PHYSICAL REVIEW D VOLUME 27, NUMBER 12 15 JUNE 1983

Derivation of the Planck radiation spectrum as an interpolation formula in classical electrodynamics with classical electromagnetic zero-point radiation

Timothy H. Boyer
Department of Physics, The City College of the City University of New York, New York, New York 10031
(Received 6 August 1982)

A closely argued derivation of Planck's spectrum for blackbody radiation is presented within classical electrodynamics with classical electromagnetic zero-point radiation. The presence of temperature-independent random classical radiation invalidates the ideas of traditional classical statistical mechanics which become valid only in the low-frequency or high-temperature limit where they lead to the Rayleigh-Jeans law for the thermal spectrum. The assumption of Lorentz invariance for the zero-point radiation determines the high-frequency part of the classical random radiation spectrum. The blackbody problem of classical physics with classical zero-point radiation as considered here is the derivation of an interpolation formula between these high- and low-frequency limits. Here we take advantage of the surprising diamagnetic behavior of a classical free point charge in zero-point radiation. This diamagnetic behavior is compared with the paramagnetic behavior of a magnetic dipole rotor of large moment of inertia, which behavior is derived from the low-frequency form of the radiation spectrum. If one requires the natural simple condition that the diamagnetic behavior as a function of temperature should differ only in the sign of the magnetic moment from the paramagnetic behavior as a function of temperature, then one is led uniquely to the Planck spectrum including zero-point radiation as the equilibrium spectrum for classical random radiation.

I. INTRODUCTION

The explanation of the blackbody radiation spectrum is regarded as a crucial milestone in the history of physics[1] because some new element beyond traditional classical physics seems to be required for an understanding of the spectrum. Physicists at the present time regard the idea of discrete elements, of quanta, as the required new element. However, it has been suggested[2] that the missing element in traditional classical physics is the presence of temperature-independent random classical radiation, classical zero-point radiation. The theory of classical electrodynamics including classical electromagnetic zero-point radiation is termed random electrodynamics or stochastic electrodynamics. The theory has already been applied to the blackbody problem from two different approaches, one involving Brownian motion[3] and one involving entropy-related fluctuations[4]; in both cases the Planck spectrum including zero-point radiation is obtained. In the present work we again consider the blackbody problem within classical theory with classical electromagnetic zero-point radiation, this time from the point of view of magnetic systems. Here we derive the Planck spectrum as the interpolation formula providing the simplest thermodynamic behavior for some classical magnetic systems in classical electromagnetic zero-point radiation.

The analysis reported here assumes that the thermal radiation spectrum takes the Rayleigh-Jeans form in the low-frequency limit and becomes the zero-point radiation spectrum in the high-frequency limit. The problem to be solved is the discovery of a natural interpolation formula between these two limits. Now the author has had this interpolation problem in mind for over a decade but has rarely found any argument which seemed to single out a preferred interpolation function. The present analysis provides just such a preferred function. And the preferred function gives exactly the Planck spectrum including zero-point radiation. The analysis seems to the author both natural and compelling.

The argument considers systems in thermal radiation including zero-point radiation and involves a comparison between the diamagnetic behavior of a free charged particle and the paramagnetic behavior of a free magnetic dipole rotator of very large moment of inertia. If one requires that the diamagnetic behavior as a function of temperature should differ only in the sign of the average magnetic moment from the paramagnetic behavior as a function of temperature, then one is led uniquely to the Planck spectrum including zero-point radiation as the equilibrium spectrum for classical random radiation.

