Complex-time path integrals beyond the stationary-phase approximation: Decay of metastable states and quantum statistical metastability

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The complex-time path-integral expression for $\mathrm{Tr}(E-H)^{-1}$ is reexamined and applied to barrier-penetration effects in quantum mechanics and field theory. Shortcomings of the conventional method for low-lying states are analyzed and resolved. This is achieved by an appropriate treatment of quasisymmetry modes occurring in the path integral and by computing certain contributions to the Fourier transform of the path integral without a stationary-phase approximation. We obtain agreement with the results of the instanton method for the ground state. For growing quantum numbers our results smoothly approach those of the standard WKB approximation. We also consider a scalar field theory with a false vacuum and compute the decay rates Γ_n of n-meson states built thereon. Final emphasis is placed on the quantum-statistical metastability of a grand canonical ensemble of these mesons at temperature β^{-1}.

I. INTRODUCTION

In the last decade there has been considerable effort devoted to the calculation of the energy spectrum in quantum mechanics and quantum field theory by functional-integral methods. Dashen, Hasslacher, and Neveu[1] computed the energy spectrum of various models in the semiclassical approximation by considering a complete set of classical periodic solutions. However, since their procedure was restricted to real-time (Minkowski) solutions, it could not account for barrier-penetration effects in the spectrum. A few years later Polyakov[2] proposed to evaluate the imaginary-time (Euclidean) version of the functional integral. This computational method (instanton method) was applied with great success to the ground-state splitting of the double-well potential[3,4] and to the spontaneous decay of the false vacuum due to quantum tunneling.[3,5]

Recently, Patrascioiu[6] has discussed in detail that in general neither real- nor imaginary-time solutions are sufficient because, at a given energy, they extend only over a restricted region of the potential. However, certain quantum effects, e.g., the level splittings or decay rates of excited states, are influenced by the global structure of the potential. To obtain the correct description of such processes one must use complex-time classical solutions[7] which can probe the total relevant region of the potential at a given energy.

A very convenient starting point to compute the energy spectrum is the Fourier transform of the trace of the Minkowskian kernel

$$G(E)\equiv \mathrm{Tr}(H-E)^{-1} = \frac{i}{\hbar}\int_0^\infty dt\, e^{iEt/\hbar}\int dx\,\langle x\,|\,e^{-iHt/\hbar}\,|\,x\rangle\ . \tag{1.1}$$

The virtue of this expression is that one never has to construct a wave function which is of great advantage in field theory. When a functional-integral representation of the kernel is inserted, classical periodic solutions with complex time arise completely naturally through the stationary-phase approximation of the time integral in Eq. (1.1). This way of computing $G(E)$[6,8] leads to the standard WKB approximation both for the level splittings and for the decay rates of metastable states by barrier penetration. This result, however, is not correct for low-lying states. In particular, the results for the ground state do not agree with those obtained by the instanton method.

In this paper we analyze the functional-integral expression of $G(E)$ more thoroughly and show how to derive the correct semiclassical approximation of barrier-penetration effects, which is also valid for low-lying states. This progress is achieved by taking into account the following two reasons. (1) The correct semiclassical kernel is written as a certain convolution of standard semiclassical kernels. This is nothing but a convenient method to handle various approximate zero modes being inherent in the problem. (2) Various pieces of the times integral in Eq. (1.1) must be computed without a stationary-phase approximation.

This procedure is carried out in Sec. II and applied to the bound-state spectrum of the quantum-mechanical double-well potential. In Sec. III, we compute the decay rates of a false ground state and of excited states built thereon. We also consider the Boltzmann average of these decay rates and compare our findings with a recent work of Affleck.[9]

In Sec. IV, these methods are extended to a scalar field theory with a false vacuum and with n-particle states (mesons) built on the false vacuum. These states are analogous to the false ground state and to the excited states in the quantum-mechanics problem of Sec. III. The decay rate of the false vacuum Γ_0 is the spontaneous decay rate, whereas the additional decay rates Γ_n are induced by the mere presence of n mesons.[10] An expression for the induced vacuum decay is derived in terms of the quantities that are required to compute the spontaneous decay. Some explicit results are obtained in the limit of vanishing energy density difference between the true and the false vacuum. We finally compute the vacuum decay induced by the presence of an ideal gas of mesons at a given temperature.

II. BOUND-STATE SPECTRUM OF THE DOUBLE WELL

We consider a particle of unit mass in a symmetrical double-well potential of the form depicted in Fig. 1,

$$H=\frac{1}{2}\left[\frac{dx}{dt}\right]^2+V(x) . \quad (2.1)$$

We want to compute the level splittings of the ground state and of excited states by functional-integral methods. Our fundamental tool will be a functional-integral representation of the Minkowskian kernel $\langle x_f \mid \exp(-iHt_f/\hbar) \mid x_i \rangle$,

$$K(x_f,t_f;x_i)=\int_{x_i}^{x_f} \mathscr{D}x(t)\exp(iS/\hbar) , \quad (2.2)$$

$$S(x_f,t_f;x_i)=\int_0^{t_f} dt\,[\tfrac{1}{2}\dot{x}^2-V(x)] . \quad (2.3)$$

The action S is calculated along an arbitrary trajectory $x(t)$ which satisfies the boundary conditions $x(0)=x_i$ and $x(t_f)=x_f$.

In the semiclassical limit ($\hbar\rightarrow 0$) the functional integral (2.2) is dominated by the stationary points (and almost stationary points) of S, which we denote by $x_c(t)$,

$$\frac{\delta S}{\delta x_c}=-\ddot{x}_c-V'(x_c)=0 . \quad (2.4)$$

Let us first assume the case of only one stationary point. Then, the stationary-phase approximation of

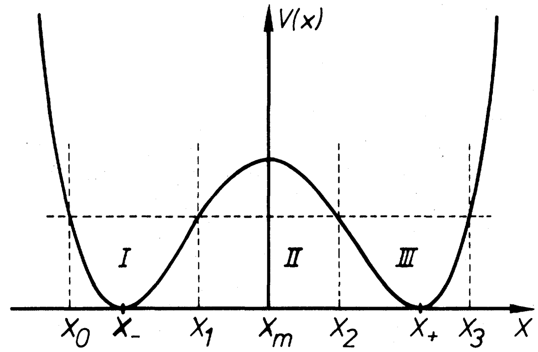

FIG. 1. The double-well potential for a quantum-mechanics theory with minima at x_- and x_+ and a maximum at x_m. The intercepts x_0, x_1, x_2, and x_3 are the turning points of the classical motion in the Minkowskian regions I and III and in the Euclidean region II, respectively.

the functional integral (2.2) gives the well-known semiclassical kernel[1]

$$K_s(x_f,t_f;x_i)=(-2\pi i\hbar)^{-1/2}\left[\dot{x}_c(0)\dot{x}_c(t_f)\frac{\partial^2 W}{\partial z^2}\right]^{-1/2}\times\exp(iS_c/\hbar) , \quad (2.5)$$

$$W=\int_{x_i}^{x_f} dx[2(z-V)]^{1/2} , \quad (2.6)$$

where $S_c=W-zt_f$ is the classical action and z is the energy associated with the classical path x_c, $z=\dot{x}_c{}^2/2+V(x_c)$.

