Atiyah-Ward two-monopole solution

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An infinite sum of integrals is performed to give the Atiyah-Ward two-monopole solution explicitly in terms of roots of a certain quartic polynomial. The Higgs fields of the two-monopole solution are given in the x-y plane and on the coordinate axes. In addition, this solution is compared with the partial solution found in the Atiyah-Drinfeld-Hitchin-Manin construction of Nahm and found to be identical up to a scale transformation.

I. INTRODUCTION

For all of the rapid progress in our understanding of systems of monopoles in various gauge theories, there is much to be done in the way of producing explicit solutions whose properties can be examined. Only the single monopole has been so completely solved.

Up to now the two-monopole system has been explicitly solved only in the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction,[1] and even there only on the axis containing the monopoles. The solution over all space is much more difficult to obtain.

The Atiyah-Ward (AW) approach, on the other hand, has had one computational difficulty which has prevented any sort of explicit solution from being worked out. There is an infinite sum of integrals which has not been expressed in closed form. In this paper, we present the closed form of the sum of integrals and give explicit expressions for the Higgs fields on the $x-y$ plane and the coordinate axes.

In the last section, we compare the results of our work with the partial ADHMN solution of Ref. 1 and after determining the scaling properties of the fields show that these solutions are the same, settling the question (insofar as our solutions go at any rate) of whether or not the two different constructions describe different multimonopoles.

We have attempted to keep the notation and usages of Ref. 2 throughout this paper.

II. THE ATIYAH-WARD FORMALISM

In the AW formalism[3–5] the self-dual gauge potentials are "coded" into a transition matrix g on a specified vector bundle. The requirements of reality and the existence of a gauge in which the potentials are time independent imply that if

$$g=\Lambda_\infty(\vec{x},\zeta)\begin{bmatrix}\zeta^2 & f(\vec{x},\zeta)\\ 0 & \zeta^{-2}\end{bmatrix}\Lambda_0(\vec{x},\zeta)\ , \qquad (2.1)$$

where $\Lambda_\infty(\vec{x},\zeta)$ is regular at $\zeta=\infty$, $\Lambda_0(\vec{x},\zeta)$ is regular at $\zeta=0$, and $f(\vec{x},\zeta)$ is a function constrained in such a way as to produce gauge fields of a specified topological charge,[4,6] in this case two, having the Laurent expansion

$$f=\sum_{l=-\infty}^{\infty}\Delta_l\zeta^{-l}\ . \qquad (2.2)$$

Then

$$A_\mu=-\frac{1}{2F}\begin{bmatrix}\eta^3_{\mu\nu}\partial_\nu F & \eta^-_{\mu\nu}\partial_\nu E\\ \eta^+_{\mu\nu}\partial_\nu G & -\eta^3_{\mu\nu}\partial_\nu F\end{bmatrix}, \qquad (2.3a)$$

where

$$\eta^a_{\mu\nu}=\epsilon_{4a\mu\nu}+\delta_{a\mu}\delta_{4\nu}-\delta_{a\nu}\delta_{4\mu}$$

and

$$\eta^\pm=\eta^1\pm i\eta^2\ . \qquad (2.3b)$$

It was found that[6,7] for

$$f=\frac{e^{\pi\hat{u}(\zeta)/2}+e^{\pi\hat{v}(\zeta)/2}}{\gamma^2+\epsilon^2}\ , \qquad (2.4a)$$

where

$$\gamma=z+\tfrac{1}{2}(x_+\zeta-x_-\zeta^{-1})\ ,$$

$$\epsilon^2=1-\frac{d^2}{4}(\zeta-\zeta^{-1})^2,\ \ x_\pm=x\pm iy\ ,$$

$$\frac{1}{\epsilon}=g_0(d)+g_+(\zeta,d)+g_-(\zeta,d)\ , \qquad (2.4b)$$

$$\hat{u}=(z_++x_+\zeta)g_0+2\gamma g_+,\ \ z_\pm=z\pm it\ ,$$

$$\hat{v}=(-z_-+x_-\zeta^{-1})g_0-2\gamma g_-\ ,$$

the most general two-monopole solution up to translations and rotations is obtained. O'Raifeartaigh *et al.*[2] have found an explicit expression for the coefficients Δ_l of f in terms of the roots of the quartic polynomial,