The analysis is made possible by the following observations. The paramagnetic behavior is known as a function of temperature because the large moment of inertia brings the rotator system in interaction with only the low-frequency part of the thermal spectrum where the Rayleigh-Jeans law holds. The diamagnetic behavior depends on the specific spectrum of thermal radiation. In the high-field (low-temperature) limit the diamagnetic behavior depends only upon the high-frequency part of the random radiation spectrum where the zero-point spectrum holds and gives a constant average magnetic moment. We can choose the magnitude of the paramagnetic moment at zero temperature to be the negative of the diamagnetic moment in zero-point radiation. Then the thermodynamic system consisting of the noninteracting diamagnetic and paramagnetic systems taken together will have zero magnetic moment at low temperature and at high temperature. If we choose the simplest interpolation between these limits, corresponding to zero-magnetic moment for the combined system at all temperatures, then we are led uniquely to the Planck spectrum including zero-point radiation for the equilibrium spectrum of random classical radiation.

ZERO-POINT RADIATION

The fundamental departure of our work from traditional classical theory is the introduction of temperature-independent random classical radiation with a Lorentz-invariant spectrum, classical electromagnetic zero-point radiation, as a boundary condition[5] on the homogeneous solution of Maxwell's equations. In earlier work[6] it has been shown that the requirement of Lorentz invariance leads to a spectrum for the zero-point radiation of the form

$$\rho_0(\omega)=\text{const}\times\omega^3 \ , \tag{1}$$

where the constant is a parameter of the theory to be determined by experiment. It turns out that experimental agreement requires the constant take the value $\hbar/2\pi^2c^3$, where $\hbar$ is Planck's constant. The expression may be more easily recognizable if we separate the number of modes per unit frequency interval and the zero-point energy $g_0(\omega)=\frac{1}{2}\hbar\omega$ per normal mode. Then we have

$$\rho_0(\omega)=\frac{\omega^2}{\pi^2c^3}g_0(\omega)=\frac{\omega^2}{\pi^2c^3}\frac{1}{2}\hbar\omega \ . \tag{2}$$

Planck's constant is introduced into the theory at this point and nowhere else. Every subsequent appearance of Planck's constant is derived from its role setting the scale of the zero-point radiation spectrum.

THERMAL RADIATION

The equilibrium spectrum $\rho(\omega,T)$ of random classical electromagnetic radiation at finite temperature thus contains two parts, the zero-point radiation $\rho_0(\omega)$ and a thermal part $\rho_T(\omega,T)$ defined by

$$\rho_T(\omega,T)\equiv\rho(\omega,T)-\rho_0(\omega) \ . \tag{3}$$

Wien's arguments involving thermodynamics and reflections from moving mirrors show that the equilibrium radiation spectrum is determined in terms of an unknown function $f(\omega/T)$, as

$$\rho(\omega)=(\omega^3/\pi^2c^3)f(\omega/T) \ , \tag{4}$$

with the total energy density $u(T)$ for the thermal part given by the Stefan-Boltzmann law

$$u(T)=\int_{\omega=0}^{\infty}d\omega\,\rho_T(\omega,T)=\sigma T^4 \ . \tag{5}$$

We note that the required functional form (4) is taken by each of the three functions: the total spectrum $\rho(\omega,T)$, the thermal part $\rho_T(\omega,T)$, and the zero-point spectrum $\rho_0(\omega)$. It turns out that the zero-point radiation spectrum is invariant in form under an adiabatic compression[7]; also zero-point radiation is assumed to exist throughout space so that Boltzmann's derivation of Stefan's law goes through unchanged despite the presence of zero-point radiation.[7]

HIGH- AND LOW-FREQUENCY LIMITS FOR THE RANDOM RADIATION SPECTRUM

The high-frequency limit of the random radiation spectrum follows immediately from Eq. (5). Since the thermal spectrum $\rho_T(\omega,T)$ has a finite integral in ω, it must decrease at high frequency. Hence we expect at high frequencies that the equilibrium spectrum of random radiation will go over to the zero-point spectrum,

$$\rho(\omega,T)\rightarrow\rho_0(\omega) \quad \text{as } \omega\rightarrow\infty \ . \tag{6}$$

The function $f(\omega/T)$ in Eq. (4) then becomes

$$f(\omega,T)=g(\omega,T)/\omega\rightarrow\hbar/2 \quad \text{as } \omega/T\rightarrow\infty \ . \tag{7}$$

In the opposite limit of low frequency the zero-point radiation vanishes, $g_0(\omega)=\frac{1}{2}\hbar\omega\rightarrow 0$ as $\omega\rightarrow 0$, and accordingly we expect the zero-point radiation to play no role. But then we expect to recover traditional classical statistical mechanics in the behavior of systems taken to the low-frequency limit.