When discussing barrier-penetration problems we also have to consider trajectories in the classically forbidden x range, where $z<V(x)$. These are solutions of $d^2x_c/d\tau^2-V'(x_c)=0$ and describe the motion $x_c(\tau)$ of a particle in the potential $U(x)\equiv -V(x)$. Here, τ is the Euclidean time being obtained from the Minkowskian time t by a formal analytic continuation $\tau=it$. In that range we conveniently use the Euclidean action which is defined as $-i$ times the analytic continuation of the Minkowskian action,

$$\tilde{S}_c=\tilde{W}+z\tau , \quad (2.7)$$

$$\tilde{W}=\int_{x_i}^{x_f} dx\,[2(V-z)]^{1/2} . \quad (2.8)$$

Let us now tackle a specific problem in the evaluation of the functional integral which arises from the classical path in the Euclidean range of the potential, where

$$\frac{dx_c}{d\tau}=[2(V-z)]^{1/2} . \tag{2.9}$$

Assuming $z=O(\hbar\omega)$, the particle will stay for a large Euclidean time close to its classical turning points x_1 and x_2 (see Fig. 1) and it will move quickly through the valley of the potential $-V(x)$. Thus, each traverse of the Euclidean region is a kinklike trajectory. Its action has an approximate time-translation symmetry. There results an approximate zero eigenvalue $\lambda_0(x_c)$ of the second variation operator $\hat{L}(x_c)=\delta^2\tilde{S}_c/\delta x_c{}^2$, so that the approximation (2.5) breaks down.

The standard way to proceed in such a situation is to treat the kink position as a collective time coordinate which serves to eliminate the quasisymmetry mode. Calculating $\text{Det}[\hat{L}(x_c)]$ and $\lambda_0(x_c)$ in a reasonable approximation, it turns out that

$$\text{Det}[\hat{L}(x_c)]/\lambda_0(x_c)\propto \text{Det}[\hat{L}(x_{c1})]\,\text{Det}[\hat{L}(x_{c2})] , \tag{2.10}$$

where x_{c1} and x_{c2} are the two sections of the kink x_c. After all, an improved semiclassical kernel is obtained in the form of a convolution of standard semiclassical kernels[11]

$$K_{is}(x_f,t_f;x_i)=\int_0^{t_f} dt\, K_s(x_f,t_f-t;x_m)\times\dot{x}_c(t)K_s(x_m,t;x_i) . \tag{2.11}$$

This expression is appropriate when x_i and x_f lie in the opposite Minkowskian regions I and III and when a single traversal of the Euclidean region II is taken into account. The position x_m is chosen somewhere in the Euclidean region II, only in due distance from the turning points x_1 and x_2. However, for the symmetrical double well it is convenient to choose x_m lying at the maximum of the barrier $V'(x_m)=0$. It should be mentioned that computing the convolution by the method of steepest descent would only reveal a semiclassical propagator of the form (2.5). Thus it is substantial to evaluate the convolution by different methods whenever a quasisymmetry is involved in the problem.

We demonstrate in the Appendix that the one-instanton contribution to the Euclidean kernel $\langle x_-\,|\exp(-HT/\hbar)\,|x_+\rangle$ is easily obtained from a convolution of the form (2.11).

We now evaluate Eq. (1.1) in the semiclassical approximation. As far as the algebra of the various path contributions is concerned, we closely follow Refs. 1 and 8. The trace operation, which appears as a periodicity condition in the path space, leads to the following contributions:

$$G(E)=G_{\text{I}}(E)+G_{\text{II}}(E)+G_{\text{III}}(E) , \tag{2.12}$$

where $G_j(E)$ ($j=\text{I,II,III}$) is determined by trajectories which begin and end in region j of Fig. 1. Let us first calculate $G_{\text{I}}(E)$,

$$G_{\text{I}}(E)=\frac{i}{\hbar}\int_{\text{I}} dx_i \int_0^\infty dt\, e^{iEt/\hbar}K_{is}(x_i,t;x_i) . \tag{2.13}$$

For closed orbits in region I K_{is} depends on the starting point only by a factor $\{2[z-V(x_i)]\}^{-1/2}$. Thus, the freedom of the starting point to be anywhere along the orbit in region I leads to a factor

$$\int_{\text{I}} dx_i\{2[z-V(x_i)]\}^{-1/2}=\frac{\partial W_1(z)}{\partial z}=T_1(z) , \tag{2.14}$$

$$W_1(z)=2\int_{x_0}^{x_1} dx\,[2(z-V)]^{1/2} , \tag{2.15}$$

where $T_1(z)$ is the period of the orbit of energy z in region I. As far as only periodic orbits in region I are considered, no dangerous quasisymmetry modes are present and we may insert $K_{is}=K_s$ in Eq. (2.13). Further, the time integral may be evaluated in the stationary-phase approximation at the stationary-phase point $z(t)=E$. Thus, one periodic orbit in region I gives the following contribution to $G_{\text{I}}(E)$:

$$A_{\text{I}1}(E)=\frac{i}{\hbar}T_1(E)U_1(E) , \tag{2.16}$$

$$U_1(E)=-\exp[iW_1(E)/\hbar] , \tag{2.17}$$

where the minus sign in Eq. (2.17) is twice the phase factor $\exp(i\pi/2)$, which is introduced by each turning point.[1] Considering multiple traverses of the basic orbit we are led to a geometrical series in $U_1(E)$,

$$A_{\text{I}}(E)=\sum_{n=1}^{\infty}A_{\text{I}n}=\frac{i}{\hbar}T_1(E)\frac{U_1(E)}{1-U_1(E)} . \tag{2.18}$$

This expression has poles with unit residue $A_{\text{I}}(E\approx E_n)=1/(E_n-E)$, where

$$W_1(E_n)=(2n+1)\pi\hbar . \tag{2.19}$$

For a single-well potential, Eq. (2.18) is the correct semiclassical result $G(E)=A_{\text{I}}(E)$.

Let us next consider a path with starting point x_i somewhere in region I which enters the Euclidean region at x_1, returns at the point x_2, and attains the starting point after a detour via the turning point x_0 (see Fig. 1). According to our previous discussion two quasisymmetry modes are present in the corresponding functional-integral expression. Thus the appropriate semiclassical approximation is given by a convolution of three standard semiclassical kernels (2.5) being joined together at x_m,

$$K_{is}(x_i,t;x_i) = -2V(x_m)\int_0^t dt_1 \int_0^{t_1} dt_2 K_s(x_i,t-t_1;x_0;x_m)K_s(x_m,t_1-t_2;x_2;x_m)K_s(x_m,t_2;x_i) , \tag{2.20}$$

where the additional parameters x_0 and x_2 are to remind us of the turning points of the path considered. In Eq. (2.20) both velocity factors at the junction x_m are approximated by $[2V(x_m)]^{1/2} \times \exp(i\pi/2)$. After insertion of Eq. (2.20) into Eq. (2.13), we obtain a contribution to $G_{\rm I}(E)$, denoted by $B_{\rm II}(E)$, being expressed in the form of a threefold time integral. The contours of the time integrals are to be deformed in the complex t plane to pass through all the stationary points of this expression. By a shift of the integration variables a convolution of the first with the third kernel in Eq. (2.20) can be isolated. This convolution, however, can be evaluated by a stationary-phase approximation since it lacks the quasisymmetry problem

$$\int dt' K_s(x_m,t''-t';x_i)K_s(x_i,t';x_0;x_m) = K_s(x_m,t'';x_0;x_m)\{2[z(t'')-V(x_i)]\}^{-1/2} . \tag{2.21}$$