$$P(\zeta)=\gamma^2+\epsilon^2 \tag{2.5a}$$

$$=\tfrac{1}{4}(x_+{}^2-d^2)\zeta^2+x_+z\zeta+1+\frac{d^2}{2}+z^2 -\tfrac{1}{2}x_+x_- -x_-z\zeta^{-1}+\tfrac{1}{4}(x_-{}^2-d^2)\zeta^{-2} \tag{2.5b}$$

$$=\tfrac{1}{4}(x_+{}^2-d^2)\zeta^{-2}(\zeta-\alpha)(\zeta-\beta) \times\left[\zeta+\frac{1}{\alpha^*}\right]\left[\zeta+\frac{1}{\beta^*}\right] \tag{2.5c}$$

(where α^* denotes the complex conjugate of α), $\hat{u}(\alpha)$, $\hat{u}(\beta)$, and their complex conjugates. The coefficients themselves are[2]

$$\Delta_l=\frac{4\alpha\beta}{x_-{}^2-d^2}[F_l(\alpha,\beta)+F_l(\beta,\alpha)]\ , \tag{2.6}$$

$$F_l(\alpha,\beta)=\frac{\alpha^{1+l}e^{\pi\hat{u}(\alpha)/2}}{(1+|\alpha|^2)(1+\alpha\beta^*)(\alpha-\beta)} -\frac{(-\alpha^*)^{1-l}e^{-\pi[\hat{u}(\alpha)]^*/2}}{(1+|\alpha|^2)(1+\alpha^*\beta)(\alpha^*-\beta^*)}\ . \tag{2.7}$$

The evaluation of g_+ and g_- has not been accomplished in closed form until now. It is to this that we now turn.

First, we observe two facts. One, it is apparent from the form of ϵ in (2.4b) that g_+ and g_- depend only on ζ^2. Two, the Ward reality requirement implies that

$$g_-(\zeta)=g_+\left[-\frac{1}{\zeta}\right]\ .$$

Using these facts, we can write

$$g_+(\zeta)=\sum_{n=1}^{\infty}g_{2n}\zeta^{2n}\ , \tag{2.8}$$

$$g_{2n}=\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{d\zeta}{\zeta}\zeta^{2n}\frac{1}{\epsilon}\ . \tag{2.9}$$

Once we compute the function g_+, the AW construction is as explicitly complete as our knowledge of the roots of $P(\zeta)$.

III. FUNCTIONS $g_+(\zeta)$ AND $\pi\hat{u}(j)/2$

We begin with the evaluation of g_0:

$$g_0=\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{d\zeta}{\zeta}\frac{1}{\epsilon} =\frac{2}{\pi}\int_0^{\pi/2}d\theta(1+d^2\sin^2\theta)^{-1/2}\ .$$

This is recognized to be the complete elliptic integral of the first kind. With the definitions

$$k^2\equiv\frac{d^2}{1+d^2}\ \text{and}\ k'^2\equiv1-k^2=\frac{1}{1+d^2}\ , \tag{3.1}$$

we have

$$g_0(d)=\frac{2}{\pi}k'K(k)\ . \tag{3.2}$$

To evaluate g_{2n}, it is useful to employ a different approach. Expressing the square root as the integration over a Gaussian, the resulting integral can be found in Ref. 8 to be

$$g_{2n}=\frac{2}{\pi^{1/2}}\int_0^{\infty}dx\,e^{-(1+\frac{1}{2}d^2)x^2}I_n(\tfrac{1}{2}d^2x^2) =\sqrt{k'}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)}\left[\frac{1-k'}{1+k'}\right]^n \times{}_2F_1\left[\tfrac{1}{2},\tfrac{1}{2};n+1;-\frac{(1-k')^2}{4k'}\right]\ . \tag{3.3}$$

