Schwinger model in curved space-time

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An exact solution to the Schwinger model in a two-dimensional Schwarzschild space-time is found. The possibility that curvature could cause a confining theory to undergo a phase transition to a nonconfining phase is also discussed.

I. INTRODUCTION

Recently there has been a good deal of interest in interacting field theories in curved space-time. This interest has been focused almost exclusively on the ultraviolet behavior of the theories. However, the infrared behavior of an interacting field theory in curved space-time is also a subject of considerable intrinsic interest.

For reasons which I will explain below, the curvature of space-time should affect confinement. With this in mind I have solved the Schwinger model in a two-dimensional mock Schwarzschild space-time.

Since confinement and curvature are both long-range phenomena which in some cases will operate over the same length scale (around a 10^{15}-g black hole for instance) it is certainly possible that curvature affects confinement.

It is believed that at high temperatures quantum chromodynamics undergoes a phase transition from a confining to a nonconfining phase[1] and it is well known that in thermodynamics curvature plays the role of a temperature.[2] This raises the question of whether curvature also plays a role similar to temperature in confinement. There are several reasons for thinking that it does.

At high temperatures QCD is believed to lose confinement because thermal excitations produce a plasma of quarks and gluons. This plasma screens the (color) electric flux and confinement is lost. Since curvature can cause particle creation it seems reasonable that in regions of very high (or very rapidly changing) curvature a plasma of quarks and gluons produced by curvature-generated particle production could screen the (color) electric flux and thus cause a loss of confinement.

In the language of the bag model confinement may be lost at high temperature due to three effects.[1] The bag constant $\mathscr{B}$ may depend on temperature. If as expected $\mathscr{B}$ decreases with increasing T then the radius of the bag will increase, if $\mathscr{B}\rightarrow 0$ then confinement will be lost. Thermal fluctuations will create a gas of bags whose density will increase as the temperature increases. At some point the bags will start to overlap and join. Eventually the bags will condense and confinement will be lost. Lastly, the chiral properties of the vacuum may depend on the temperature.

I am uncertain as to how this picture will change if one thinks about increasing or decreasing the curvature instead of the temperature. The bag constant $\mathscr{B}$ may or may not change as the curvature changes and the direction of change could conceivably depend on whether the curvature was positive or negative. In regions of high or rapidly changing curvature a gas of bags could be produced in the same way that curvature can lead to particle production. It is important to note that here I am not talking about bags produced by fluctuations of the metric. These could be important but that is a subject for quantum gravity not quantum field theory in curved space-time. Throughout this paper I work with a fixed manifold upon which I quantize the matter (and gauge) fields. As for the chiral properties of the vacuum I have no idea what, if anything, curvature does to them.

In a quantum field theory in a black-hole space-time the free field propagators are periodic in imaginary time and one gets thermal Green's functions with a temperature that is inversely proportional to the surface gravity of the black hole.[3] It is believed that the Green's functions remain thermal even in the presence of interactions.[4] In light of this one would certainly expect that if QCD loses confinement at high temperatures it will lose confinement at high curvatures.

Aside from being an interesting theoretical problem there are two places where the loss of confinement due to curvature might be important: near a small primordial black hole and in the early universe.

If QCD becomes unconfined at high curvatures then a small black hole might Hawking radiate free gluons and in the last stage of its life a few free light quarks. One must be careful here about talking about particles since the radius of a 10^{15}-g black hole is about a fermi. However, one can still talk about the quark and gluon fields and of course for an observer some distance from the hole the particle concept makes sense. By a small primordial black hole I mean a "Hawking mass" black hole. The Hawking mass is the mass of a black hole that evaporates in the present age of the universe. The Hawking mass has been computed by Page[5-7] to be $5\times10^{14}\ \text{g} \lesssim M_H \lesssim 7\times10^{14}$ g. Such a black hole has a very high surface gravity. A very crude and speculative estimate of when curvature could cause a loss of confinement can be obtained as follows. Assume for the moment that QCD loses confinement when the temperature is of the order 100 MeV or 10^{12} K. Then assuming that QCD also loses confinement when the temperature which characterizes the black hole (which is the temperature that appears in the thermal Green's functions for fields near the hole) reaches 10^{12} K tells one that confinement will be lost when the surface gravity κ is $(2\pi k/\hbar)10^{12}$ K. Any black hole of mass 2×10^{15} g or less will have a surface gravity this large or larger. Thus it looks like a Hawking-mass black hole will do the trick. Once colored particles are emitted from a black hole they will remain unconfined since away from the hole they can interact only with color singlets. In light of our present ignorance about QCD at high curvatures and about primordial black holes (see Blandford and Thorne[8] for a review of primordial black holes) it is impossible to say if colored Hawking radiation from primordial black holes might be detectable but the situation does not look hopeful.

