Chiral formulation of Yang-Mills equations: A geometric approach

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The chiral formulation of gauge theories is considered from a geometric point of view. Geometric chiral equations are explicitly written down and the associated distributional problems are pointed out. A new continuous set of conserved quantities is obtained within a Lagrangian formulation.

I. INTRODUCTION

Since the observations by Polyakov[1] on the formal analogy between Yang-Mills equations and chiral equations in loop space, there have been a number of papers[1–5] on the realization of the attractive original Polyakov program. All these works treat the loop in a parametric form and, although general conditions are imposed to guarantee invariance under reparametrization, the geometric content of chiral equations is somewhat hidden.

On the other hand, the relevance of the group of loops as the basic underlying geometric structure of gauge theories has been emphasized by a number of authors.[3–8] It has been explicitly shown that the kinematic content of any Yang-Mills theory is a direct consequence of the geometric structure of the group of loops.[6]

In this paper the chiral description of gauge theories is considered from a completely geometric point of view. All differential operators are related to infinitesimal generators of the group of loops, and a parametrization is never introduced.

The relevant generators may be expressed in terms of the path-dependent differential operator $\Delta_{\mu\nu}(P)$ which is the generalization of the "area derivative" first considered by Mandelstam.[9]

Let δL be the infinitesimal loop $P\,\delta u\,\delta v\,\delta\bar{u}\,\delta\bar{v}\,\bar{P}$. $\Delta_{\mu\nu}(P)$ is defined for any function on the group of loops as

$$\Psi(\delta L\,L)=[1+\tfrac{1}{2}(\delta u^{\mu}\delta v^{\nu}-\delta u^{\nu}\delta v^{\mu})\Delta_{\mu\nu}(P)]\Psi(L)\;. \tag{1.1}$$

$\Delta_{\mu\nu}(P)$ is a noncommutative operator. From geometrical considerations alone its commutation relations may be found to be

$$\begin{aligned}[\Delta_{\mu\nu}(P)\Delta_{\alpha\beta}(P')]&=\Delta_{\mu\nu}(P)\{\Delta_{\alpha\beta}(P')\}\\&=-\Delta_{\alpha\beta}(P')\{\Delta_{\mu\nu}(P)\}\\&=\tfrac{1}{2}\Delta_{\mu\nu}(P)\{\Delta_{\alpha\beta}(P')\}\\&\quad-\tfrac{1}{2}\Delta_{\alpha\beta}(P')\{\Delta_{\mu\nu}(P)\}\;,\end{aligned} \tag{1.2}$$

where $\Delta_{\mu\nu}(P)\{\Delta_{\alpha\beta}(P')\}$ means the differential action of the first operator on the path dependence of the second.

The Δ operators are constrained by the Bianchi identities

$$D_{\lambda}\Delta_{\mu\nu}(P)+D_{\mu}\Delta_{\nu\lambda}(P)+D_{\nu}\Delta_{\lambda\mu}(P)=0\;, \tag{1.3}$$

where D_{μ} is the familiar Mandelstam covariant derivative.

The geometrical commutators (1.2) could be considered as the defintion of the "algebra" of the group of loops. A formal analogy with conventional Lie algebras may be obtained by writing the right-hand side (rhs) as a sum over paths of Δ generators with path-dependent "structure constants." A conventional Yang-Mills theory is obtained by mapping this geometric structure into a specific semisimple compact Lie group. For any choice, a necessarily unfaithful representation of the group of loops is implied, resulting in a contraction of its rich geometric structure. Surpassing the level of compact Lie groups is necessary to incorporate general relativity to the Yang-Mills picture; hence the study of the group of loops itself, its faithful representations, or the search for natural geometric contractions could result in a deeper understanding of the role of the Yang-Mills idea in physics.

