Field-operator decomposition in the Lee model

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The decomposition of the meson field operator into internal and external parts is shown to provide a new, relatively simple approximation scheme that gives approximate eigenvalues, eigenvectors, and scattering phase shifts in all sectors of the Lee model. The form of the internal meson mode functions involves an energy parameter ϵ that is determined in a physical way that is peculiar to systems in which the ground-state expectation value of the meson source current operator is required by the internal symmetry of the system to be zero.

I. INTRODUCTION

A simplified model for the interaction of charged mesons with a source was proposed by Lee[1]; the Lee-model Hamiltonian is

$$H=\frac{\Delta}{2}\tau_0+\int \omega(k)a^\dagger(k)a(k)dk +\tau_+\int \mu^*(k)a(k)dk +\tau_-\int \mu(k)a^\dagger(k)dk\ , \quad (1)$$

where

$$\tau_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ \tau_+=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \tau_-=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}. \quad (2)$$

In this Hamiltonian the source has two states, an N state $|N\rangle$ at $-\Delta/2$ belonging to the eigenvalue -1 of τ_0 and a V state $|V\rangle$ at $\Delta/2$ belonging to $+1$. When a meson is emitted the source changes from V to N, and when it is absorbed the source changes from N to V. Thus there is a conserved charge $\hat{Q}$,

$$\hat{Q}=\tfrac{1}{2}(1+\tau_0)+\int a^\dagger(k)a(k)dk\ . \quad (3)$$

The eigenvalues of $\hat{Q}$ are the non-negative integers, and the Hilbert space of the system splits into subspaces each of which belongs to a definite eigenvalue Q of $\hat{Q}$. The form factor $\mu(k)$ describes the momentum dependence of the meson emission and absorption. The dimension of the vector k is arbitrary. It will be assumed in the following that the minimum value of $\omega(k)$ occurs at $k=0$, that is, $\omega(0)$ will be used to represent the minimum value of $\omega(k)$.

In the $Q=0$ subspace there is only the N state of the source. The $Q=1$ subspace is easily solved. The $Q=2$ subspace has been solved exactly,[2] but the solution is quite complicated. It is likely that Q subspaces with higher values of Q can also be solved exactly, but the solutions are undoubtedly so complicated as to be unusable. This paper presents a general approximation method that is relatively simple and can be applied to all Q subspaces with discrete ground states of the Lee model described by the Hamiltonian of Eq. (1). The method is based on the decomposition[3–5] of the meson field operator $a(k)$ into two parts:

$$a(k)=A\phi(k)+a_\perp(k)\ , \quad (4)$$

where $\phi(k)$ is a normalized internal mode of the meson field with associated annihilation operator A and $a_\perp(k)$ is the external meson field; the external field is orthogonal to the internal mode $\phi(k)$:

$$\int \phi^*(k)a_\perp(k)dk=0\ . \quad (5)$$

The mode function ϕ will be tailored to give the best results for the particular value of Q under consideration.

Section II discusses how the operator decomposition of Eq. (4) splits the Hamiltonian into various pieces, some of which can be treated perturbatively. The function ϕ is chosen initially in Sec. II to simplify the Hamiltonian; this criterion determines ϕ to within a single parameter ϵ. Section III displays the "unperturbed" Hamiltonian and describes its eigenstates. Section IV considers various approximations to the ground-state energy in the Q subspace and corresponding choices of the parameter ϵ.

Reference 3 gives a discussion of the one-meson sector in general static models with discrete ground states. This work shows that the considerations of Ref. 3 are incomplete, in that an important categorization of static meson fields was neglected there, namely, the subdivision into fields that can and cannot have a ground-state expectation value (GSEV). Note that this categorization also depends on the particular ground state under consideration, that is, a particular meson field can have a GSEV in some subspaces and not in others. For example, the p-wave pion field can have an expectation value in the nucleon ground state, but not in the α-particle ground state. In cases in which the meson field under consideration can have a GSEV, Ref. 3 shows that zero is the appropriate choice for the parameter ϵ; the remainder of Ref. 3 completes the discussion of this case. When, as in the Lee model, the meson field cannot have a GSEV, the form of the ϕ function, in particular, the value of the parameter ϵ, must be determined from the considerations of this paper.

II. APPLICATION OF FIELD-OPERATOR DECOMPOSITION

Substitution of Eq. (4) into the Hamiltonian of Eq. (1) gives

$$\begin{aligned} H &= H_A + H_\perp + H_I + H_I^\dagger \, , \\ H_A &= W A^\dagger A + V(\tau_- A^\dagger + \tau_+ A) \, , \\ H_\perp &= \int \omega(k) a_\perp^\dagger(k) a_\perp(k) dk \, , \\ H_I &= \tau_+ \int \mu^*(k) a_\perp(k) dk \\ &\quad + A^\dagger \int \omega(k)\phi^*(k) a_\perp(k) dk \, , \end{aligned} \tag{6}$$

where the parameters W and V are given in terms of the internal-mode function ϕ by

$$\begin{aligned} W &= \int \omega(k)\phi^*(k)\phi(k) dk \, , \\ V &= \int \mu^*(k)\phi(k) dk \, . \end{aligned} \tag{7}$$