If curvature affects confinement then the curvature of the early universe should affect T_c, the temperature at which QCD undergoes a phase transition from an unconfining to a confining phase. It is difficult to say how large this shift might be. Strictly speaking, in the presence of curvature one should probably talk about a critical line parametrized by T and a curvature parameter; however, in the simplest models of the early universe the curvature and the temperature are not independent so it makes sense to talk about a T_c.

All this is very interesting but also very speculative. To try to learn something concrete I have been studying the Schwinger model[9,10] in curved space-time. The Schwinger model confines in flat space-time and has the advantage of having been extensively studied in flat space-time.[11-15] The disadvantage of the Schwinger model is that it is confining at high temperatures[16-18] and thus one would expect it to be confining at high curvatures as well. The Schwinger model is also superrenormalizable which will make it less interesting in curved space-time than a renormalizable model would be. I will say more about this point later on. It turns out however, that the Schwinger model in curved space-time is difficult enough for a first step.

In Sec. II the model is set up and I show that the vector current satisfies the Klein-Gordon equation. I also discuss why it is difficult to prove confinement in curved space-time.

In Sec. III I set up and solve the equations for the Green's functions and show that the solution is more complicated in curved space-time.

In Sec. IV I discuss some open problems and suggest some directions for further research.

II. THE MODEL

In flat space-time the Lagrangian for the Schwinger model is

$$\mathscr{L}=\frac{i}{2}\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+e\bar{\psi}\gamma^{\mu}\psi A_{\mu} -\frac{1}{2\beta}(\partial_{\mu}A^{\mu})^2\,, \tag{2.1}$$

where the $1/\beta$ term is a gauge-fixing term.

With my conventions the flat—space-time metric $\eta_{\alpha\beta}$ is $\eta_{00}=-\eta_{11}=1$ and

$$\gamma^0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \gamma^1=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$

and

$$\gamma^5=\gamma^0\gamma^1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

The curved—space-time γ matrices are denoted by $\underline{\gamma}^{\mu}(x)$ and satisfy

$$\{\underline{\gamma}^{\mu}(x),\underline{\gamma}^{\nu}(x)\}=2g^{\mu\nu}(x)\,. \tag{2.2}$$

The curved—space-time γ matrices are related to the flat—space-time γ matrices by

$$\underline{\gamma}_{\mu}(x)=b^{\alpha}_{\mu}(x)\gamma_{\alpha}\,, \tag{2.3}$$

where the vierbein (really zweibein in two dimensions) fields are defined by

$$\eta_{\alpha\beta}=b^{\mu}_{\alpha}(x)b^{\nu}_{\beta}(x)g_{\mu\nu}(x)\,. \tag{2.4}$$

In curved space-time the Lagrangian is

$$\mathscr{L}=i\bar{\psi}\underline{\gamma}^{\mu}(x)\nabla_{\mu}\psi-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+e\bar{\psi}\underline{\gamma}^{\mu}(x)\psi A_{\mu}(x) -\frac{1}{2\beta}(\nabla_{\mu}A^{\mu})^2\,, \tag{2.5}$$

where ∇_{μ} is the covariant derivative and

$F_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu$. The covariant derivative of ψ is

$$\nabla_\mu\psi=(\partial_\mu-\Gamma_\mu)\psi\ , \tag{2.6}$$

where $\Gamma_\mu(x)$ is the spinorial affine connection defined by

$$\Gamma_\mu(x)=\tfrac{1}{4}\gamma_\alpha\gamma_\beta b^\beta_\lambda(x)g^{\lambda\sigma}(x)\nabla_\mu b^\alpha_\sigma(x) \tag{2.7}$$

with

$$\nabla_\mu b^\alpha_\sigma(x)=\partial_\mu b^\alpha_\sigma-\Gamma^\lambda_{\mu\sigma}b^\alpha_\lambda(x)\ . \tag{2.8}$$