The blackbody spectrum within traditional classical statistical mechanics can be determined[8] from consideration of a charged harmonic dipole oscillator, which we may think of as a point particle of charge e and mass m at the end of a spring of constant $K=m\omega_0^2$. The charge interacts with the random radiation, being forced into random oscillation by electric forces from the random radiation and being damped by radiation reaction. This dipole oscillator comes into equilibrium with the radiation with an average energy $U=\langle \frac{1}{2}mv^2+\frac{1}{2}m\omega_0^2x^2\rangle$,

$$\begin{aligned} U&=g(\omega_0)+O(e^2)\\ &=\rho(\omega_0)/(\omega_0^2/\pi^2c^3)+O(e^2)\ , \end{aligned} \tag{8}$$

where $g(\omega_0)$ is the average energy per normal mode in the radiation field at the natural frequency $\omega_0=\sqrt{K/m}$ of the oscillator. If we apply traditional classical statistical mechanics to the oscillator, then we find it has an average energy $U=kT$, and this average energy is unchanged as we take the low-frequency limit $\omega_0\to 0$. This suggests that Eq. (8) gives the low-frequency limit to the blackbody spectrum,

$$\rho(\omega,T)\to(\omega^2/\pi^2c^3)kT \quad \text{as } \omega\to 0\ , \tag{9}$$

which is the Rayleigh-Jeans law. The function $f(\omega/T)$ in Eq. (4) must behave as

$$f(\omega/T)=g(\omega,T)/\omega\to kT/\omega \quad \text{as } \omega/T\to 0\ . \tag{10}$$

The problem of the blackbody spectrum within classical physics with classical electromagnetic zero-point radiation as considered here is the determination of the appropriate interpolation function $f(\omega/T)$ between the limits in Eqs. (7) and (10).

PARAMAGNETIC SYSTEM

In order to carry out the required interpolation we compare the thermodynamics of two simple magnetic systems. The first is a paramagnetic system consisting of a magnetic dipole of fixed moment μ. In order to be specific we may think of this magnetic moment incorporated into a planar rigid rotor of very large moment of inertia I and placed in a magnetic field $\vec{B}$ in the plane of the rotor so that the mechanical system has an energy

$$H=\tfrac{1}{2}I\dot{\theta}^2-\mu B\cos\theta\ , \tag{11}$$

where θ is the angle between the magnetic moment $\vec{\mu}$ and the magnetic field $\vec{B}$.

In the presence of random electromagnetic radiation the magnetic moment will be forced into rotation by the random magnetic field and will be damped by radiation reaction. This is a mechanical system which can be analyzed using action-angle variables, in the limit of small rates of energy pickup and loss using the perturbation analysis of van Vleck.[9] We now substitute the magnetic moment $\vec{\mu}$ and magnetic field $\vec{B}$ for the electric dipole moment $e\vec{r}$ and elastic field $\vec{E}$ in van Vleck's work. The limit of large moment of inertia $I\to\infty$ decreases the oscillation frequency of the system indefinitely, thus both decreasing indefinitely the rates of energy exchange with the random radiation, and also bringing the system into interaction with only the lowest frequencies of the radiation spectrum where the Rayleigh-Jeans spectrum becomes appropriate. It has been shown[10] that for such a multiply periodic system in interaction with the Rayleigh-Jeans spectrum of random radiation, the equilibrium distribution $P(\tilde{J})$ on phase space for the mechanical system becomes

$$P(\tilde{J})=\text{const}\times\exp[-H(\tilde{J})/kT]\ , \tag{12}$$

where $\tilde{J}=J/2\pi$ with J the action variable of the mechanical system. In other words the distribution becomes the Boltzmann distribution of traditional classical statistical mechanics. We emphasize that this result is derived from purely classical electromagnetic arguments involving random classical electromagnetic radiation where the radiation spectrum takes the Rayleigh-Jeans form at low frequency. Yet this result is precisely what we could obtain if we applied traditional statistical mechanics to a planar rigid magnetic dipole rotor in a magnetic field $\vec{B}$ with a corresponding energy given by Eq. (11).