The parameter V has been made real by adjusting the phase of the annihilation operator A. Owing to Eq. (5) the term H_I can also be written as

$$\begin{aligned} H_I &= \tau_+ \int \mu_\perp^*(k) a_\perp(k) dk \\ &\quad + A^\dagger \int [\omega(k)\phi^*(k)]_\perp a_\perp(k) dk \, . \end{aligned} \tag{8}$$

This term takes its simplest form if the internal mode is chosen so that

$$\mu_\perp(k) \propto [\omega(k)\phi(k)]_\perp \tag{9}$$

or, equivalently,

$$\mu(k) = G[\omega(k) - \epsilon]\phi(k) \, , \tag{10}$$

where ϵ is an arbitrary parameter that is required to be less than $\omega(0)$. Thus,

$$\phi(k) = \frac{\mu(k)}{G[\omega(k) - \epsilon]} \, , \tag{11}$$

where G is both the normalization constant for ϕ and a dimensionless coupling constant given by

$$G^2 = \int \frac{|\mu(k)|^2}{[\omega(k) - \epsilon]^2} dk \, . \tag{12}$$

With ϕ given by Eq. (11) the parameters W and V become

$$\begin{aligned} W &= \frac{1}{G^2} \int \frac{\omega |\mu|^2}{(\omega - \epsilon)^2} dk \, , \\ V &= \frac{1}{G} \int \frac{|\mu|^2}{\omega - \epsilon} dk = G(W - \epsilon) \, , \end{aligned} \tag{13}$$

and the term H_I in the Hamiltonian takes the form

$$H_I = (A^\dagger + G\tau_+) \int \chi^*(k) a_\perp(k) dk \, , \tag{14}$$

where the source function for the external meson field is $\chi(k)$:

$$\begin{aligned} \chi(k) &= \chi_\perp(k) = (\omega\phi)_\perp = \frac{1}{G}\mu_\perp \\ &= [\omega(k) - W]\phi(k) \, . \end{aligned} \tag{15}$$

The parameters W, V, and G and the function $\chi(k)$ all depend on ϵ. Note that ϕ and χ are independent of the strength of the coupling function $\mu(k)$ and depend only on its shape.

III. UNPERTURBED HAMILTONIAN

The idea now is to treat the terms H_A and $H_\perp$ as the unperturbed Hamiltonian H_U:

$$H_U = H_A + H_\perp \tag{16}$$

and H_I and $H_I^\dagger$ as the perturbation. The parameter ϵ is to be chosen so as to minimize the effects of the perturbation term. Of course the full Hamiltonian H is independent of the choice of the parameter ϵ.

As in Refs. 3 and 5, the Q subspace of the Hilbert space will be subdivided according to the number of external field quanta present. Clearly H_U commutes with the operator for the number of external quanta

$$\int a_\perp^\dagger(k) a_\perp(k) dk \, .$$

The n-external-meson subspace of the Q subspace will be called the $n\text{EM}_Q$ subspace; n is restricted by the condition

$$0 \le n \le Q \quad \text{in } n\text{EM}_Q \, . \tag{17}$$

For $Q=0$ there is only the single state $|N\rangle$; in the following it is assumed that $Q>0$. The 0EM_Q subspace is spanned by the two states $|Q,a\rangle$ and $|Q,b\rangle$:

$$|Q,a\rangle=\frac{1}{\sqrt{Q!}}(A^\dagger)^Q|N\rangle\ ,$$
$$|Q,b\rangle=\frac{1}{\sqrt{(Q-1)!}}(A^\dagger)^{Q-1}|V\rangle\ . \tag{18}$$

The states in the 1EM_Q subspace are $a_\perp^\dagger(k)|Q-1,a\rangle$ and $a_\perp^\dagger(k)|Q-1,b\rangle$; similar constructions give all the $n\text{EM}_Q$ subspaces. The $n\text{EM}_Q$ sector is defined as the union of the $m\text{EM}_Q$ subspaces for $m\le n$.

Obviously the first step is to diagonalize the Hamiltonian within the 0EM_Q subspace, where H is equivalent to H_A. The two eigenstates of H_A in the 0EM_Q subspace will be denoted $|Q,g\rangle$ and $|Q,*\rangle$. The Hamiltonian H_A of Eq. (6) is easily diagonalized using the basis of (18); the resulting eigenvalues $E_{Q,g}$ and $E_{Q,*}$ of H_A are

$$E_{Q,g}=(Q-\tfrac{1}{2})W-\left[\left(\frac{W-\Delta}{2}\right)^2+QV^2\right]^{1/2},$$
$$E_{Q,*}=(Q-\tfrac{1}{2})W+\left[\left(\frac{W-\Delta}{2}\right)^2+QV^2\right]^{1/2}. \tag{19}$$

Note that the eigenvalues are functions of the parameter ϵ. Each value of ϵ gives a full spectrum consisting of two states for every value of Q.