Now observe that the freedom of the starting point to be anywhere in region I is again given by Eq. (2.14). Further, the remaining t'' integral has contributions both from the Minkowskian region I and Euclidean region II. The former contribution, however, can be evaluated by a stationary-phase approximation. Thus we obtain

$$B_{\rm II}(E) = -\frac{i}{\hbar}T_1 U_1 Y_1 Y_2 , \tag{2.22}$$

where Y_j $(j=1,2)$ is the contribution of half a cycle in region II with turning point x_j,

$$Y_j(E) = [2V(x_m)]^{1/2} \times \int_\alpha dt\, e^{iEt/\hbar} K_s(x_m,t;x_j;x_m) . \tag{2.23}$$

Again, α is the integration contour suitably deformed as discussed above. Note that the factorization Y_1Y_2 in Eq. (2.22) arises from the two quasisymmetry modes being taken into account. Next we insert Eq. (2.5) and the appropriate turning point factors and change over to the Euclidean time variable $\tau = it$. There results

$$Y_2(E) = -i(2\pi\hbar)^{-1/2}\int_\alpha d\tau \left|\frac{\partial^2 \widetilde{W}_2}{\partial z^2}\right|^{-1/2} \exp\{-[\widetilde{W}_2(z)+(z-E)\tau]/\hbar\} , \tag{2.24}$$

$$\widetilde{W}_2(z) = 2\int_{x_m}^{x_2} dx\,[2(V-z)]^{1/2} , \tag{2.25}$$

and an analogous expression for $Y_1(E)$. Here, the running energy $z=z(\tau)$ is determined by resolving the relation $\tau = -\partial\widetilde{W}_2(z)/\partial z$. The saddle point of the integrand in Eq. (2.24) is on the positive real axis of the complex τ plane at $\tau_s \equiv \tau(E)$ and the direction of steepest descent is perpendicular to the real axis.

Prior to the further evaluation of the integral (2.24) we now first complete the sum over the various path contributions to $G(E)$.

We have to attach an arbitrary number of cycles in the Minkowskian regions I and III to the basic trajectory which has led us to Eq. (2.22) and have to sum over all of them. Thus we obtain

$$B_{\rm I}(E) = \frac{i}{\hbar}T_1\frac{U_1}{1-U_1}R , \qquad R = -\frac{Y_1Y_2}{(1-U_1)(1-U_3)} , \tag{2.26}$$

where U_3 is the contribution of one cycle in region III and is defined analogously to Eq. (2.17). $B_{\rm I}(E)$ comprises only one cycle in region II. Summing up multiple cycles in region II generates a geometrical series in R. Thus we arrive at

$$G_{\rm I}(E) = \frac{i}{\hbar}T_1\frac{U_1}{1-U_1}\frac{1}{1-R} . \tag{2.27}$$

An analogous expression is obtained for $G_{\rm III}(E)$. Further, it turns out that $G_{\rm II}(E)$ is suppressed by an extra factor $\exp[-\widetilde{W}_2(E)/\hbar]$ as compared to $G_{\rm I}$ and $G_{\rm III}$ and therefore may be neglected. Thus the semiclassical approximation of $G(E)$ assumes the form

$$G(E) = \frac{i}{\hbar}\frac{[T_1U_1(1-U_3)+T_3U_3(1-U_1)]}{(1-U_1)(1-U_3)+Y_1Y_2} , \tag{2.28}$$

which holds for a general double-well potential. In the case of a symmetrical double well we have $T_1 = T_3$, $U_1 = U_3$, and $Y_1 = Y_2$.

Assuming that Y_2 is small we may find the poles of Eq. (2.28) iteratively.[8] For $E \approx E_n$, where E_n is determined by Eq. (2.19), there results from Eq. (2.28)

$$G(E \approx E_n) = \frac{1}{E_n + \Delta E_n/2 - E} + \frac{1}{E_n - \Delta E_n/2 - E} , \tag{2.29}$$

where ΔE_n is splitting of the nth excited state

$$\Delta E_n = \frac{2\hbar}{T_1(E_n)} Y_2(E_n) . \tag{2.30}$$

Equation (2.30) is the semiclassical result of the level splitting in a symmetrical double well by barrier penetration.

Let us now return to the further evaluation of $Y_2(E_n)$. In recent papers,[6,8] the Euclidean time integrations are done by the method of steepest descent. Assuming that this is correct we obtain

$$Y_2(E_n) = Y_2^{(s)}(E_n) \equiv \exp[-\widetilde{W}_2(E_n)/\hbar] . \tag{2.31}$$

Thus we arrive at the standard WKB approximation

$$\Delta E_n^{(s)} = \frac{2\hbar}{T_1(E_n)} \exp[-\widetilde{W}_2(E_n)/\hbar] . \tag{2.32}$$

It is well known that Eq. (2.32) is not correct for low-lying states $E = O(\hbar\omega)$, where $\omega^2 = V''(x_+)$. We wish to emphasize, however, that our result (2.30) is more general and also holds in that energy range, where Eq. (2.32) fails. Rather it is the saddle-point approximation of the integral in Eq. (2.24) which breaks down for $E = O(\hbar\omega)$. In this case considerable contributions to the integral come from those z values for which the smoothness assumption of $\partial^2\widetilde{W}_2/\partial z^2$ is not valid.

In order to obtain explicit results we now solve Eq. (2.24) in a reasonable approximation. With the method of the Appendix [see Eq. (A8)], we obtain

$$\widetilde{W}_2(z) = S_0 - \frac{z}{\omega} + \frac{z}{\omega} \ln\left[\frac{2z}{A^2}\right] , \tag{2.33}$$

where $S_0 = \widetilde{W}_2(0)$ and the constant A is determined by the asymptotic behavior (A7) of the instanton trajectory $\bar{x}(\tau) = x_+ - (A/2\omega)\exp(-\omega\tau)$. Insertion of Eq. (2.33) in Eq. (2.32) gives

$$\Delta E_n^{(s)} = \frac{\hbar\omega}{\pi} \left[\frac{A^2 e}{\hbar\omega(2n+1)}\right]^{n+1/2} \exp(-S_0/\hbar) , \tag{2.34}$$

where we have restricted ourselves to the harmonic approximation in the wells, $E_n = \hbar\omega(n + \frac{1}{2})$.