The average values for the energy and magnetic moment may be evaluated conveniently using the canonical phase-space variables of angle θ and angular momentum $L=I\dot{\theta}$ rather than the alternative but more complicated action-angle variables J and w which were preferred in the electromagnetic analysis.[10] For example, the average energy is

$$\begin{aligned} \langle H\rangle&=\frac{\int_0^\infty dL\int_0^{2\pi}d\theta(\frac{1}{2}L^2/I-\mu B\cos\theta)\exp[-(\frac{1}{2}L^2/I-\mu B\cos\theta)/kT]}{\int_0^\infty dL\int_0^{2\pi}d\theta\exp[-(\frac{1}{2}L^2/I-\mu B\cos\theta)/kT]}\\ &=\tfrac{1}{2}kT-\mu B[\coth(\mu B/kT)-kT/\mu B]\ . \end{aligned} \tag{13}$$

The average magnetic moment in the z direction given by the magnetic field is found similarly,

$$\langle M_z \rangle = \langle \mu \cos\theta \rangle = \mu[\coth(\mu B/kT) - kT/\mu B] \ . \qquad (14)$$

It is noteworthy that the value of the moment of inertia I does not appear in the final values for $\langle H \rangle$ and $\langle M_z \rangle$ so that the $I \to \infty$ limit needed for the approximations above does not disturb these results.

If we look upon our magnetic dipole rotation as a thermodynamic system dependent upon the magnetic field B, then we note that the energy is

$$\langle H \rangle = -\langle M_z \rangle B + \tfrac{1}{2}kT \qquad (15)$$

with the magnetic work done by the system

$$\Delta W = \langle M_z \rangle \Delta B \ . \qquad (16)$$

The magnetic moment determined in Eq. (14) takes the form of the magnetic moment μ times a function of $\mu B/kT$ with limiting values

$$\langle M_z \rangle \to \begin{cases} \mu, & T/B \to 0 \\ 0, & T/B \to \infty \ . \end{cases} \qquad (17)$$

DIAMAGNETIC SYSTEM

The second system we will use for our interpolation is a diamagnetic system consisting of a point particle of mass m and charge e in random radiation. Just as for the paramagnetic system we must start with certain additional parameters at finite values and then simplify our system by carrying these parameters to limiting values. In the paramagnetic case the moment of inertia I was chosen as finite and then carried to the limiting value $I \to \infty$ so that the system interacted with only the lowest frequencies of radiation. The residual behavior of the moment of inertia is still to be found in the contribution of kinetic energy of amount $\frac{1}{2}kT$ appearing in Eqs. (13) and (15). For our diamagnetic system we start with our point particle bound in an isotropic harmonic potential of natural frequency $\omega_0 = \sqrt{K/m}$ and also in the presence of a magnetic field $\vec{B} = \hat{k}B$ along the z axis. The free-particle limit is obtained by taking the limit of no binding $\omega_0 \to 0$ for finite B.

This diamagnetic system was analyzed[11] in detail, and the interested reader should consult the earlier detailed description and analysis. The Hamiltonian is given by

$$H = [\vec{p} - (e/c)\vec{A}]^2/2m + \tfrac{1}{2}m\omega_0^2 r^2 \ , \qquad (18)$$

where the vector potential may be chosen as $\vec{A} = -\frac{1}{2}(\hat{i}y - \hat{j}x)B$. The system is a linear multiply periodic system and interacts with the random radiation only near the natural frequencies of the system and not at any of the harmonics. The system in interaction with random classical radiation can be solved exactly[12] in the dipole approximation giving the average values[13]