In Sec. II the chiral formulation of gauge theories is considered. The whole chiral approach is based on the concept of the tangent vector on a point of a loop. This is necessarily a parameter-dependent object in the conventional formulations. Therefore, emphasis is made in the introduction of a geometric distributional object which generalized the idea of a tangent of the loop to a distributional object defined for every point in R^4. Differential operators and chiral functions are introduced in the same spirit. It will become apparent that the distributional problems of the formulation and the intricacies of its integrability conditions are most distinctly pointed out within this geometric approach. The strong suggestion that chiral objects acquire a tensorial character in the sense of continuous indices taking values in R^4 also develops itself in this formulation, and the possibility of introducing large symmetry groups of loop-dependent continuous linear transformations becomes an interesting issure to explore.

In Sec. III the developed formulation is applied to the search of conserved quantities via Noether's theorem. A continuous set of conservation law is found which cannot be considered within the conventional parametric approach since, as pointed out in Sec. II, larger invariance groups could also be considered. This section contains essentially an example of the naturality of the formulation where expressed in geometric terms. Important questions such as the need to introduce subsidiary conditions to the chiral equations in order to achieve complete equivalence with the Yang-Mills system are but slightly touched upon.

II. NONPARAMETRIC CHIRAL EQUATIONS IN LOOP SPACE

The formal analogy between Yang-Mills equations and classical chiral equations in loop space was first noticed by Polyakov.[1] The introduction of a formulation based on equations in loop space would agree with the expectation that closed strings should play the role of elementary excitations in the confining phase of a gauge theory.[10,11] Also, the possibility that inverse-scattering equations could be formulated and infinite numbers of conserved quantities obtained raised the hope that important information about the nonperturbative sector of Yang-Mills theories could be obtained. Recently, a Lagrangian approach along the same lines was worked out to some extent by Dolan.[4,5] However, some complications associated with the intricate integrability conditions for the chiral equations in loop space could provide a natural obstruction for the whole program.[2]

Let us start this section by building up a geometric formulation of the chiral equations in loop space. It will become apparent that the distributional problems associated with the chiral equations are more easily recognized using this nonparametric approach.

Chiral equations depend heavily on the concept of the tangent vector at a given point of loop L. A geometric realization of this object may be introduced as

$$T_\mu(x,L)=\oint_L dy_\mu \delta^4(x-y) \; . \tag{2.1}$$

By combining this object with the loop derivative (1.1) one may obtain a natural representation for a contact deformation of a Loop L at a point. Explicitly, one may introduce the distribution-valued differential operator

$$\delta_\mu(z)\Psi(L)=\oint_L dy^\lambda \delta^4(z-y)\Delta_{\lambda\mu}(\not{L}^y)\Psi(L) \; , \tag{2.2}$$

where $\not{L}^y$ is that portion of L from the origin to y. This operator satisfies the orthogonality relation

$$T_\mu(z,L)\delta^\mu(z)=0 \tag{2.3}$$

in close analogy with Polyakov's functional derivative operator.

Let us now introduce the chiral object

$$\begin{aligned} f_\mu(Z,L) &= [\delta_\mu(z)W(L)]W^{-1}(L) \\ &= \oint_L dy^\lambda \delta^4(z-y)F_{\lambda\mu}(\not{L}^y) \; , \end{aligned} \tag{2.4}$$

where $W(L)$ is the familiar phase factor of Yang and $F_{\mu\nu}(P)$ is the ordinary path-dependent field strength.

As a first step to obtain $\delta_\nu(x)f_\mu(z,L)$ let us compute

$$\begin{aligned} \Delta_{\alpha\beta}(\not{L}^{y'})\left[\oint_L dy^\lambda \delta^4(z-y)F_{\lambda\mu}(\not{L}^y)\right] &= \delta^4_{,\beta}(z-y')F_{\alpha\mu}(\not{L}^{y'})-\delta^4{}_{,\alpha}(z-y')F_{\beta\mu}(\not{L}^{y'}) \\ &\quad +\delta^4(z-y')\left[D_\alpha F_{\beta\mu}(\not{L}^{y'})+D_\beta F_{\mu\alpha}(\not{L}^{y'})\right] \\ &\quad +\oint_L dy^\lambda \delta^4(z-y)\theta_L(y-y')\Delta_{\alpha\beta}(\not{L}^{y'})F_{\lambda\mu}(\not{L}^y) \; , \end{aligned} \tag{2.5}$$

where D_α stands for Mandelstam's covariant derivative and where we have introduced the symbol

$$\theta_L(y-y')=\begin{cases} 1 & \text{for } y' \text{ before } y \text{ on } L \; , \\ 0 & \text{for } y' \text{ after } y \text{ on } L \; . \end{cases} \tag{2.6}$$