However, it is possible to evaluate the integral (2.24) exactly when Eq. (2.33) is inserted. Changing over to the integration variable $u = (A^2/2\hbar\omega) \times \exp(-\omega\tau)$, we arrive at the expression

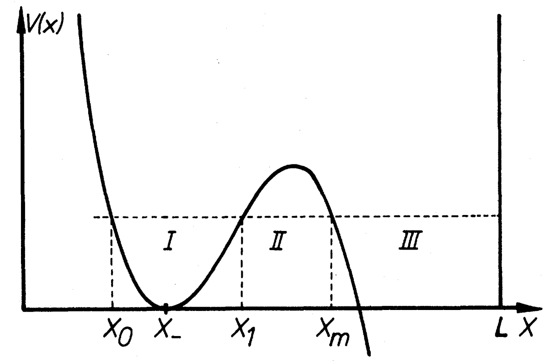

FIG. 2. The potential for a quantum-mechanics theory with a metastable ground state.

$$Y_2(E) = \sqrt{2\pi} \left[\frac{A^2}{2\hbar\omega}\right]^{E/\hbar\omega} \times \exp(-S_0/\hbar) \frac{1}{2\pi i} \int_\beta du\, u^{-\nu} e^u , \tag{2.35}$$

where $\nu = E/\hbar\omega + \frac{1}{2}$. Since $\nu \geq 1$, the integration contour β can be continuously deformed to encircle the negative real axis of the complex u plane in the counterclockwise sense. The integral in Eq. (2.35) thus obtained is Hankel's contour integral of the inverse Γ function $1/\Gamma(\nu)$. Thus we obtain from Eq. (2.30)

$$\Delta E_n = \frac{1}{n!} \left[\frac{A^2}{2\hbar\omega}\right]^n \Delta E_0 , \tag{2.36}$$

where the splitting of the ground state ΔE_0 is given by Eq. (A11). For $n = 0$ our result (2.36) is identical to the result of the instanton method Eq. (A11). For $n \gg 1$, we may approximate $n!$ in Eq. (2.36) by the asymptotic Stirling formula. Just in this way the standard WKB result (2.34) is obtained from Eq. (2.36).

Summarizing, there are three basic ingredients in our derivation of the final result (2.36): first, convolutions of semiclassical propagators in order to deal with the quasisymmetry modes; second the sum over multiple traverses of the basic orbits in the regions I, and II, and III; and third a treatment of the Euclidean time integrals without resort to a saddle-point approximation.

Let us at the end of this section shortly discuss a slightly asymmetric double-well potential. We add to the symmetrical potential $V_s(x)$ of Fig. 1 a linear term

$$V_{as}(x) = V_s(x) + \Delta V \frac{x}{x_+} \tag{2.37}$$

and wish to consider ΔV small compared to $\hbar\omega$. The case of ΔV large is the problem of the false vacuum decay and will be discussed in Sec. III. According to our choice $\Delta V \ll \hbar\omega$, we may further assume $Y_1 = Y_2$ in Eq. (2.28). By the insertion of

$$U_1(E \approx E_n) = 1 + \frac{i}{\hbar} T_1(E - E_n - \Delta V) ,$$
$$U_3(E \approx E_n) = 1 + \frac{i}{\hbar} T_1(E - E_n + \Delta V) , \tag{2.38}$$

in Eq. (2.28) we immediately obtain the poles of $G(E)$ at

$$E = E_n \pm [\tfrac{1}{4}\Delta E_n{}^2 + \Delta V^2]^{1/2} , \tag{2.39}$$

where ΔE_n is given by Eq. (2.30). For $n = 0$ the same result has been obtained in a recent work by Levine.[12] This author used instanton methods and exploited an analogy with an Ising model in an external field.

III. DECAY WIDTHS OF UNSTABLE STATES AND QUANTUM-STATISTICAL METASTABILITY

In this section we demonstrate how to apply the above methods to a system with a false ground state and with excited states built thereon. We calculate the individual decay rates of these quantum states. Also the Boltzmann average of these decay rates is considered.

We again consider a problem which is described by the Hamiltonian (2.1), where now the potential has a form as depicted in Fig. 2. Such a system has well-localized quantum states in the well around $x = x_-$ with finite escape probabilities through the barrier. It is convenient to assume a large normalization box with an infinite well at $x = L$ in region III and to consider the limit of L going to infinity. Then, complex time periodic solutions exist in regions I, II, and III completely analogous to the problem in Sec. II and $G(E)$ is again given by Eq. (2.28).

Since the Hamiltonian is Hermitian, $G(E)$ can have poles only on the real axis. It is therefore impossible to obtain the decay rates directly from $G(E)$. It has been stressed in the literature,[6,8] that one should rather consider the smoothed density of states[13]

$$\rho_\gamma(E) = \frac{1}{\pi} \operatorname{Im} G(E + i\gamma) . \tag{3.1}$$

Here, the discrete states in the box are smoothed with a Lorentzian width γ which must be chosen (i) large compared to the level spacing δ in the quasicontinuum and (ii) small compared to the widths of the unstable states. Under the first condition periodic trajectories in region III contribute negligibly to $\rho_\gamma(E)$, since there holds[8] $| U_3(E + i\gamma) | \approx \exp(-\pi\gamma/\delta)$. The second condition guarantees that the imaginary part γ is still negligible in the corresponding expressions for regions I and II. Thus we obtain, with Eq. (2.28),

$$\rho_\gamma(E) = \frac{T_1}{\pi\hbar} \operatorname{Re} \frac{U_1}{1 - U_1 + X_2} , \tag{3.2}$$

where $X_2(E)$ is the contribution of one periodic orbit in region II to $\rho_\gamma(E)$. The expression $X_2(E)$ differs from $Y_1 Y_2$ in Eq. (2.28) by two subtleties.

(1) Since a periodic orbit in region II bounces off the turning point x_2, only one quasisymmetry mode is associated with that trajectory. Thus, Eq. (2.20) can be replaced by an expression which involves only one convolution of semiclassical propagators. Most conveniently these propagators are joined together at $x_m = x_2$.

(2) It is well known that there is one fluctuation mode about the bounce trajectory which has a large negative eigenvalue. It has been discussed in detail in the literature[3,14] that this behavior gives rise to an additional factor $\frac{1}{2}$ in the semiclassical propagator. Thus, $X_2(E)$ assumes the form

$$X_2(E) = -\frac{i}{2}(2\pi\hbar)^{-1/2} \int_\alpha d\tau \left[\frac{\partial^2 \widetilde{W}_2}{\partial z^2} \right]^{-1/2} \exp\{-[\widetilde{W}_2(z) + (z - E)\tau]/\hbar\} , \tag{3.3}$$

$$\widetilde{W}_2(z) = 2 \int_{x_1}^{x_2} dx\, [2(V - z)]^{1/2} , \tag{3.4}$$

where $\tau = -\partial \widetilde{W}_2(z)/\partial z$, and α is again a contour which passes through the saddle point in the direction of steepest descent.

For $\widetilde{W}_2 \gg \hbar$, the smoothed density of states, Eq. (3.2), has poles at complex energies $z_n = E_n - i\hbar\Gamma_n/2$, such that

$$W_1(E_n) = (2n + 1)\pi\hbar , \tag{3.5}$$

$$\Gamma_n = \frac{2}{T_1(E_n)} X_2(E_n) . \tag{3.6}$$

Equation (3.6) is the semiclassical result for the decay width of the quantum state n. The further discussion of our result (3.6) is much the same as in the preceding section below Eq. (2.30). Evaluating Eq.