Here $\Gamma^\lambda_{\mu\nu}$ is the usual Christoffel symbol. The field $\bar\psi$ is defined to be $\bar\psi=\psi^\dagger\gamma_0$. In curved space-time the anticommutator between ψ and $\bar\psi$ is[19]

$$\delta(n[x-x'])\{\psi(x),\bar\psi(y)\}=\frac{\gamma^1\delta^n(x-y)}{(g_x)^{1/2}}\ , \tag{2.9}$$

where n is a timelike unit vector lying in the forward light cone. The curved—space-time field equations are

$$i\underline{\gamma}^\mu(x)[\nabla_\mu-ieA_\mu(x)]\psi(x)=0 \tag{2.10}$$

and

$$\nabla_\nu F^{\mu\nu}(x)=ej^\mu(x)\ , \tag{2.11}$$

where

$$j^\mu(x)=\bar\psi(x)\underline{\gamma}^\mu(x)\psi(x)\ . \tag{2.12}$$

The current conservation equation is of course

$$\nabla_\mu j^\mu(x)=0\ . \tag{2.13}$$

The equation on $F^{\mu\nu}$ can be simplified to the more useful form

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\nu}(\sqrt{g}\ F^{\mu\nu})=ej^\mu(x)\ . \tag{2.14}$$

I now wish to specialize to a "mock Schwarzschild" space-time. In two dimensions there are no curved space-times which satisfy the Einstein field equations. For this reason I work in a space-time obtained by restricting the Schwarzschild metric to two dimensions. It, of course, does not satisfy the Einstein field equations. The metric is given by

$$ds^2=\left[1-\frac{2MG}{r}\right]dt^2-\left[1-\frac{2MG}{r}\right]^{-1}dr^2\ . \tag{2.15}$$

This metric has $\sqrt{g}=1$, a singularity at $r=0$, an event horizon at $r=2MG$, and the scalar curvature R is

$$R=\frac{4MG}{r^3}\ . \tag{2.16}$$

Although I have specialized to a two-dimensional Schwarzschild space-time most of what follows will be applicable to any time-independent space-time provided the space-time is asymptotically flat (AF). It is difficult to define confinement in nonasymptotically flat space-times[20] although on physical grounds one expects to be able to define confinement operationally.

My strategy in solving the Schwinger model in curved space-time is to follow Brown's[11] flat—space-time solution as closely as possible.

The first step is to solve for $F_{\mu\nu}$. From Eq. (2.14) I have

$$\frac{\partial F^{01}}{\partial r}=ej^0(r,t)\ . \tag{2.17}$$

This equation is solved by

$$F^{01}=-e\partial^1\int_0^\infty dr'\mathscr{D}(r,r',t)j^0(r',t)\ , \tag{2.18}$$

where $\mathscr{D}(r,r',t)$ satisfies

$$\frac{-d}{dr}\left[g^{11}(r)\frac{d}{dr}\right]\mathscr{D}(r,r',t)=\delta(r-r')\ . \tag{2.19}$$

The solution to Eq. (2.19) is

$$\mathscr{D}(r,r')=\begin{cases}-\frac{1}{2}(r-r')+MG\ln\left[\dfrac{r'-2MG}{r-2MG}\right], & r<r'\\[2ex] \frac{1}{2}(r-r')+MG\ln\left[\dfrac{r-2MG}{r'-2MG}\right], & r>r'\ .\end{cases} \tag{2.20}$$

Note that for $M\to0$ and $r,r'\to\infty$, $\mathscr{D}(r,r')\to\frac{1}{2}|r-r'|$ which is the correct flat—space-time result. Also note that $\mathscr{D}(r,r')=\mathscr{D}(r',r)$. Using Eq. (2.18) for F^{01} and current conservation it is easy to show that F^{10} satisfies the correct field equation.

At this point I need to pick a gauge. The most convenient gauge is the Coulomb gauge and from this point on I will work in the Coulomb gauge unless otherwise noted. In the Coulomb gauge $A_1=0$ thus $F^{01}=\partial^1A^0(x)$ and so using Eq. (2.18) I have

$$A^0(x)=-e\int_0^\infty dr'\mathscr{D}(r,r')j^0(r',t)+f(x^0)\ . \tag{2.21}$$

In flat space-time $f(x^0)$ is usually set to zero and for an AF space-time $\lim_{x^1\to\infty} A^0(x)^{\rm curved} = A^0(x)^{\rm flat}$ so $f(x^0)$ is zero in an AF space-time if it is zero in flat space-time. Thus I will set $f(x^0)=0$. I now want to construct the Hamiltonian. Now

$$H=\int_0^\infty dr\,\mathscr{H}$$

and

$$\mathscr{H}=\frac{\partial L}{\partial(\nabla_0\psi)}\nabla_0\psi-\mathscr{L}\ , \tag{2.22}$$

so in order to compute H I need to evaluate $\Gamma_\mu(x)$. A tedious calculation gives

$$\Gamma_\mu(x)=\begin{cases}\dfrac{-\gamma_5 MG}{2r^2}, & \mu=0\\ 0, & \mu=1\ .\end{cases} \tag{2.23}$$