The geometric action of $\Delta_{\alpha\beta}(\not{L}^{y'})$ on $f_\mu(z,L)$ comes directly from its definition and may easily be recognized in (2.5). The first two lines in this equation contain the perturbation of the infinitesimal parallelogram $\delta\sigma^{\alpha\beta}$ at the point y' on the $\delta^4(z-y)$ and $F_{\lambda\mu}(\not{L}^y)$ elements contained in $f_\mu(z,L)$. The last line measures the overall differential action of the Δ operator on the remaining L dependence of $f_\mu(z,L)$. It is easy to see that no effect is produced if y' is posterior to y along L, and this is taken conveniently into account with the introduction of the $\theta_L(y-y')$ loop-dependent step function.

Then according to (2.5) the following is immediately obtained:

$$\begin{aligned}\delta_\nu(x)f_\mu(z,L)=&\delta^4{}_{,\nu}(z-x)f_\mu(x,L)+\oint_L dy'^\rho\delta^4{}_{,\rho}(z-y')\delta^4(x-y')F_{\mu\nu}(\not{L}^{y'})\\&+\delta^4(z-x)\oint_L dy'^\rho\delta^4(x-y')[D_\nu F_{\mu\rho}(\not{L}^{y'})+D_\rho F_{\nu\mu}(\not{L}^{y'})+D_\mu F_{\rho\nu}(\not{L}^{y'})]\\&+\delta^4(z-x)\oint_L dy'^\rho\delta^4(x-y')D_\mu F_{\nu\rho}(\not{L}^{y'})\\&+\oint_L\oint_L dy^\lambda dy'^\rho\theta_L(y-y')\delta^4(z-y)\delta^4(x-y')\Delta_{\nu\rho}(\not{L}^{y'})F_{\mu\lambda}(\not{L}^{y})\,.\end{aligned}\tag{2.7}$$

From this expression and after some trivial redistribution of terms one easily arrives at

$$\begin{aligned}\delta_\nu(x)f_\mu(z,L)-\delta_\mu(z)f_\nu(x,L)+[f_\mu(z,L),f_\nu(x,L)]\\=&\delta^4{}_{,\nu}(z-x)f_\mu(x,L)-\delta^4{}_{,\mu}(x-z)f_\nu(z,L)\\&-\oint_L dy'^\rho\delta^4(z-y')\delta^4(x-y')[D_\rho F_{\mu\nu}(\not{L}^{y'})+D_\mu F_{\nu\rho}(\not{L}^{y'})+D_\nu F_{\rho\mu}(\not{L}^{y'})]\\&+\oint_L\oint_L dy^\lambda dy'^\rho\delta^4(z-y)\delta^4(x-y')\{\Delta_{\nu\rho}(\not{L}^{y'})F_{\mu\lambda}(\not{L}^{y})-[F_{\nu\rho}(\not{L}^{y'}),F_{\mu\lambda}(\not{L}^{y})]\}\,.\end{aligned}\tag{2.8}$$

Therefore, if $F_{\mu\nu}(P)$ satisfies the Bianchi identities and Mandelstam equations one has the improper chiral equation

$$\delta_\nu(x)f_\mu(z,L)-\delta_\mu(z)f_\nu(x,L)+[f_\mu(z,L),f_\nu(x,L)]=\delta^4{}_{,\nu}(z-x)f_\mu(x,L)-\delta^4{}_{,\mu}(x-z)f_\nu(z,L)\tag{2.9}$$

with distributional counterterms.