(3.3) by the method of steepest descent gives $X_2(E)=X_{2s}(E)$, where

$$X_{2s}(E)\equiv\exp[-\widetilde{W}_2(E)/\hbar]/2 .$$

Thus we arrive at the standard WKB result[8]

$$\Gamma_n^{(s)}=\frac{1}{T_1(E_n)}\exp[-\widetilde{W}_2(E_n)/\hbar] . \quad (3.7)$$

Further, in the approximation analogous to Eq. (A8) we have

$$\widetilde{W}_2(E)=S_b-\frac{E}{\omega}+\frac{E}{\omega}\ln\left[\frac{2E}{A^2}\right] , \quad (3.8)$$

where $S_b=\widetilde{W}_2(0)$ is the action of the bounce $\bar{x}(\tau)$ and A is a characteristic constant occurring in the asymptotic behavior of $\bar{x}(\tau)$, having bounced off the turning point x_2 at $\tau=0$,

$$\bar{x}(\tau)=x_-+\frac{A}{2\omega}\exp(-\omega\tau) . \quad (3.9)$$

By insertion of Eq. (3.8) in Eq. (3.7), we obtain

$$\Gamma_n^{(s)}=\frac{\omega}{2\pi}\left[\frac{A^2e}{(2n+1)\hbar\omega}\right]^{n+1/2}\exp(-S_b/\hbar) , \quad (3.10)$$

where we have confined ourselves to the harmonic approximation $E_n=(n+\frac{1}{2})\hbar\omega$.

Like in the problem of Sec. II, the standard WKB result (3.7) or (3.10) is not correct for low-lying states. Again, this failure is caused by a breakdown of the saddle-point approximation of Eq. (3.3), which has led us to Eq. (3.7). However, by use of Eq. (3.8) we can evaluate Eq. (3.3) exactly and arrive at the correct semiclassical result

$$\Gamma_n=\frac{1}{n!}\left[\frac{A^2}{2\hbar\omega}\right]^n\Gamma_0 , \quad (3.11)$$

$$\Gamma_0=\left[\frac{\omega}{\pi\hbar}\right]^{1/2}\frac{A}{2}\exp(-S_b/\hbar) . \quad (3.12)$$

For $n=0$ and $n=1$ our result is identical to the findings of Affleck and De Luccia,[10] who evaluated the functional integral of the Euclidean propagator. For $n\gg1$, one may insert the asymptotic Stirling formula for $n!$ in Eq. (3.11), thus returning to $\Gamma_n^{(s)}$ again.

Let us next apply our result (3.11) to the decay of a system at a finite temperature. The equilibrium decay rate at temperature β^{-1} is determined by taking a Boltzmann average of Γ_n,

$$\Gamma=Z^{-1}\sum_{n=0}^{\infty}\Gamma_n\exp(-\beta E_n) , \quad (3.13)$$

$$Z=\sum_{n=0}^{\infty}\exp(-\beta E_n) . \quad (3.14)$$

At temperatures β^{-1} small compared to $\hbar\omega$ the particle is mainly in the low-lying metastable states, so that we may insert $E_n=(n+\frac{1}{2})\hbar\omega$ and our result (3.11) for Γ_n. Then we can perform the sums in Eqs. (3.13) and (3.14) explicitly and obtain

$$\Gamma=(1-e^{-\beta\hbar\omega})\exp\left[\frac{A^2}{2\hbar\omega}e^{-\beta\hbar\omega}\right]\Gamma_0 , \quad (3.15)$$

which exhibits a rapidly growing decay probability when the temperature is raised.

We conclude this section with some remarks on a recent treatment of the same problem by Affleck.[9] This work has three basic ingredients.

(i) The sum in Eq. (3.13) is replaced by an integral,

$$\Gamma=Z^{-1}\int_0^{\infty}dE\,\rho(E)\Gamma(E)\exp(-\beta E) . \quad (3.16)$$

(ii) The standard WKB result

$$\Gamma(E)=\frac{\omega}{2\pi}\exp[-\widetilde{W}_2(E)/\hbar] \quad (3.17)$$

is inserted in Eq. (3.16).

(iii) At low temperature the integral is dominated by energies close to a stationary point,

$$\beta\hbar=-\partial\widetilde{W}_2/\partial E\equiv T(E) . \quad (3.18)$$

The result is

$$\Gamma=Z^{-1}\left|2\pi\hbar\frac{\partial T}{\partial E}\right|^{-1/2}\exp[-(\widetilde{W}_2+ET)/\hbar] . \quad (3.19)$$

In order to compare Affleck's result (3.19) with our result (3.15), we have to use the expression (3.8) for $\widetilde{W}_2(E)$. The stationary point is at

$$E=E_s\equiv\frac{A^2}{2}\exp(-\beta\hbar\omega) . \quad (3.20)$$

We find that the resulting expression for Γ is identical with Eq. (3.15). Thus, to leading order in $\hbar$ the same value of Γ is obtained by both methods.

Finally, it should be mentioned that Affleck's calculation is valid in the semiclassical limit $\hbar\to0$ with $\beta\hbar\omega$ held fixed. In this limit we have

$$E_s\gg\hbar\omega . \quad (3.21)$$

Affleck's approximations are justified exactly under this condition. The interesting point, however, is that the resulting expressions (3.19) or (3.15) are still valid in the limit $\beta\hbar\omega\to\infty$. This property has been already mentioned by Affleck.

IV. INDUCED VACUUM DECAY IN FIELD THEORY

In the preceding section we explained how to use complex-time functional integrals to study the decay rates of a quantum-mechanics problem. In this section we will apply these methods to a scalar field theory with a false vacuum.

We consider the theory of a single scalar field in ν-dimensional space-time with dynamics defined by the Lagrangian density

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{4.1}$$

with a potential $V(\phi)$ as depicted in Fig. 3,

$$V(\phi)=\frac{\lambda}{8}\left[\phi^2-\frac{\omega^2}{\lambda}\right]^2+\epsilon\frac{\sqrt{\lambda}}{2\omega}\left[\phi-\frac{\omega}{\sqrt{\lambda}}\right] . \tag{4.2}$$

In fact, this is the only renormalizable theory in $\nu=4$ dimensions. The false vacuum $\phi=\phi_+ \equiv\omega/\sqrt{\lambda}$ differs from the true vacuum state $\phi=\phi_- \equiv -\omega/\sqrt{\lambda}$ by an amount of energy density $\Delta V=\epsilon$. The mesons built on the false vacuum have mass $m=\hbar\omega$. The state $\phi=\phi_+$ is a stable classical equilibrium. However, in the quantum theory it is unstable by barrier penetration.