In computing $\Gamma_\mu(x)$ one needs the zweibein fields. With my conventions these are as follows. For $b^\mu_\alpha(x)$:

$$b^0_0(x)=\sec\theta\left[1-\frac{2MG}{r}\right]^{-1/2},\quad b^1_0(x)=\tan\theta\left[1-\frac{2MG}{r}\right]^{1/2},$$

$$b^0_1(x)=\tan\theta\left[1-\frac{2MG}{r}\right]^{-1/2},\quad b^1_1(x)=\sec\theta\left[1-\frac{2MG}{r}\right]^{1/2}.$$

For $b^\alpha_\mu(x)$:

$$b^0_0(x)=\sec\theta\left[1-\frac{2MG}{r}\right]^{1/2},\quad b^0_1(x)=-\tan\theta\left[1-\frac{2MG}{r}\right]^{-1/2},$$

$$b^1_0(x)=-\tan\theta\left[1-\frac{2MG}{r}\right]^{1/2},\quad b^1_1(x)=\sec\theta\left[1-\frac{2MG}{r}\right]^{-1/2}.$$

Here θ is an angle which determines the orientation of the zweibein fields. I have chosen $\theta=0$.

Using the result for $\Gamma_\mu(x)$ in Eq. (2.22) gives

$$\mathscr{H}=-i\bar\psi\underline{\gamma}^1(r)\nabla_1\psi+\frac{e^2}{2}j_0(r,t)\int_0^\infty dr'\mathscr{D}(r,r')j^0(r',t)\ . \tag{2.24}$$

In deriving the equation for $\mathscr{H}$ I have used the fact that $\mathscr{L}$ can be rewritten as

$$\begin{aligned}\mathscr{L}&=i\bar\psi\underline{\gamma}^\mu\nabla_\mu\psi+\tfrac{1}{2}(\partial_1A_0)(\partial^1A^0)+ej_0(x)A^0(x)\\&=i\bar\psi\underline{\gamma}^\mu\nabla_\mu\psi+\frac{e}{2}j_0(x)A^0(x)+\tfrac{1}{2}\partial_1(A_0\partial^1A^0)\ .\end{aligned}$$

I have then thrown $\partial_1(A_0\partial^1A^0)$ away because it is a surface term. This is a delicate point. The contribution from the $r\to\infty$ part of $\partial_1(A_0\partial^1A^0)$ will be the same as in flat space-time but the contribution from the other end point is more difficult. Due to the singularity at $r=0$ the space-time is the half plane $0\le r\le\infty$ and $-\infty\le t\le\infty$ not the full plane $-\infty\le x\le\infty, -\infty\le t\le\infty$ as it is in flat space-time. Thus $\partial_1(A_0\partial^1A^0)$ contributes a piece which comes from the $r=0$ end point of the integral, not the $x\to-\infty$ end point as in flat space-time. I have thrown this piece away and this procedure can be thought of as defining my Hamiltonian. Such surface terms will be thrown away throughout this work.

The axial-vector current is defined as

$$j_5^\mu(x)=\bar\psi\underline{\gamma}_5(x)\underline{\gamma}^\mu(x)\psi\ . \tag{2.25}$$

The axial-vector current may also be defined with γ_5 instead of $\underline{\gamma}_5$ since $\gamma_5=\underline{\gamma}_5$. It is easy to show that j_5^μ and j^μ are related by

$$j_5^\mu(x)=\epsilon^{\mu\nu}j_\nu(x)\ , \tag{2.26}$$

where $\epsilon^{01}=-\epsilon^{10}=1$. In order to derive the wave equation on $j_\mu(x)$ I need to know the time development of j_5^0. In curved space-time the time development of an operator $\mathscr{O}$ is given by

$$\mathscr{L}_{\partial/\partial t}\mathscr{O}=-i[\mathscr{O},H]\ , \tag{2.27}$$

where $\mathscr{L}$ is the Lie derivative. For a Schwarzschild space-time

$$\mathscr{L}_{\partial/\partial t} j_5^\mu(x) = \frac{\partial j_5^\mu(x)}{\partial t} , \tag{2.28}$$

thus using Eq. (2.25) I get

$$\frac{\partial j_5^0(x)}{\partial t} = -i[j^1(x),H] . \tag{2.29}$$

Using Eq. (2.24) gives

$$\frac{\partial j_5^0(x)}{\partial t} = -\left[j^1(x), \int_0^\infty dr\, \bar{\psi}\underline{\gamma}^1(r)\nabla_1^r \psi \right] - \frac{ie^2}{2}\left[j^1(x), \int_0^\infty dr\, j_0(r) \int_0^\infty dr' \mathscr{D}(r,r') j^0(r',t) \right] . \tag{2.30}$$