Moreover, if the dynamical Yang-Mills equation

$$D^\mu F_{\mu\rho}(P)=0\tag{2.10}$$

is satisfied one obtains an ill-defined second chiral equation of the form

$$\lim_{x\to z}\delta_\mu(x)f^\mu(z,L)=\lim_{x\to z}\delta^4{}_{,\mu}(z-x)f^\mu(x,L)\,.\tag{2.11}$$

One distributional problem associated with the formulation is unambiguously emphasized by these improper chiral equations. This problem may also be related to the noncommutative character of δ operators. In fact, by using the definition (2.4) and the kinematical equations of the Yang-Mills field, one immediately obtains

$$[\delta_\mu(z),\delta_\nu(x)]W(L)=[\delta^4{}_{,\mu}(x-z)\delta_\nu(z)-\delta^4{}_{,\nu}(z-x)\delta_\mu(x)]W(L)\,,\tag{2.12}$$

which shows explicitly the troublesome distributional terms arising from the commutator of δ operators.

Nevertheless, it is remarkable that a redefinition of the differential operators of the formulation eliminates all distributions to obtain proper chiral equations.

Explicitly, let the $\Delta_\mu(z)$ operator be defined as follows:

$$\Delta_\rho(z)W(L)=\delta_\mu(z)W(L)\tag{2.13}$$

which implies

$$f^\mu(z,L)=[\Delta^\mu(z)W(L)]W^{-1}(L)\,,\tag{2.14}$$

$$\Delta_\mu(x)f_\nu(z,L)=\delta_\mu(x)f_\nu(z,L)+\delta^4{}_{,\mu}(x-z)f_\nu(x,L)\,.\tag{2.15}$$

Then it is immediately proved that the kinematic Yang-Mills equations imply

$$\Delta_\nu(x)f_\mu(z,L)-\Delta_\mu(z)f_\nu(x,L)+[f_\mu(z,L),f_\nu(x,L)]=0\,,\tag{2.16}$$

and the dynamical equations (2.10) lead to

$$\lim_{x\to z}\Delta_\mu(x)f^\mu(z,L)=0\,.\tag{2.17}$$

The commutator of the Δ operators may also be worked out:

$$[\Delta_\mu(z),\Delta_\nu(x)]W(L)=0 \ . \tag{2.18}$$

It is also interesting to notice that the second chiral equation (2.17) holds more properly in the form

$$\int d^4z\, \Delta_\mu(z) f^\mu(z,L)=0 \ , \tag{2.19}$$

which shows clearly that Yang-Mills equations lead to an infinite-dimensional chiral systems where $f^\mu(z,L)$ should be considered as a vector field with a direct Lorentz index μ and a continuous index z, so (2.19) furnishes the total divergence of the object. Therefore, a more appropriate notation should be

$$f^\mu(z,L) \equiv f^{(\mu z)}(L) \ , \tag{2.20}$$

where (μz) represents the continuous Lorentz index for the loop-space vector field $f^\mu(z,L)$.

Although the introduction of the operator $\Delta_\mu(z)$ produces appropriate chiral equations, it is apparent that it has only been defined by its action on $W(L)$ by (2.13) and on $f_\mu(z,L)$ by (2.20). It is obviously highly desirable to provide a general definition of this object for its differential action on functions with arbitrary dependence on continuous variables and loops. Equations (2.13) and (2.15) seems to indicate that $\Delta_\mu(z)$ recognizes the tensorial character of the function on which it is acting. This operator behaves then like the covariant derivative in an anholonomic system of coordinates. It is an interesting point to explore whether this analogy may lead to the desired general definitions of $\Delta_\mu(z)$ as a differential operator. Nevertheless, we shall show in the next section that the partial definition contained in (2.13) and (2.15) is sufficient to produce an infinite set of conserved currents within a Lagrangian approach.

III. LAGRANGIAN FORMULATION AND CONSERVED QUANTITIES

A Lagrangian formulation for the loop-dependent chiral system was recently proposed by Dolan.[5] The analogous object in terms of the present geometric operators is simply

$$\mathscr{L}(L)=\int d^4x\, \mathrm{Tr}[\Delta_\mu(x)W(L)\Delta^\mu(x)W^{-1}(L)] \ . \tag{3.1}$$