We now first consider the bounce $\bar{\phi}(\vec{x})$, a solution in ν-dimensional Euclidean space of

$$\partial^2\phi=V'(\phi) \tag{4.3}$$

with $\bar{\phi}(\vec{x})\to\phi_+$ as $|\vec{x}|\equiv r\to\infty$. The bounce of lowest action is spherically symmetric,[15]

$$\left[\frac{d^2}{dr^2}+\frac{(\nu-1)}{r}\frac{d}{dr}\right]\bar{\phi}(r)=V'(\bar{\phi}) . \tag{4.4}$$

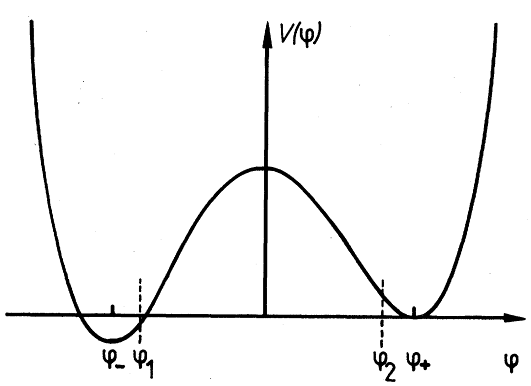

FIG. 3. The potential for a field theory with a metastable ground state. The intercepts ϕ_1 and ϕ_2 denote the boundary values of the distorted bounce solution considered in Eq. (4.15).

This is the equation for a particle of position $\bar{\phi}$ at time r in the potential $-V$ with a time-dependent damping force. In the limit $\epsilon\lambda/\omega^4 \ll 1$, explicit expressions for the bounce are available. The particle must start with zero velocity very close to ϕ_-, stays close to ϕ_- for some very large time $r=R$, then quickly moves through the valley with negligible friction and comes to rest at ϕ_+ at time infinity.[3] Thus, the bounce looks like a large ν-dimensional spherical bubble of true vacuum separated by a thin wall from the false vacuum without. The bubble wall is a solution of Eq. (4.4) for $\nu=1$,

$$\bar{\phi}(r\approx R)=\phi_+\tanh\left[\frac{\omega(r-R)}{2}\right] . \tag{4.5}$$

In the thin-wall approximation, the action S_b has a contribution from the bubble volume and a contribution from the bubble wall

$$S_b=-\frac{2\pi^{\nu/2}}{\nu\Gamma(\nu/2)}R^\nu\epsilon+\frac{2\pi^{\nu/2}}{\Gamma(\nu/2)}R^{\nu-1}s_1 , \tag{4.6}$$

where s_1 is the one-dimensional action

$$s_1=\int_{-\infty}^{+\infty}dr\left[\tfrac{1}{2}\bar{\phi}'^2+V(\bar{\phi})\right]=\frac{2}{3}\frac{\omega^3}{\lambda} . \tag{4.7}$$

We obtain R by minimizing S_b, $dS_b/dR=0$,

$$R=(\nu-1)\frac{s_1}{\epsilon} , \tag{4.8}$$

$$S_b=\frac{2\pi^{\nu/2}}{\nu\Gamma(\nu/2)}R^{\nu-1}s_1 . \tag{4.9}$$

The bounce $\bar{\phi}$ approaches ϕ_+ according to

$$\bar{\phi}(r)-\phi_+ \underset{r\to\infty}{\sim} -AF_\nu(r) , \tag{4.10}$$

where $F_\nu(r)$ is a solution in ν-dimensional Euclidean space of

$$(-\partial^2+\omega^2)F_\nu(r)=\delta^\nu(\vec{x}) . \tag{4.11}$$

The constant A is chosen in accordance with Ref. 10. The solution of Eq. (4.11) is

$$F_\nu(r)=\frac{\omega^{\nu-2}}{(2\pi)^{\nu/2}}\frac{K_\mu(\omega r)}{(\omega r)^\mu} , \tag{4.12}$$

where K_μ is a modified Bessel function with index $\mu\equiv\nu/2-1$. Thus we obtain

$$\bar{\phi}(r)-\phi_+ \underset{r\to\infty}{\sim} -\frac{A}{2\omega}\left[\frac{\omega}{2\pi r}\right]^{(\nu-1)/2}e^{-\omega r} . \tag{4.13}$$

The constant A can be evaluated in the thin-wall approximation. There results from matching Eq. (4.13) to Eq. (4.5) just outside the wall,

$$A=4\frac{\omega^{3-\nu}}{\sqrt{\lambda}}(2\pi\omega R)^{(\nu-1)/2}e^{\omega R} . \tag{4.14}$$

Let us next consider the following superposition of widely separated bounces lying on a string orientated in some distinct direction $\vec{e}_y$ in the ν-dimensional Euclidean space:

$$\phi(\vec{x})=\bar{\phi}(r)-\eta(|\vec{x}-\vec{y}|)-\eta(|\vec{x}+\vec{y}|) , \tag{4.15}$$

$$\eta(|\vec{x}\pm\vec{y}|)=\phi_+-\bar{\phi}(|\vec{x}\pm\vec{y}|) . \tag{4.16}$$

Since $\phi-\bar{\phi}$ is only important in the harmonic range of the potential, $V(\phi)\approx\omega^2\phi^2/2$, $\phi(\vec{x})$ is an approximate solution of Eq. (4.3) with the property grad $\phi(\vec{x}=\pm\vec{y}/2)=0$. Thus, $\phi(\vec{x})$ is a slightly distorted bounce which for $\vec{x}=0$ bounces off the boundary of the Euclidean region at $\phi_1=\phi(0)$ near ϕ_- and reaches for a finite value of $\vec{x}$, $\vec{x}=\pm\vec{y}/2$, the boundary of the Euclidean region near $\phi_+,\phi_2=\phi(\pm\vec{y}/2)$ (see Fig. 3). Thus, $\phi(\vec{x})$ is analogous to a solution of Eq. (2.9) for $z>0$.

The Euclidean action of $\phi(\vec{x})$ is given by

$$S(y)=\int d^{\nu-1}\vec{x}_\perp\int_{-y/2}^{y/2}dx_{||}[\tfrac{1}{2}(\partial\phi)^2+V(\phi)]$$
$$=S_b+S_{\rm I}(y) , \tag{4.17}$$

where we have decomposed $\vec{x}$ into vectors parallel and perpendicular to $\vec{y}$, $\vec{x}=\vec{x}_{||}+\vec{x}_\perp$. Expanding to second order in η we obtain, with Eqs. (4.15) and (4.3),

$$S_{\rm I}(y)=2\int d^{\nu-1}\vec{x}_\perp\bar{\phi}\left[\vec{x}_\perp,x_{||}=\frac{y}{2}\right]\times\frac{d}{dx_{||}}\bar{\phi}\left[\vec{x}_\perp,x_{||}=\frac{y}{2}\right] \tag{4.18}$$

$$=-A^2F_\nu(|\vec{y}|) . \tag{4.19}$$

By means of a Fourier transform with respect to the $\nu-1$ space coordinates $\vec{s}$ an expression merely depending on the Euclidean time τ is obtained,

$$\tilde{S}_{\rm I}(\tau)=-\frac{A^2}{V}\int d^{\nu-1}\vec{s}\,e^{i\vec{q}\cdot\vec{s}}F_\nu[(\tau^2+\vec{s}^{\,2})^{1/2}] , \tag{4.20}$$

where V is the volume of the $(\nu-1)$-dimensional coordinate space. By use of Eq. (4.11), the Fourier integration can be performed, thus obtaining

