2}{B^2r^2}\right)+\frac{d}{dr}\left(\frac{A'}{B^2Ar}\right)+\frac{1}{B^2A^2}\frac{d}{dr}\left(\frac{A'A}{r}\right)=0 \quad . \tag{5}$$

With the addition of an equation of state to this system we obtain a second coupled differential equation, which could be solved simultaneously with (5).[2]

For slowly rotating fluid spheres it is well known that the metric is given as

$$ds^2=A^2(r)dt^2-B^2(r)dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)+2r^2\sin^2\theta\,\Omega(r)d\phi\, dt \quad , \tag{6}$$

where $\Omega(r)$ represents the dragging of inertial frames. In this case the field equations to be solved are the same three equations (2)–(4) plus an additional equation to be solved for $\Omega(r)$, which is given by

$$\Omega''+\left(\frac{4}{r}-\frac{B'}{B}-\frac{A'}{A}\right)\Omega'=16\pi B^2(P+\rho)(\Omega-\omega) \quad , \tag{7}$$

where $\omega(r)=d\phi/dt$ is the rotation function of the fluid. For a given static solution and a rotation function the coefficients in the above second-order linear, inhomogeneous differential equation can be determined and the differential equation for $\Omega(r)$ can be solved, with the appropriate boundary conditions.

However, following Whitman's interesting paper[4] we will show that, if the metric is split up in a special form, then any static solution can be immediately used to obtain a slowly rotating solution with a particular form of the rotation function. We write the metric given in (6) in the form

$$ds^2=ug(u)dt^2-\frac{du^2}{4f(u)g(u)u^2}-u(d\theta^2+\sin^2\theta\, d\phi^2)+2u\sin^2\theta\,\Omega(u)d\phi\, dt \quad , \tag{8}$$

where $u=r^2$. Metric coefficients $A(r)$ and $B(r)$ redefined in this way put Eqs. (5) and (7) into the following forms, respectively:

$$\frac{d^2g}{du^2}+\frac{1}{2}\left[\frac{d}{du}\ln(u^5f)\right]\frac{dg}{du}+\left[\frac{1}{fu}\frac{df}{du}\right]g=-\frac{1}{2u^3f} \tag{9}$$

and

$$\frac{d^2\Omega}{du^2}+\frac{1}{2}\left[\frac{d}{du}\ln(u^5f)\right]\frac{d\Omega}{du}+\left[\frac{1}{fu}\frac{df}{du}\right]\Omega=\frac{1}{fu}\frac{df}{du}\omega(u) \quad , \tag{10}$$

where $u=r^2$. As seen, any solution of (9) with a given equation of state is also a solution of the equation for the dragging of inertial frames with the rotation function of the star equal to

$$\omega(u)=-1/2u^2(df/du) \ . \tag{11}$$

As an example we can take the solution for the $P=\alpha\rho$ equation of state, which is given as[1]

$$\begin{aligned} A(r)&=C_1 r^{2\alpha/(\alpha+1)} \ , \\ B^2(r)&=D/(1+\alpha), \quad D=(1+\alpha)^2+4\alpha \ , \\ P(r)&=\frac{\alpha^2}{2\pi D r^2} \ , \\ \rho(r)&=\frac{\alpha}{2\pi D r^2} \ , \end{aligned} \tag{12}$$

This solution can immediately be generalized to the slow-rotation case, where

$$\begin{aligned} \omega(r)&=\frac{DC_1^2}{4\alpha(1+\alpha)} r^{(2\alpha-2)/(\alpha+1)} \\ &\text{and} \\ \Omega(r)&=C_1^2 r^{(2\alpha-2)/(\alpha+1)} \ . \end{aligned} \tag{13}$$

Similarly all known static solutions in the literature can be generalized to slow differential rotation immediately.

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