$$\langle H \rangle = \sum_{i=1}^{3} g(\omega_i, T) \ , \qquad (19)$$

$$\langle M_z \rangle = \frac{e}{2mc}\left[-\frac{g(\omega_1,T)}{\omega_s} + \frac{g(\omega_2,T)}{\omega_s} \right] , \qquad (20)$$

where

$$\omega_1 = \omega_s + \omega_L, \quad \omega_2 = \omega_s - \omega_L, \quad \omega_3 = \omega_0 \qquad (21)$$

with

$$\omega_L = \tfrac{1}{2}\omega_B = eB/2mc \qquad (22)$$

and

$$\omega_s = (\omega_0^2 + \omega_L^2)^{1/2} \ . \qquad (23)$$

In the limit of no binding $\omega_0 \to 0$ for fixed magnetic field B, we find

$$\omega_1 \to \omega_B, \quad \omega_2 \cong \omega_0^2/\omega_B \to 0, \quad \omega_s \to \tfrac{1}{2}\omega_B \ . \qquad (24)$$

Also, we recall that $g(\omega_i, T) \to kT$ as $\omega_i \to 0$ for $i = 2,3$. Thus in the limit of a free point charge in a magnetic field, we obtain from (19) and (20) the average values

$$\langle H \rangle = g(\omega_B, T) + 2kT = [g(\omega_B,T) - kT] + 3kT \qquad (25)$$

and

$$\langle M_z \rangle = -\frac{e}{2mc}\left[\frac{g(\omega_B,T)}{\frac{1}{2}\omega_B} - \frac{kT}{\frac{1}{2}\omega_B} \right] = -[g(\omega_B,T) - kT]/B \ . \qquad (26)$$

The harmonic binding parameter ω_0 no longer appears in this limit but the residual effect can be found in Eq. (25) in the contribution to $3kT$ to the average energy.

The results of the electromagnetic analysis thus give us a thermodynamic system where the average energy can be written as

$$\langle H \rangle = -\langle M_z \rangle B + 3kT \ , \qquad (27)$$

and the work done by the system during a change of the magnetic field B is

$$\Delta W = \langle M_z \rangle \Delta B \ . \qquad (28)$$

If we could evaluate $\langle H \rangle$ or $\langle M_z \rangle$ in Eqs. (27) and

(26) as functions of temperature or of magnetic field, then we would have $g(\omega_B,T)$ and hence the radiation spectrum

$$\rho(\omega_B,T)=(\omega_B{}^2/\pi^2c^3)g(\omega_B,T) .$$

Now we do know the high- and low-frequency values for the radiation spectrum as recorded in Eqs. (7) and (10). Substituting from these into Eq. (26) and noting (22), we see that $\langle M_z\rangle$ takes the form

$$\langle M_z\rangle=-\frac{e}{mc}\left[f\left[\frac{\omega_B}{T}\right]-k\left[\frac{\omega_B}{T}\right]^{-1}\right], \tag{29}$$

which has e/mc multiply a function of ω_B/T with the limiting values

$$\langle M_z\rangle\to\begin{cases}-e\hbar/2mc, & T/B\to 0\\ 0, & T/B\to\infty .\end{cases} \tag{30}$$

But these forms except for a sign reversal are strikingly reminiscent of the limits for $\langle M_z\rangle$ in Eq. (17) for the paramagnetic system.

CHOOSING THE INTERPOLATION FUNCTION

Indeed, suppose we were to choose a paramagnetic system of magnetic moment μ equal in magnitude to $e\hbar/2mc$,

$$\mu=e\hbar/2mc , \tag{31}$$

and then combine the paramagnetic and diamagnetic systems, separated far enough to have negligible magnetostatic interaction, into a single thermodynamic system. The total average energy $\langle H_{\text{total}}\rangle$ is given by the sum of the energies in Eqs. (15) and (27), the total work done $\langle \Delta W_{\text{total}}\rangle$ by the system is given by the sum of the work done in Eqs. (16) and (28), and the total magnetic moment $\langle M_{\text{total}\,z}\rangle$ is given by the sum of Eqs. (14) and (29),