$$\tilde{S}_{\rm I}(\tau)=-\frac{A^2}{2V\omega_q}\exp(-\omega_q\tau) , \tag{4.21}$$

$$\omega_q=(\omega^2+\vec{q}^{\,2})^{1/2} . \tag{4.22}$$

Moving on now to the abbreviated action

$$\tilde{W}(z)=S_b+\tilde{S}_{\rm I}(\tau)-z\tau , \tag{4.23}$$

$$\tau=-\frac{\partial\tilde{W}}{\partial z} , \tag{4.24}$$

there results from Eq. (4.21)

$$\tilde{W}(z)=S_b-\frac{z}{\omega_q}+\frac{z}{\omega_q}\ln\left[\frac{2zV}{A^2}\right] , \tag{4.25}$$

$$z=\frac{A^2}{2V}e^{-\omega_q\tau} . \tag{4.26}$$

Thus we have obtained an "effective" abbreviated action which is identical in form with the corresponding expression (3.8) of the quantum-mechanical problem. Observe the subtlety that the dimensionless constant A^2 in Eq. (3.8) has been replaced in Eq. (4.25) by the dimensionless constant A^2/V of the field-theoretical problem.

Before tackling the general problem of the induced vacuum decay with the methods of the preceding section, we briefly state the results obtained from the functional integral of the Euclidean kernel,

$$\langle\phi_+|\exp(-HT/\hbar)|\phi_+\rangle = \int\mathscr{D}\phi(\vec{x})\exp[-S(\phi)/\hbar] . \tag{4.27}$$

After saturating the functional integral by a dilute gas of bounces, one obtains for the spontaneous decay rate of the false vacuum per unit volume,[3,5,10]

$$\frac{\Gamma_0}{V}=\left[\frac{S_b}{2\pi\hbar}\right]^{\nu/2}\left|\frac{{\rm Det}'[-\partial^2+V''(\bar{\phi})]}{{\rm Det}[-\partial^2+\omega^2]}\right|^{-1/2}\times\exp(-S_b/\hbar) , \tag{4.28}$$

where Det′ denotes the determinant with the ν translational zero modes omitted. The singularity structure of the functional determinant in Eq. (4.28) for $\epsilon\to0$ has been considered in the literature.[16]

The one-meson decay rate Γ_1 has been computed by Affleck and De Luccia[10] by evaluating the functional integral of the Euclidean propagator

$$D(\vec{s},T)=\langle0|[\phi(\vec{s})-\phi_+]\exp(-HT/\hbar)\times[\phi(0)-\phi_+]|0\rangle . \tag{4.29}$$

This method yields

$$\frac{\Gamma_1}{V}=\frac{A^2}{2E_1V}\frac{\Gamma_0}{V} , \tag{4.30}$$

where A is the constant occurring above in Eq. (4.13) and $E_1=\hbar(\omega^2+\vec{k}^2)^{1/2}$ is the energy of a meson with momentum $\hbar\vec{k}$.

In order to compute the general induced vacuum decay, we follow our approach developed in Sec. III. It is obvious that now the number of quasisymmetry modes is higher by a factor ν as compared to the quantum-mechanical case. Furthermore, if we were ambitious to compute the absolute numbers of the decay rates Γ_n, we would have to master all those difficulties which are connected with the computation of the functional determinant in Eq. (4.28) for Γ_0/V.[16] However, this can be avoided if one is only interested in the enhancement of the induced decay over the spontaneous decay. In the following, the enhancement factors $\alpha_n=\Gamma_n/\Gamma_0$ are easily obtained by exploiting some analogy with the quantum-mechanical case in Sec. III.

Our starting point is the quantum-mechanical result

$$\frac{\Gamma(E)}{\Gamma(E-\hbar\omega)}=\frac{X_2(E)}{X_2(E-\hbar\omega)}=\frac{A^2}{2\hbar\omega}\frac{\hbar\omega}{E-\hbar\omega/2}\,, \tag{4.31}$$

which follows from Eqs. (3.6) and (3.11), where the last expression is obtained by use of Eq. (3.8). Now observe that the "effective" action $\widetilde{W}(z)$ in Eq. (4.25) is identical in form with the quantum-mechanical action $\widetilde{W}_2(z)$ in Eq. (3.8). Furthermore, $\widetilde{W}(z)$ enters into an expression $X(E)$ which is defined analogous to Eq. (3.3). All additional factors of the functional determinant being omitted in $X(E)$ cancel if the ratio of decay rates is considered. Thus we obtain, for our field-theoretical model,

$$\frac{\Gamma_n(E^{(n)})}{\Gamma_{n-1}(E^{(n)}-E_i)}=\frac{A^2}{2E_iV}\frac{E_i}{E^{(n)}}\,, \tag{4.32}$$

where $E^{(n)}$ is the total energy of an n-meson state and the zero-point energy has been subtracted,

$$E^{(n)}=\sum_{i=1}^{n}E_i=\sum_{i=1}^{n}\hbar(\omega^2+\vec{k}_i^{\,2})^{1/2}\,. \tag{4.33}$$

For $n=1$ the decay rate of the one-meson state is obtained from Eq. (4.32),

$$\Gamma_1(E_1)=\frac{A^2}{2E_1V}\Gamma_0\,, \tag{4.34}$$

which is identical with the result of Affleck and De Luccia,[10] Eq. (4.30).

For the computation of the two-particle decay rate Γ_2, we first symmetrize $\Gamma_2(E^{(2)})$ with respect to the single-particle energies $E^{(2)}=E_1+E_2$,

$$\Gamma_2(E^{(2)})=\frac{1}{2}\left[\frac{\Gamma_2(E^{(2)})}{\Gamma_1(E_1)}\Gamma_1(E_1)+\frac{\Gamma_2(E^{(2)})}{\Gamma_1(E_2)}\Gamma_1(E_2)\right] \tag{4.35}$$

and insert Eq. (4.32) for Γ_2/Γ_1 and Eq. (4.34) for Γ_1. There results

$$\Gamma_2(E^{(2)})=\frac{1}{2}\frac{A^2}{2E_1V}\frac{A^2}{2E_2V}\Gamma_0\,. \tag{4.36}$$

The n-particle decay rate is obtained by induction,

$$\Gamma_n(E^{(n)})=\frac{1}{n}\sum_{i=1}^{n}\frac{\Gamma_n(E^{(n)})}{\Gamma_{n-1}(E^{(n)}-E_i)}\times\Gamma_{n-1}(E^{(n)}-E_i)\,, \tag{4.37}$$

where Γ_n/Γ_{n-1} is given by Eq. (4.32) and Γ_{n-1} has been obtained in the preceding step. There results the simple expression

$$\Gamma_n(E^{(n)})/V=\frac{1}{n!}\left[\frac{A^2}{2V}\right]^n\prod_{i=1}^{n}\frac{1}{E_i}\Gamma_0/V\,. \tag{4.38}$$

Equation (4.38) is our final result for the decay rate per unit volume of the false vacuum induced by the presence of n noninteracting mesons with total energy $E^{(n)}$. It contains the correct time dilatation factors $1/E_i$ associated with the motion of the mesons. The expression (4.38) is renormalized by replacing Γ_0 by its renormalized expression[3] and by writing A and E_i in terms of the renormalized parameters ω_R and λ_R.