The first term can be evaluated in a straightforward manner and doing so gives the axial-vector divergence as

$$\nabla_\mu j_5^\mu(x) = -\frac{ie^2}{2} \left\{ \int_0^\infty dr [j_1(x^0,x^1), j_0(r,t)] \int_0^\infty dr' \mathscr{D}(r,r') j^0(r,t) + \int_0^\infty dr\, j^0(r,t) \int_0^\infty dr' \mathscr{D}(r,r') [j_1(x,t), j_0(r',t)] \right\} . \tag{2.31}$$

To evaluate this expression I need to evaluate commutators of the form $[j_\mu(t,x), j_\nu(t,y)]$. It is sufficient to evaluate this commutator for currents with flat—space-time γ matrices since the space-time factors in the curved—space-time γ matrices can be pulled out of the commutator. Since the commutator is ill-defined at $x=y$ it must be evaluated by point separation. Doing this gives

$$[j'_\mu(t,x), j'_\nu(t,y)] = -\underset{\epsilon,\epsilon'\to 0}{\text{s-lim}} \{ [\bar{\psi}(y+\epsilon')\gamma_\nu\psi(y-\epsilon'), \bar{\psi}(x+\epsilon)]\gamma_\mu\psi(x-\epsilon) + \psi(x+\epsilon)\gamma_\mu[\bar{\psi}y+\epsilon')\gamma_\nu\psi(y-\epsilon'), \psi(x-\epsilon)] \} , \tag{2.32}$$

where the primes denote currents defined with γ_μ and where s-lim stands for the symmetric limit with x and y approaching each other along a spacelike path. Using the canonical commutation relation (CCR) [Eq. (2.9)] gives

$$[j_1(t,x), j_0(t,y)] = \underset{\epsilon,\epsilon'\to 0}{\text{s-lim}} [\bar{\psi}(y+\epsilon')\delta(y-x-\epsilon-\epsilon')\gamma_1\psi(x-\epsilon) - \bar{\psi}(x+\epsilon)\gamma_1\delta(x-y-\epsilon-\epsilon')\psi(y-\epsilon)] . \tag{2.33}$$

If I assume (as is done in flat space-time) that the commutator is a c number then

$$[j'_1(t,x), j'_0(t,y)] = \underset{\epsilon,\epsilon'\to 0}{\text{s-lim}} [\langle 0 | \bar{\psi}(y+\epsilon)\gamma_1\psi(x-\epsilon) | 0 \rangle \delta(y-x-\epsilon-\epsilon') - \langle 0 | \bar{\psi}(x+\epsilon)\gamma_1\psi(y-\epsilon) | 0 \rangle]\delta(x-y-\epsilon-\epsilon')] . \tag{2.34}$$

Furthermore I will assume that

$$\lim_{x\to y} \langle 0 | \bar{\psi}(x)\gamma_1\psi(y) | 0 \rangle = \langle 0 | \bar{\psi}(x)\gamma_1\psi(y) | 0 \rangle^{\text{free}} . \tag{2.35}$$

This is a standard assumption in flat space-time (see Schwinger[10] for a justification of this assumption) and since it is a short-distance limit it should hold in curved space-time as well. In the limit $\epsilon,\epsilon'\to 0$ $[j_1, j_0]$ would vanish if it were not for the poles in $\langle 0 | \bar{\psi}\gamma_1\psi | 0 \rangle^{\text{free}}$. This means that I only need the pole part of the free propagator and I can use a Riemann normal coordinate expansion to obtain this part. The vacuum expectation value (VEV) that I need is related to the fermion propagator by

$$\langle 0 | \bar{\psi}(x)\gamma_1\psi(y) | 0 \rangle = \text{Tr}[\gamma_1 S(x,y)] , \tag{2.36}$$

where $S(x,y)$ is the fermion propagator. The expansion for the free fermion propagator in Riemann normal coordinates has been worked out by Bunch and Parker[21] and is