In this representation that diagonalizes H_A, H_U is also diagonal. In the 0EM_Q subspace its eigenvectors are $|Q,g\rangle$ and $|Q,*\rangle$, with eigenvalues $E_{Q,g}$ and $E_{Q,*}$, respectively, and in the 1EM_Q subspace there are $a_\perp^\dagger(k)|Q-1,g\rangle$ and $a_\perp^\dagger(k)|Q-1,*\rangle$ with eigenvalues $E_{Q-1,g}+\omega(k)$ and $E_{Q-1,*}+\omega(k)$, respectively. Further eigenstates and eigenvalues of H_U can be constructed analogously.

IV. APPROXIMATIONS AND $\epsilon(Q)$

The simplest approximate ground-state energy value and state vector are obviously $E_{Q,g}$ and $|Q,g\rangle$. At this level the only reasonable choice for $\epsilon(Q)$ is the value that minimizes $E_{Q,g}(\epsilon)$. The Q-subspace approximate ground state is discrete if and only if the minimizing value of ϵ is less than $\omega(0)$.

The 1EM_Q states couple directly to the state $|Q,g\rangle$ with matrix elements proportional to $\langle Q-1,g|A+G\tau_-|Q,g\rangle$ and $\langle Q-1,*|A+G\tau_-|Q,g\rangle$. The natural choice of ϵ is the value that prevents coupling to the lower states $a_\perp^\dagger(k)|Q-1,g\rangle$, that is, the value of ϵ that makes the corresponding matrix element vanish:

$$\langle Q-1,g|A+G\tau_-|Q,g\rangle=0\ . \tag{20}$$

From the commutator

$$[A,H_A]=WA+V\tau_- \tag{21}$$

it follows that

$$(E_{Q,g}-E_{Q-1,g}-W)\langle Q-1,g|A+G\tau_-|Q,g\rangle$$
$$=(E_{Q,g}-E_{Q-1,g}-\epsilon)\langle Q-1,g|\tau_-|Q,g\rangle\ , \tag{22}$$

so that the appropriate value of ϵ is

$$\epsilon=E_{Q,g}-E_{Q-1,g}\ . \tag{23}$$

This implicit equation is to be solved for $\epsilon(Q)$ with $E_{Q,g}$ given by Eq. (19) and with W and V from Eqs. (13). Again, Eq. (23) has a solution that is less than $\omega(0)$ if and only if the approximate ground state is discrete.

The derived value of $\epsilon(Q)$ given by Eq. (23) has the advantage that ϵ is a meaningful energy difference of eigenstates of H_A, a feature that is lacking in the previously mentioned variational choice of $\epsilon(Q)$. With ϵ chosen so that Eq. (20) is satisifed, the only state that couples directly to $|Q,g\rangle$ is the state $|Q-1,*\rangle$, so that a next approximation to the energy of the ground state in the Q subspace is the solution of

$$0=E-E_{Q,g}-|\langle Q-1,*|A+G\tau_-|Q,g\rangle|^2\times\int\frac{|\chi(k)|^2}{E-E_{Q-1,*}-\omega(k)}dk\ . \tag{24}$$

For the special case $Q=1$, there is no state $|Q-1,*\rangle$, and therefore no state that is coupled to the state $|Q,g\rangle$ by the interaction Hamiltonian; hence, for $\epsilon(Q=1)$, chosen according to Eq. (23), the ground-state energy in the $Q=1$ subspace is exactly $E_{1,g}$ of Eq. (19) with ϵ set equal to $\epsilon(1)$. It can be verified that in this case the condition of Eq. (23) is in fact equivalent to the standard simple expression for the ground-state energy; the details are left to the interested reader. The phase of the right-hand side of Eq. (24) for $E>E_{Q-1,*}+\omega(0)$ is an approximation to the phase shift for meson scattering by the $Q-1$ excited state in the usual way. There is no meson scattering by the $Q-1$ ground state in this approximation.[3]

Finally, H can be diagonalized within the entire 1EM_Q sector as in Ref. 3. In this case it is again appropriate to use a value of $\epsilon(Q)$ that minimizes the resulting approximate ground-state energy; discreteness of the approximate ground state is characterized as above. Again, an approximation to the scattering phase shift is obtained; in this case there is (weak) scattering by the $Q-1$ ground state, as well as excitation of the $Q-1$ excited state.

Since, as was noted above, ϵ is an artificial parameter, in that H does not depend on ϵ, it seems most suitable to use the more physical value of $\epsilon(Q)$ given by Eq. (23) in preference to the variational values of $\epsilon(Q)$, whatever they may be.

V. SUMMARY

The decomposition of the meson field operator into internal and external parts has been shown to provide a new relatively simple approximation scheme that gives approximate eigenvalues, eigenvectors, and scattering phase shifts in all sectors of the Lee model. The form of the internal-meson-mode functions involves an energy parameter ϵ that is determined in a physical way that is peculiar to systems in which the ground-state expectation value of the meson source current operator is required by the internal symmetry of the system to be zero. This parameter has previously[3] been shown to be zero in systems in which there is no internal symmetry that constrains the ground-state expectation value of this current operator.

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