Now, taking $n=0$ and using the relationship between hypergeometric functions and elliptic integrals,[9] we see that

$$g_0=\frac{2}{\pi}\sqrt{k'}K\left[i\frac{1-k'}{2\sqrt{k'}}\right]\ ,$$

and therefore

$$K\left[i\frac{1-k'}{2\sqrt{k'}}\right]=\sqrt{k'}K(k)\ . \tag{3.4a}$$

The analogous relationship for elliptic integrals of the second kind is

$$E\left[i\frac{1-k'}{2\sqrt{k'}}\right]=\frac{1}{2\sqrt{k'}}[E(k)+k'K(k)]\ . \tag{3.4b}$$

To perform the sum and obtain g_+, we realized that g_+ is a generating function and used the same procedure to find it as is used to find the generating function for Bessel functions. We define

$$b \equiv \frac{(1+k')^2}{4k'} > 1 \text{ and } c = b-1 > 0 ,$$

$$\xi \equiv \zeta^2 \left[\frac{c}{b} \right]^{1/2} ,$$

$$f_n \equiv \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};n+1;c) , \qquad (3.5)$$

$$G_+ \equiv \sum_{n=1}^{\infty} \xi^n f_n .$$

Then we have $g_+(\zeta)=(k'/\pi)^{1/2}G_+(\xi,c)$. Turning the appropriate recursion relation for hypergeometric functions into one for the f_n's, multiplying by ξ^n, and summing gives

$$\xi(1-\xi)(b\xi-c)\frac{dG_+}{d\xi} - \tfrac{1}{2}(b\xi^2-c)G_+ = [(2b-1)f_1 - \tfrac{3}{2}cf_2]\xi^2 - \tfrac{1}{2}cf_1\xi . \qquad (3.6)$$

According to Ref. 2, we are interested only in the region

$$|\zeta| \le 1 \text{ or } |\xi| \le \left[\frac{c}{b}\right]^{1/2} < 1 .$$

It is evident that the sum diverges for

$$\xi = \frac{c}{b} < \left[\frac{c}{b}\right]^{1/2} ,$$

and therefore our solution will only be valid for

$$0 \le \xi < \frac{c}{b} \text{ or } 0 \le \zeta^2 < \left[\frac{c}{b}\right]^{1/2} < 1 .$$

There may be a way to treat the remaining region, but the present solution covers more than enough of space to be interesting and will allow us to make contact with the ADHMN solution. On that note we proceed to integrate the differential equation (truly a laborious process) to obtain

$$g_+ = -\frac{1}{\pi}K(k)\left[\frac{\mathrm{cn}u}{\mathrm{sn}u}Z(u)+k'-\mathrm{dn}u\right] , \qquad (3.7)$$

where

$$\mathrm{sn}^2 u = \frac{1}{k^2}\left[1-k'^2\left[\frac{1+\zeta^2}{1-\zeta^2}\right]^2\right]$$

and

$$0 < |\mathrm{sn}^2 u| \le 1 , \qquad (3.8)$$

and $Z(u)$ is the Jacobi ζ function.

Substituting (3.2) and (3.7) into (2.4b) and simplifying, we find

$$\frac{\pi}{2}\hat{u}(\zeta) = \tfrac{1}{2}K(k)\left[(x_+\zeta + x_-\zeta^{-1})k' - (2z + x_+\zeta - x_-\zeta^{-1}) \times \left[\frac{\mathrm{cn}u}{\mathrm{sn}u}Z(u) - \mathrm{dn}u\right]\right] . \qquad (3.9)$$

IV. TWO-MONOPOLE HIGGS FIELDS

Presented here are the results of our study on the x-y plane and explicit expressions for the Higgs fields on the axes.