$$\langle H_{\text{total}}\rangle=-\langle M_{\text{total}\,z}\rangle B+\tfrac{7}{2}kT , \tag{32}$$

$$\Delta W_{\text{total}}=\langle M_{\text{total}\,z}\rangle\Delta B , \tag{33}$$

$$\begin{aligned}\langle M_{\text{total}\,z}\rangle&=\frac{e\hbar}{2mc}\left[\coth\left[\frac{\hbar\omega_B}{2kT}\right]-\frac{2kT}{\hbar\omega_B}\right]\\&\quad-\frac{e}{mc}\left[f\left[\frac{\omega_B}{T}\right]-\frac{kT}{\omega_B}\right]\\&=\frac{e\hbar}{2mc}\left[\coth\left[\frac{\hbar\omega_B}{2kT}\right]-\frac{2}{\hbar}f\left[\frac{\omega_B}{T}\right]\right] .\end{aligned} \tag{34}$$

What we would like to do is to evaluate the average magnetic moment $\langle M_{\text{total}\,z}\rangle$ for this thermodynamic system. What we actually have available to us from the present analysis is the functional dependence given in Eq. (34) and the limiting forms obtained from $f(\omega_B/T)$ in Eqs. (7) and (10) or taken directly from Eqs. (17) and (30). We see that

$$\langle M_{\text{total}\,z}\rangle\to\begin{cases}0, & T/B\to 0\\ 0, & T/B\to\infty .\end{cases} \tag{35}$$

What can we suggest about $\langle M_{\text{total}\,z}\rangle$ as a function of T/B? The interpolation choice with the least possible structure corresponds to $\langle M_{\text{total}\,z}\rangle=0$ for all values of T/B. Any other choice introduces unnecessary structure into the thermodynamics of the system. We therefore propose, from (34), that

$$f(\omega/T)=\tfrac{1}{2}\hbar\coth(\hbar\omega/2kT) , \tag{36}$$

and that

$$\begin{aligned}\rho(\omega,T)&=(\omega^3/\pi^2c^3)f(\omega/T)\\&=(\hbar\omega^3/2\pi^2c^3)\coth(\hbar\omega/2kT)\\&=\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\exp(\hbar\omega/kT)-1}+\frac{1}{2}\frac{\hbar\omega^3}{\pi^2c^3} ,\end{aligned} \tag{37}$$

which is exactly the Planck spectrum of thermal radiation along with the zero-point radiation spectrum.

The choice $\langle M_{\text{total}\,z}\rangle=0$ for all temperatures is equivalent to requiring that diamagnetic behavior as a function of temperature for a free particle should differ only in the sign of the average magnetic moment from the paramagnetic behavior of a free magnetic dipole rotator. If one thinks of heuristic ideas of information theory in which a magnetic moment can have different spatial orientations, then symmetry between diamagnetic and paramagnetic behavior which involves only a sign reversal seems natural. Also, this choice is natural for other combined systems; if one combines a paramagnetic rotator of any size magnetic moment with a noninteracting free point charge, then the diamagnetic behavior of the point charge effectively diminishes the magnetic moment of the paramagnetic rotator in just such a way that a separation into distinct paramagnetic and diamagnetic contributions is impossible from the thermodynamics of the system. Thus the choice in Eqs. (34) and (35) of $\langle M_{\text{total}\,z}\rangle=0$ for all temperatures seems natural from considerations of simplicity, though it does not seem to follow from any more general thermodynamic principle. It is, of course, most interesting that the simplest natural choice leads to Planck's spectrum, which agrees with experiment.