$$S(x,x')=i\int\frac{d^nk}{(2\pi)^n}e^{ik(x-x')}\left[\frac{ik_\alpha\gamma^\alpha-m}{k^2+m^2}+\frac{1}{4}\frac{R(ik_\alpha\gamma^\alpha-m)}{(k^2+m^2)^2}\right.$$
$$\left.-\frac{2}{3}\frac{R_{\sigma\tau}k^\sigma k^\tau(i\gamma_\alpha k^\alpha-m)}{(k^2+m^2)^3}+\frac{i}{8}\frac{R^{\alpha\beta}{}_{\mu\nu}[\gamma_\alpha,\gamma_\beta]\gamma^\mu k^\nu}{(k^2+m^2)^2}\right]\gamma^{-1}\,, \tag{2.37}$$

where γ is defined by $\gamma\gamma_\alpha\gamma^{-1}=-\gamma_\alpha^\dagger$ and $R,R_{\sigma\tau}$, and $R^{\alpha\beta}{}_{\mu\nu}$ are evaluated at x'. I have chosen $\gamma=\gamma^0$. I am interested only in the $m=0$ case and in two dimensions the only divergent term is the $k_\alpha\gamma^\alpha/k^2$ term. There are no curvature-dependent divergences. This is because the Schwinger model is superrenormalizable.

The fact that in flat space-time it is really $[j_\mu,j_\nu]$ that determines the confining properties of the Schwinger model suggests that the best place to look for loss of confinement due to curvature is in field theories that are renormalizable but not superrenormalizable. Since the only divergence is the flat—space-time piece I get

$$\langle 0\,|\,\bar\psi(x)\gamma_1\psi(y)\,|\,0\rangle=\frac{i}{2\pi}\frac{\mathrm{Tr}[\gamma_1\gamma_\mu](x-y)^\mu}{(x-y)^2+i\epsilon}+\text{finite terms}$$
$$=-\frac{i}{\pi}\frac{1}{(x-y)+i\epsilon}+\text{finite terms}\,. \tag{2.38}$$

The two propagators I am interested in are $\langle 0\,|\,\bar\psi(x+\epsilon)\gamma_1\psi(y-\epsilon')\,|\,0\rangle$ and $\langle 0\,|\,\bar\psi(y+\epsilon')\gamma_1\psi(x-\epsilon)\,|\,0\rangle$. Since I care only about the divergent part which is the same as in flat space-time I can write

$$\langle 0\,|\,\bar\psi(x+\epsilon)\gamma_1\psi(y-\epsilon')\,|\,0\rangle=\langle 0\,|\,\bar\psi(y+\epsilon')\gamma_1\psi(x-\epsilon)\,|\,0\rangle=\langle 0\,|\,\bar\psi(2[\epsilon+\epsilon'])\gamma_1\psi(0)\,|\,0\rangle\,. \tag{2.39}$$

Equation (2.39) is of course only true for the pole terms. It is not true for the finite part of $\langle 0\,|\,\bar\psi\gamma_1\psi\,|\,0\rangle$, but I only need the pole for Eq. (2.34). Using Eq. (2.38) in Eq. (2.34) gives

$$[j_1'(t,x),j_0'(t,y)]=\frac{1}{i\pi}\operatorname*{s-lim}_{\epsilon,\epsilon'\to 0}\left[\frac{\delta(x-y+\epsilon+\epsilon')-\delta(x-y-\epsilon-\epsilon')}{2(\epsilon+\epsilon')}\right]\,. \tag{2.40}$$

Letting $\eta/2=\epsilon+\epsilon'$ and taking the limit $\eta\to 0$ gives

$$[j_1'(t,x),j_0'(t,y)]=\frac{1}{i\pi}\frac{\partial}{\partial x}\delta(x-y)\,. \tag{2.41}$$

In terms of j^μ instead of $j^{\mu'}$ Eq. (2.41) is

$$[j^1(t,x^1),j^0(t,y^1)]=\frac{1}{i\pi}\left[1-\frac{2MG}{r_x}\right]^{-1/2}\left[1-\frac{2MG}{r_y}\right]^{1/2}\frac{\partial}{\partial x_1}\delta(x^1-y^1)\,. \tag{2.42}$$