A. The x-y plane

The reality condition of Ward implies that if ζ is a root of $P(\zeta)$, then so is $(-\zeta^*)^{-1}$. We used this fact to choose the roots whose magnitudes were less than 1 (at least in some region of space). Calling these α and β after Ref. 2, we have

$$\alpha = -\beta = \left[\frac{d^2+y^2-x^2+2ixy}{2+d^2-s^2+2(1+d^2-s^2-d^2y^2)^{1/2}}\right]^{1/2} , \qquad (4.1)$$

$$s^2 = x^2+y^2$$

$$(\alpha^*)^{-1} = -(\beta^*)^{-1} = \left[\frac{d^2+y^2-x^2+2ixy}{2+d^2-s^2-2(1+d^2-s^2-d^2y^2)^{1/2}}\right]^{1/2} . \qquad (4.2)$$

In the regions we have chosen to explore it is true that $\alpha=-\beta$. Then, in the x-y plane, Eq. (3.9) implies that

$$\frac{\pi}{2}\hat{u}(\alpha) = -\frac{\pi}{2}\hat{u}(-\alpha) . \qquad (4.3)$$

This in turn implies that

$$\Delta_0=\frac{4\alpha^2}{d^2-x_-{}^2}\left[\frac{1}{1-|\alpha|^4}\right]\left[\cosh\frac{\pi}{2}\hat{u}(\alpha)+\cosh\frac{\pi}{2}[\hat{u}(\alpha)]^*\right], \tag{4.4a}$$

$$\Delta_1=\Delta_{-1}=\frac{4\alpha^2}{d^2-x_-{}^2}\left[\frac{1}{1-|\alpha|^4}\right]\left[\alpha\sinh\frac{\pi}{2}\hat{u}(\alpha)+(\alpha^*)^{-1}\sinh\frac{\pi}{2}[\hat{u}(\alpha)]^*\right], \tag{4.4b}$$

$$D=\frac{16}{|d^2-x_-{}^2|}\left[\frac{4|\alpha|^4}{(1-|\alpha|^4)^2}\cos^2\mathrm{Im}\frac{\pi}{2}\hat{u}(\alpha)+\frac{|\alpha|^2}{(1+|\alpha|^2)^2}\left[\sin^2\mathrm{Im}\frac{\pi}{2}\hat{u}(\alpha)+\sinh^2\mathrm{Re}\frac{\pi}{2}\hat{u}(\alpha)\right]\right]. \tag{4.4c}$$

Then, to compute the Higgs field, A_4, decompose it into fields as usual by

$$A_4=\phi_+\sigma_++\phi_-\sigma_-+\phi_3\sigma_3\,, \tag{4.5}$$

where $\sigma_\pm=\frac{1}{2}(\sigma_1\pm i\sigma_2)$ and $\phi_\pm=\phi_1\mp i\phi_2$. According to (2.3a) and (2.3b), we have

$$\phi_+=-\frac{D}{\Delta_0}\partial_+\left[\frac{\Delta_1}{D}\right], \tag{4.6a}$$

$$\phi_-=-\frac{D}{\Delta_0}\partial_-\left[\frac{\Delta_{-1}}{D}\right], \tag{4.6b}$$

$$\phi_3=\frac{D}{2\Delta_0}\partial_3\left[\frac{\Delta_0}{D}\right]. \tag{4.6c}$$

The special dependence of the function f of (2.2) implies relations between the Δ_l summarized by

$$\partial_{z_+}\Delta_l=\partial_+\Delta_{l-1}\,, \tag{4.7a}$$

$$\partial_{z_-}\Delta_l=-\partial_-\Delta_{l+1}\,, \tag{4.7b}$$

where $\partial_\pm\equiv\frac{1}{2}(\partial_x\mp i\partial_y)$ and $\partial_{z_\pm}\equiv\frac{1}{2}(\partial_z\mp i\partial_t)$. It should also be noted that this same dependence gives

$$i\partial_t\Delta_l=-\frac{\pi}{2}g_0\Delta_l\,. \tag{4.8}$$

Using all of this, ϕ_3 can be rewritten entirely in terms of derivatives in the x-y plane as follows:

$$\phi_3=\frac{1}{2}\left[\frac{\partial_+\Delta_{-1}-\partial_-\Delta_1}{\Delta_0}\right.$$
$$\left.+\frac{\Delta_0{}^2}{D}\left[\partial_-\left[\frac{\Delta_1}{\Delta_0}\right]-\partial_+\left[\frac{\Delta_{-1}}{\Delta_0}\right]\right]\right]. \tag{4.9}$$

This completes the AW construction in the x-y plane. By specializing to the coordinate axes explicit expressions for the Higgs fields can be written down.