Let us finally apply our result (4.38) to the decay of the false vacuum in the presence of an ideal gas of mesons being in contact with a particle reservoir and with a heat bath of temperature $1/\beta$. In the following, we confine ourselves to a (3 + 1)-dimensional Minkowskian space. The equilibrium decay rate at temperature $1/\beta$ is determined by the expression

$$\Gamma=Z^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{A^2}{2V}\right]^n e^{\beta\mu n}P^n\Gamma_0\,, \tag{4.39}$$

$$P=\int_m^{\infty}dE\frac{\rho(E)}{E}\,, \tag{4.40}$$

where $m=\hbar\omega$ is the energy of one meson at rest, μ is the chemical potential, and ρ is the single-meson density

$$\rho(E)=\frac{V}{2\pi^2\hbar^3}E(E^2-m^2)^{1/2}e^{-\beta E}\,. \tag{4.41}$$

By use of the factorization property of the n-particle partition function,

$$Q_n = \frac{1}{n!} Q_1{}^n , \tag{4.42}$$

$$Q_1 = \int_m^\infty dE\, \rho(E) , \tag{4.43}$$

the grand partition function Z assumes the form

$$Z = \sum_{n=0}^{\infty} e^{\beta\mu n} Q_n = \exp(e^{\beta\mu} Q_1) . \tag{4.44}$$

After computing the integrals in Eqs. (4.40) and (4.43) and eliminating μ in favor of the mean particle number $\bar{N}$, we finally obtain

$$\frac{\Gamma}{V} = \exp(-\bar{N}) \exp\left[\frac{A^2\bar{N}}{2Vm}\frac{K_1(\beta m)}{K_2(\beta m)}\right]\frac{\Gamma_0}{V} , \tag{4.45}$$

where $K_\nu(z)$ is a modified Bessel function. At low temperatures, $\beta m \gg 1$, this expression becomes

$$\frac{\Gamma}{V} = \exp(-\bar{N}) \exp\left[\frac{A^2\bar{N}}{2Vm}\right]\frac{\Gamma_0}{V} , \tag{4.46}$$

and at temperatures much higher than the mass, $\beta m \ll 1$,

$$\frac{\Gamma}{V} = \exp(-\bar{N}) \exp\left[\frac{A^2\bar{N}}{2Vm}\frac{\beta m}{2}\right]\frac{\Gamma_0}{V} . \tag{4.47}$$

Thus, for fixed $\bar{N}$, the decay probability decreases when the temperature is raised. This somewhat peculiar behavior is explained by the time dilatation factors $1/E_i$ in Eq. (4.38) and by the increase of the mean particle energy. However, at high temperatures particle-creation processes induced by the nonlinear terms of the potential are important. These processes cause an increase of $\bar{N}$ and finally also an increase of the decay probability when the temperature is raised.

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APPENDIX

We briefly consider the functional integral of the Wick-rotated Euclidean kernel

$$K \equiv \langle x_+ \,|\, \exp(-HT/\hbar) \,|\, x_- \rangle$$

in the semiclassical limit and for large T. Since now we are completely restricted to the Euclidean domain, the tilde above the Euclidean quantities is omitted. The stationary points of the corresponding functional-integral expression are solutions of Eq. (2.9), where now $z \equiv -u < 0$ is required,

$$\frac{dx_c}{d\tau} = [2(V+u)]^{1/2} \tag{A1}$$

in order to satisfy the boundary conditions $x_c(0) = x_-$ and $x_c(T) = x_+$. According to our discussion in Sec. II, the one-instanton contribution to the kernel requires one convolution of standard semiclassical kernels,

$$K^{(1)}(x_+, T; x_-) = [2V(x_m)]^{1/2} \times \int_0^T d\tau\, K_s(x_+, \tau; x_m) \times K_s(x_m, T-\tau; x_-) , \tag{A2}$$

$$K_s(x_+, \tau; x_m) = (2\pi\hbar)^{-1/2} \left| \dot{x}_c(0)\dot{x}_c(\tau)\frac{\partial^2 W}{\partial u^2} \right|^{-1/2} \times \exp[-(W - u\tau)/\hbar] , \tag{A3}$$

$$W(u) = \int_{x_m}^{x_+} dx\, [2(V+u)]^{1/2} , \tag{A4}$$

where $\tau = \partial W/\partial u$. For $u \ll V(x_m)$, we can evaluate Eq. (A4) in a reasonable approximation. Let us define a quantity $\Delta W(u) \equiv W(u) - W_h(u)$, where W_h is given by Eq. (A4) with V replaced by the harmonic potential $V_h = \omega^2(x - x_+)^2/2$, $\omega^2 = V''(x_+)$. Now observe that the expansion of ΔW in powers of u will rapidly converge. Thus we obtain in a reasonable approximation

$$W(u) = W_h(u) + W(0) - W_h(0) + u \lim_{u\to 0} [W'(u) - W_h'(u)] . \tag{A5}$$

An explicit result is obtained by use of the instanton trajectory $\bar{x}(\tau)$,

$$\tau = \int_{x_m}^{\bar{x}} dx\, (2V)^{-1/2} , \tag{A6}$$

and of its asymptotic behavior,

$$\bar{x}(\tau) = x_+ - \frac{A}{2\omega} e^{-\omega\tau} , \tag{A7}$$

where the constant A is chosen in accordance with Ref. 10. There results

$$W(u)=\tfrac{1}{2}S_0+\frac{u}{2\omega}\left[1+\ln\left[\frac{A^2}{2u}\right]\right], \tag{A8}$$

where $S_0=2W(0)$. On substituting $W(u)$ into Eq. (A3), we find

$$K_s(x_+,\tau;x_m)=\left[\frac{\omega A}{2\pi\hbar}\right]^{1/2}[2V(x_m)]^{-1/4}e^{-\omega\tau/2}\times\exp\left[-\left[\frac{S_0}{2}+\frac{A^2}{4\omega}e^{-2\omega\tau}\right]\Big/\hbar\right]. \tag{A9}$$

An analogous expression is obtained for $K_s(x_m,T-\tau;x_-)$. Now observe that in the limit $T\to\infty$ the convolution integral in Eq. (A2) gives a factor T up to exponentially small corrections. Thus the one-instanton contribution to the kernel is given by

$$K^{(1)}(x_+,T;x_-)=\left[\frac{\omega}{\pi\hbar}\right]^{1/2}e^{-\omega T/2}\frac{\Delta E_0}{2\hbar}T, \tag{A10}$$

$$\Delta E_0=\hbar\left[\frac{\omega}{\pi\hbar}\right]^{1/2}A\exp(-S_0/\hbar). \tag{A11}$$

The generalization to multiple traversals of the valley of the potential $-V(x)$ is obvious. Summing up all configurations which satisfy the boundary conditions, we obtain

$$K(x_+,T;x_-)=\left[\frac{\omega}{\pi\hbar}\right]^{1/2}e^{-\omega T/2}\sinh\left[\frac{\Delta E_0}{2\hbar}T\right], \tag{A12}$$

verifying that the ground-state splitting is given by Eq. (A11).

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