Plugging this into Eq. (2.31) for $\nabla_\mu j_5^\mu(x)$ gives

$$\nabla_\mu j_5^\mu(x)=-\frac{e^2}{2\pi}\left[\int_0^\infty dr\left[1-\frac{2MG}{r_x}\right]^{-1/2}\left[1-\frac{2MG}{r}\right]^{1/2}\frac{\partial}{\partial r_x}\delta(r_x-r)\int_0^\infty dr'\mathscr{D}(r,r')j^0(r',t)\right.$$
$$\left.+\int_0^\infty dr\,j^0(r,t)\int_0^\infty dr'\mathscr{D}(r,r')\left[1-\frac{2MG}{r_x}\right]^{-1/2}\left[1-\frac{2MG}{r'}\right]^{1/2}\frac{\partial}{\partial r_x}\delta(r_x-r')\right]\,, \tag{2.43}$$

where j^0 is a component of the vector $\vec{j}=j^\mu\hat{e}_\mu$ with $\hat{e}_\mu$ a basis vector such that $\hat{e}_\mu\cdot\hat{e}_\nu=g_{\mu\nu}$. Thus

$$\nabla_\mu j_5^\mu(x)=-\frac{e}{2\pi}\left[\frac{\partial}{\partial r_x}\int_0^\infty dr'\mathscr{D}(r_x,r')j^0(r',t)+\frac{\partial}{\partial r_x}\int_0^\infty dr\,j^0(r,t)\mathscr{D}(r,r_x)\right]$$
$$=-\frac{e^2}{\pi}\int_0^\infty dr'\mathscr{D}(r_x,r')j^0(r',t)\,, \tag{2.44}$$

where in the last line I have used the fact that $\mathscr{D}(r_x,r')=\mathscr{D}(r',r_x)$. Using Eq. (2.18) gives

$$\nabla_\mu j_5^\mu(x) = -\frac{e^2}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \tag{2.45}$$

which has the same form as the flat—space-time equation. Using Eq. (2.45) along with $j_5^\mu(x)=\epsilon^{\mu\nu}j_\nu(x)$ and $\nabla_\mu j^\mu(x)=0$ gives

$$\left[\Box+\frac{e^2}{\pi}\right]j_\mu(x)=0\ , \tag{2.46}$$

where $\Box$ is the curved—space-time d'Alembertian. Equation (2.46) is quite complicated. Writing it out in components gives

$$g^{\alpha\beta}\partial_\alpha\partial_\beta j_1(x)-\frac{4MG}{r^2}\frac{\partial j_1(x)}{\partial r}-\frac{2MG}{r^2}\left[1-\frac{2MG}{r}\right]^{-2}\frac{\partial j_0(x)}{\partial t}+\left[\frac{2MG}{r^3}-\left[\frac{MG}{r^2}\right]^2\left[1-\frac{2MG}{r}\right]^{-1}\right]j_1(x)+\frac{e^2}{\pi}j_1(x)=0 \tag{2.47}$$

and

$$g^{\alpha\beta}\partial_\alpha\partial_\beta j_0(x)-\frac{2MG}{r^2}\frac{\partial j_1(x)}{\partial t}-\frac{2MG}{r^3}j_0(x)+\frac{e^2}{\pi}j_0(x)=0\ . \tag{2.48}$$

Clearly these equations cannot be solved analytically. One must either use a computer or be content to study them in the limit $MG/r \ll 1$ or $(1-2MG/r)\ll 1$. Fortunately I will not need the solution to Eq. (2.46). In flat space-time it is easy to show that Eq. (2.46) implies confinement.[11] One first shows that the current-current correlation function satisfies the spectral representation

$$\langle j^\mu(x)j^\nu(x')\rangle = (\eta^{\mu\nu}\partial^2-\partial^\mu\partial^\nu)\int d\sigma^2 r(\sigma^2)\Delta^+(x-x';\sigma^2)\ , \tag{2.49}$$

where

$$\Delta^+(x-x';\sigma^2)=\int \frac{d^nk}{(2\pi)^n}\theta(k^0)\delta(k^2+\sigma^2)e^{ik(x-x')}$$

and $r(\sigma^2)$ vanishes for $\sigma^2<0$ and is real and positive semidefinite for $\sigma^2 \geq 0$. Applying Eq. (2.46) to Eq. (2.49) then gives

$$\langle 0\,|\,j^\nu(x)j^\mu(x')\,|\,0\rangle = (\eta^{\mu\nu}\partial^2-\partial^\mu\partial^\nu)\mu^2\Delta^+(x-x';\mu^2)\ , \tag{2.50}$$

where $\mu^2=e^2/\pi$. This shows that there is a massive boson in the model and that this gives the only intermediate-state contribution to $\langle j^\mu j^\nu\rangle$. Unfortunately things are not so simple in curved space-time. The trouble lies in deriving the analog of Eq. (2.49) in curved space-time. The starting point for the derivation of Eq. (2.49) was writing $\langle j^\mu j^\nu\rangle$ as

$$\langle j^\mu(x)j^\nu(x')\rangle = \sum_\alpha \langle 0\,|\,j^\mu(0)\,|\,\alpha\rangle e^{-ip_\alpha(x-x')}\langle \alpha\,|\,j^\nu(0)\,|\,0\rangle\ .$$

One cannot do this in curved space-time because one no longer has translational invariance. As a result I have not yet been able to prove that the Schwinger model confines in curved space-time although I suspect that it does.