Because of the D_{2h} symmetry of these solutions[2] the determinant, D, is a function of x^2, y^2, and z^2, and consequently,

$$\partial_x D\,|_{x=0}=\partial_y D\,|_{y=0}=\partial_z D\,|_{z=0}=0\,. \tag{4.10a}$$

Furthermore, Δ_0 must have this same behavior. From this it can be shown that

$$\partial_x\Delta_1\,|_{x=0}=\partial_x\Delta_{-1}\,|_{x=0}$$

and (4.10b)

$$\partial_y\Delta_1\,|_{y=0}=-\partial_y\Delta_{-1}\,|_{y=0}\,.$$

These results imply that ϕ_2 and ϕ_3 vanish on the x and y axes. Putting all of this together, we can specialize to the axes.

On the x axis the results are, for $x\le d$,

$$\alpha=-\beta=\frac{(d^2-x^2)^{1/2}}{1+(1+d^2-x^2)^{1/2}}\,, \tag{4.11a}$$

$$\mathrm{sn}u=\frac{k'}{k}x\,, \tag{4.11b}$$

$$\frac{\pi}{2}\hat{u}(\alpha)=K(k)Z(u)\,, \tag{4.11c}$$

$$\Delta_0=\frac{2k'}{\mathrm{dn}u}\cosh KZ\,, \tag{4.11d}$$

$$\Delta_1=\Delta_{-1}=\frac{2k'}{k\,\mathrm{cn}u}\sinh KZ\,, \tag{4.11e}$$

$$D=\Delta_0{}^2-\Delta_1\Delta_{-1}$$
$$=\frac{4k'^2}{k^2}\left[\frac{\mathrm{dn}^2u-k'^2\cosh^2 KZ}{\mathrm{dn}^2u\,\mathrm{cn}^2u}\right], \tag{4.11f}$$

$$\phi_1=\frac{1}{2}k'K-\frac{k'}{\mathrm{dn}^2u-k'^2\cosh^2KZ}$$
$$\times\left[\frac{k'^2\,\mathrm{sn}u\,\sinh 2KZ}{2\,\mathrm{cn}u\,\mathrm{dn}u}+K\frac{dZ}{du}\right], \tag{4.11g}$$

$$\phi_2=\phi_3=0\,. \tag{4.11h}$$

While on the y axis, we obtain for $y\le 1$

$$\alpha=-\beta=\frac{(d^2+y^2)^{1/2}}{(1+d^2)^{1/2}+(1-y^2)^{1/2}}\,, \tag{4.12a}$$

$$\mathrm{sn}u=\frac{iy}{k(1-y^2)^{1/2}}\,, \tag{4.12b}$$

$$\frac{\pi}{2}\hat{u}(\alpha)=K\left[k^2\frac{\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u}-\frac{\mathrm{sn}^2u}{\mathrm{cn}^2u}Z(u)\right]$$

$$\equiv iK(k)T(u)\ , \tag{4.12c}$$

where the function $T(u)$ is purely real. Then the Δ_l's and fields are

$$\Delta_0=2k'\,\mathrm{dn}u\,\cos KT\ , \tag{4.13a}$$

$$\Delta_1=-\Delta_{-1}=-\frac{2ik'^2\,\mathrm{dn}u}{k\,\mathrm{cn}u}\sin KT\ , \tag{4.13b}$$

$$D=4k'^2\,\mathrm{dn}^2u\left[1-\frac{\mathrm{dn}^2u}{k^2\mathrm{cn}^2u}\sin^2 KT\right]\ , \tag{4.13c}$$