CONCLUDING REMARKS

The theoretical context for our present derivation of Planck's spectrum is that provided by classical electromagnetism in which one introduces classical electromagnetic zero-point radiation. The theory is sometimes termed random electrodynamics or stochastic electrodynamics and gives some results in agreement with quantum theory.[2]

Now the Planck spectrum has been derived[3,4] earlier within classical physics with classical electromagnetic zero-point radiation. However, one derivation[14] contains a loophole in its argument, and the second[15] depends upon a suggestive but *ad hoc* designation of the fluctuations to be associated with thermodynamic entropy. Moreover, these derivations seem to be contradicted by several derivations[10,16–18] of the Rayleigh-Jeans spectrum within classical physics using not traditional statistical mechanics but rather perturbative expansions for nonlinear systems in random classical radiation.

It is partly because of these contradictory conclusions within classical theory that we have presented this interpolation derivation of the Planck spectrum within classical physics with classical electromagnetic zero-point radiation. The present derivation involves two classical mechanical systems neither of which has harmonics, and hence each of which comes to equilibrium in any isotropic spectrum of random radiation. The systems can not individually force any particular spectrum for the equilibrium radiation. However, considered together they lead to a natural interpolation between the high- and low-frequency limits of thermal radiation within classical electrodynamics with classical electromagnetic zero-point radiation. The analysis given here is intended to be careful and to avoid dubious perturbative approximations. The interpolation is intended to be unambiguous. If classical physics with classical zero-point radiation has a consistent thermodynamics, then we believe that this interpolation derivation makes it extremely likely that the equilibrium spectrum for classical thermal radiation is the Planck spectrum including zero-point radiation.

ACKNOWLEDGMENTS

I wish to thank Professor Harry Soodak, Professor Myriam Sarachik, and Professor Harold Falk for helpful conversations regarding the thermodynamic behavior of magnetic systems. This work was supported by The City University of New York PSC-BHE Research Award Program, Grant No. 13966.

[1]See the historical account by Thomas S. Kuhn, *Black-body Theory and the Quantum Discontinuity 1894–1912* (Oxford University Press, New York, 1978).

[2]See the review of classical electron theory with classical electromagnetic zero-point radiation given by T. H. Boyer, in *Foundations of Radiation Theory and Quantum Electrodynamics,* edited by A. O. Barut (Plenum, New York, 1980), p. 49.

[3]T. H. Boyer, Phys. Rev. 182, 1374 (1969).

[4]T. H. Boyer, Phys. Rev. 186, 1304 (1969). See also O. Theimer, Phys. Rev. D 4, 1597 (1971); O. Theimer and P. R. Peterson, *ibid.* 10, 3962 (1974).

[5]See T. H. Boyer, Phys. Rev. D 11, 790 (1975).

[6]T. W. Marshall, Proc. Cambridge Philos. Soc. 61, 537 (1965); see also Ref. 3.

[7]See Ref. 4, p. 1309.

[8]See, for example, M. Born, *Atomic Physics,* 7th edition (Hafner, New York, 1966), p. 254.

[9]J. H. van Vleck, Phys. Rev. 24, 330 (1924); 24, 347 (1924).

[10]T. H. Boyer, Phys. Rev. A 18, 1228 (1978).

[11]T. H. Boyer, Phys. Rev. A 21, 66 (1980).

[12]See the appendix of T. H. Boyer, Phys. Rev. D 11, 809 (1975) and the discussion of Ref. 11.

[13]See Ref. 11, Eqs. (10), (36), (37), (39), and (41). The discussion of Ref. 11 immediately substitutes the Planck spectrum in passing from Eqs. (10) and (36) to Eqs. (37) and (39).

[14]See Ref. 3, p. 1378. The assumption that $\langle vJ \rangle$ is temperature independent restricts the derivation to a small number of systems. It appears possible to make a refinement of the model which should allow explicit calculation of corresponding to the energy loss at the walls.

[15]See T. H. Boyer in Ref. 4, p. 1350.

[16]T. H. Boyer, Phys. Rev. D 13, 2832 (1976).

[17]L. Pesquera and E. Santos (unpublished).

[18]R. Blanco, L. Pesquera, and E. Santos, Phys. Rev. D 27, 1254 (1983).