The action for the system should be written as

$$S=\int DL\, \mathscr{L}(L) \ , \tag{3.2}$$

where the sum over loop space is supposed to have the appropriate measure to assure the Gauss theorem for total Δ derivatives. In such a case the following are immediately obtained as equations of motion:

$$\int d^4x\, \Delta^\mu(x) f_\mu(x,L)=0 \tag{3.3}$$

with

$$f_\mu(x,L)=[\Delta_\mu(x)W(L)]W^{-1}(L) \ . \tag{3.4}$$

The commutativity of Δ derivatives produces immediately the kinematic chiral equation

$$\Delta_\nu(x) f_\mu(z,L)-\Delta_\mu(z) f_\nu(x,L) +[f_\mu(z,L),f_\nu(x,L)]=0 \ . \tag{3.5}$$

It is important to remark that (3.4) and (3.5) do not imply the Yang-Mills equations. They certainly would if $S(L)$ were Yang's phase factor. However, this information cannot be furnished in the Lagrangian approach where $W(L)$ must be the field variable.

However, from the differential structure of $\Delta_\mu(x)$ and definition (3.4) one may infer the structure

$$f_\mu(z,L)=\oint_L dy^\lambda \delta^4(z-y) F_{\lambda\mu}(y,L) \ , \tag{3.6}$$

where $F_{\lambda\mu}(y,L)$ is a loop-dependent antisymmetric Lorentz tensor. If $F_{\lambda\mu}(y,L)$ depends only on $\not{L}^y$, one recovers the form (2.4) and chiral equations imply Yang-Mills equations for $F_{\lambda\mu}$ in this case. A necessary and sufficient condition to ensure this fact is

$$\theta_L(y'-y)\Delta_{\alpha\beta}(\not{L}^{y'})F_{\mu\nu}(y,L)=0 \ , \tag{3.7}$$

which simply states that $F_{\mu\nu}$ does not depend on that portion of the loop after y. By projecting this equation with the geometrical tangent (2.1) one may obtain the subsidiary condition

$$\Delta_\mu(x) f_\nu(z,L)=0 \tag{3.8}$$

for x,z on L and x posterior to z.

The need of subsidiary conditions on $f_\mu(z,L)$ agrees with the remark by Dolan[5] that the chiral Lagrangian does not imply Yang-Mills equations. An interesting possibility could be to incorporate subsidiary conditions into the chiral Lagrangian in order to obtain equivalence between Yang-Mills and chiral equations.

In any case, since the chiral equations are implied by Yang-Mills equations, any conserved quantity obtained from (3.1) will obviously produce a conserved quantity for the Yang-Mills system.

The geometric formulation of the chiral system allows one to easily obtain an infinity of conserved quantities which cannot be obtained within the conventional parametric approach.

For instance, let us consider the total variation on $W(L)$ induced by a "local translation" $\epsilon^\mu(x)$:

$$\delta W(L)=\int d^4x\, \epsilon^\mu(x)\Delta_\mu(x)W(L) \ . \tag{3.9}$$

By using the obvious extension of Noether's theorem it is easily to obtain the conserved quantities

$$T^{\mu}{}_{\nu}(x,y,L)=2\,\mathrm{Tr}[f_{\nu}(x,L)f^{\mu}(y,L)]-\delta^{\mu}{}_{\nu}\,\delta^{4}(x-y)\mathrm{Tr}\int d^{4}z\, f_{\lambda}(z,L)f^{\lambda}(z,L)\ . \qquad (3.10)$$

Within the usual parametric approach only global translations ϵ^{μ} may be incorporated in a natural way. The corresponding conserved quantities could be obtained from (3.9) by integrating the y dependence. Hence (3.9) contains a wide generalization of the conserved quantities obtained from translational invariance.

These quantities are conserved in the "total-divergence" sense:

$$\int d^{4}x\, \Delta_{\mu}(x)T^{\mu}{}_{\nu}(x,y,L)=0\ . \qquad (3.11)$$

Let us also notice that (3.10) is automatically a symmetric object. This property does not follow in the parametric Lagrangian approach.

We have already pointed out in the preceding section the analogy between x dependence in the chiral objects with a continuous vector index. The exciting possibility to consider appropriate loop-dependent linear transformations in the x indices as symmetry groups to obtain new conserved quantities remains to be explored.

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