$$\phi_1=-\tfrac{1}{2}k'K+\frac{ik'\mathrm{dn}^2u}{k^2\mathrm{cn}^2u-\mathrm{dn}^2u\,\sin^2KT}\times\left[\frac{\mathrm{dn}u\,\mathrm{sn}u\,(\mathrm{cn}^2u-\mathrm{sn}^2u)}{2\,\mathrm{cn}^3u}\sin 2KT+K\frac{dT}{du}\right]\ , \tag{4.13d}$$

$$\phi_2=\phi_3=0\ . \tag{4.13e}$$

B. The z axis

On the z axis relations (4.6), (4.7), and (4.10) give

$$\phi_+=-\frac{1}{\Delta_0}\left[\partial_z\Delta_2+\frac{\pi}{2}g_0\Delta_2\right]\ , \tag{4.14a}$$

$$\phi_-=-\frac{1}{\Delta_0}\left[-\partial_z\Delta_{-2}+\frac{\pi}{2}g_0\Delta_{-2}\right]\ , \tag{4.14b}$$

$$\phi_3=-\frac{1}{2\Delta_0}\partial_3\Delta_0\ . \tag{4.14c}$$

Now we need the additional functions Δ_2 and Δ_{-2}. They are easily computed by performing the contour integral, as in Ref. 2:

$$\Delta_2=\frac{4\alpha\beta}{x_-{}^2-d^2}[F_2(\alpha,\beta)+F_2(\beta,\alpha)]+\frac{4}{x_+{}^2-d^2}e^{-(\pi/2)g_0z_-}\ , \tag{4.15a}$$

$$\Delta_{-2}=\frac{4\alpha\beta}{x_-{}^2-d^2}[F_{-2}(\alpha,\beta)+F_{-2}(\beta,\alpha)]+\frac{4}{x_-{}^2-d^2}e^{(\pi/2)g_0z_+}\ , \tag{4.15b}$$

where $F_2(\alpha,\beta)$ and $F_{-2}(\alpha,\beta)$ are as defined in (2.7).

On the z axis it is also true that $\alpha=-\beta$, where α,β are the roots of $P(\zeta)$ that are less than unity in magnitude:

$$\alpha=-\beta=\frac{1}{d}[(1+d^2-z^2)^{1/2}-(1+z^2)^{1/2}]\ , \tag{4.16a}$$

$$\mathrm{sn}u=\frac{z}{(1+z^2)^{1/2}}\ , \tag{4.16b}$$

$$\frac{\pi}{2}\hat{u}(\alpha)=-K\left[Z(u)-\frac{\mathrm{sn}u}{\mathrm{cn}u}\mathrm{dn}u\right]$$

$$\equiv K(k)S(u)\ . \tag{4.16c}$$

Now, however, Eq. (3.9) implies that

$$\frac{\pi}{2}\hat{u}(-\alpha)=\frac{\pi}{2}\hat{u}(\alpha)\ .$$

Combining this with (2.6) and (4.16) gives

$$\Delta_0=-\frac{2k'\mathrm{cn}^2u}{\mathrm{dn}u}\cosh KS\ , \tag{4.17a}$$

$$\Delta_1=\Delta_{-1}=0\ , \tag{4.17b}$$

$$\Delta_2=-\frac{2k'\mathrm{cn}^2u}{\mathrm{dn}u}\cosh(KS+\ln\alpha^2)-\frac{4k'^2}{k^2}e^{-k'K\mathrm{sn}u/\mathrm{cn}u}\ , \tag{4.17c}$$

$$\Delta_{-2}=-\frac{2k'\mathrm{cn}^2u}{\mathrm{dn}u}\cosh(KS-\ln\alpha^2)-\frac{4k'^2}{k^2}e^{k'K\mathrm{sn}u/\mathrm{cn}u}\ . \tag{4.17d}$$

Substituting these results into the expressions (4.14), we find the Higgs fields

$$\phi_1=-k'K\left[\mathrm{sn}^2u-\mathrm{cn}^2u+\frac{2}{k^2}\frac{dZ}{du}\right]\ , \tag{4.18a}$$

$$\phi_2=i\left[\frac{k^2\mathrm{cn}^3u\,\mathrm{sn}u}{\mathrm{dn}^2u}+K\tanh KS\times\left[\frac{k^2\mathrm{cn}^2u}{\mathrm{dn}u}(\mathrm{dn}^2u\,\mathrm{sn}^2u-\mathrm{cn}^2u)+\frac{k'^2+\mathrm{dn}^2u}{k^2\mathrm{dn}u}\frac{dZ}{du}\right]\right]\ , \tag{4.18b}$$

$$\phi_3=\frac{\mathrm{cn}u\,\mathrm{sn}u}{2\,\mathrm{dn}^2u}(k'^2+\mathrm{dn}^2u)-\tfrac{1}{2}K\left[\frac{k'^2+k^2\mathrm{cn}^4u}{\mathrm{dn}u}-\frac{\mathrm{cn}^2u}{\mathrm{dn}u}\frac{dZ}{du}\right]\ . \tag{4.18c}$$

V. COMPARISON WITH THE ADHMN TWO-MONOPOLE SOLUTION

In Ref. 1, the two-monopole solution is found on the axis containing the monopoles. It has been an open question whether the AW and ADHMN solutions describe the same monopole systems. Direct comparison shows that they are indeed the same.

To compare the two solutions, we must ascertain the relations between their coordinates and normalization. In (2.4b) we see that $\hat{u}(\zeta)$ is invariant under

$$\vec{x} \to \vec{x}' = \lambda \vec{x} \text{ and } \epsilon \to \epsilon' = \lambda \epsilon , \tag{5.1a}$$

which implies that

$$g_0 \to g_0' = \frac{1}{\lambda} g_0 . \tag{5.1b}$$

Since g_0 is proportional to the normalization of the Higgs fields at infinity, we see that

$$\left| A_4 \left[\frac{1}{\lambda} g_0, \lambda \vec{x} \right] \right| = \frac{1}{\lambda} | A_4(g_0, \vec{x}) | . \tag{5.2}$$

It seems that the normalization of the Higgs field fixes the scale of the coordinates.

Comparing (6.13) of Ref. 1 and (4.11g), we see that the two Higgs normalizations differ by a factor of $\frac{1}{2}$. We therefore expect that

$$| \vec{x}_{\mathrm{AW}} | = 2 | \vec{x}_{\mathrm{ADHMN}} | . \tag{5.3}$$

Aside from this scale factor, there is a rotation between the two sets of coordinates. The x axis of the AW solution is the x_2 or y axis of the ADHMN solution. In both constructions the z or x_3 axis is the axis of symmetry when the separation parameter vanishes.

Now to compare the two solutions we see that (6.8) of Ref. 1 can be rewritten as

$$4{x_2}^2 = \frac{k^2}{k'^2} \mathrm{sn}^2 t . \tag{5.4}$$

This is precisely (4.11b) with the factor of 2 as in (5.1). At this point it should be mentioned that the k of this paper was defined the same way as in Ref. 1. Looking back to the scaling properties of $P(\zeta)$ and ϵ shows that k and k' are invariant under these transformations even though, by necessity, the parameter d scales as the x_i's. The presence of such invariants is extremely convenient when comparing the solutions.

Taking (6.11) of Ref. 1 and substituting into (6.13) leads, after much algebra, to

$$| (A_4)_{11} | = 2\phi_1 ,$$

where we used the ϕ_1 of (4.11g) and identified sn$t$ with sn$u$ subject to the scale change (5.1).

VI. CONCLUDING REMARKS

Two concluding remarks are in order.

The function g_+ is a universal function for multimonopole solutions with a single free parameter. Accordingly, for the conditions of Ref. 10 all of those solutions can now be explicitly obtained.

At this writing, Panagopoulos[11] has obtained the ADHMN solution over all space. In a forthcoming article his work will be presented. Comparisons between the two solutions show the same sort of agreement that we have seen so far. In particular, both solutions depend on the roots of $P(\zeta)$.

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