III. CONSTRUCTION OF THE GREEN'S FUNCTIONS

The next and last step in solving the model is to construct the Green's functions. Let

$$G(x_1,x_2)=\langle 0\,|\,T[\psi(x_1)\bar\psi(x_2)]\,|\,0\rangle\ , \tag{3.1}$$

where I will worry about which vacuum state is appropriate later. I want to compute $i\underline{\gamma}^\mu\nabla_\mu G$. Now

$$i\underline{\gamma}^\mu(x_1)\nabla_\mu^{x_1}G(x_1,x_2)=i\underline{\gamma}^\mu(x_1)\{[\nabla_\mu^{x_1}\theta(t_1-t_2)]\langle 0\,|\,\psi(x_1)\bar\psi(x_2)\,|\,0\rangle-[\nabla_\mu^{x_1}\theta(t_2-t_1)]\langle 0\,|\,\bar\psi(x_2)\psi(x_1)\,|\,0\rangle\}$$
$$+i\underline{\gamma}^\mu(x_1)\langle 0\,|\,T[\nabla_\mu^{x_1}\psi(x_1)\bar\psi(x_2)\,|\,0\rangle\ . \tag{3.2}$$

To make further progress I need to evaluate $\nabla_\mu^{x_1}\theta(t_1-t_2)$. The easiest way to calculate $\nabla_\mu^{x_1}\theta(t_1-t_2)$ is to assume that

$$\nabla_\mu^x\theta(x_0-x_0')=f(x)\delta(n[x_0-x_0']) ,$$

and then to use

$$[i\gamma^\mu(x)\nabla_\mu^x-m]S_F(x,x')=[g(x)]^{-1/2}\delta^n(x-x') , \tag{3.3}$$

where S_F is the free fermion propagator, to fix $f(x)$. Doing this gives

$$\nabla_\mu^x\theta(t-t')=g_{00}{}^{1/2}\delta(n[t-t']) . \tag{3.4}$$

One can then check for consistency by showing that the use of Eq. (3.4) gives

$$[\Box_x+m^2+\xi R(x)]G_F(x,x')=[g(x)]^{-1/2}\delta^n(x-x') , \tag{3.5}$$

where G_F is the free propagator for a scalar particle. Using Eq. (3.4) in Eq. (3.2) gives

$$\underline{\gamma}^\mu(x_1)\nabla_\mu^{x_1}G(x_1,x_2)=\delta^2(x_1-x_2)+ie\langle 0|T[\underline{\gamma}^\mu(x_1)A_\mu(x_1)\psi(x_1)\overline{\psi}(x_2)]|0\rangle , \tag{3.6}$$

where I have also used Eq. (2.10). In the Coulomb gauge, Eq. (3.6) becomes

$$\underline{\gamma}^\mu(x_1)\nabla_\mu^{x_1}G(x_1,x_2)=\delta^2(x_1-x_2)+ie\langle 0|T[\underline{\gamma}_0(x_1)A^0(x_1)\psi(x_1)\overline{\psi}(x_2)]|0\rangle . \tag{3.7}$$

Inserting the result for A^0 from Eq. (2.21) gives

$$\underline{\gamma}^\mu(x_1)\nabla_\mu^{x_1}G(x_1,x_2)=\delta^2(x_1-x_2)-ie^2\int_0^\infty dy^1\mathscr{D}(x_1^1,y^1)\langle 0|T[\underline{\gamma}^0(x_1)j^0(y^1,x_1^0)\psi(x_1)\overline{\psi}(x_2)]|0\rangle . \tag{3.8}$$

In order to get a closed equation for $G(x_1,x_2)$ I need to compute the effect of the Klein-Gordon operator on $\langle 0|T[j^\mu\psi\overline{\psi}]|0\rangle$. An easy but long calculation gives

$$\begin{aligned}\nabla_\nu^x\langle 0|T[j^\mu(x)\psi(x_1)\overline{\psi}(x_2)]|0\rangle=&\langle 0|T[\nabla_\nu j^\mu(x)\psi(x_1)\overline{\psi}(x_2)]|0\rangle\\&-g_{00}{}^{1/2}(x)\delta^2(x-x_1)\gamma^0\underline{\gamma}^\mu(x)\langle 0|T[\psi(x_1)\overline{\psi}(x_2)]|0\rangle\delta_\nu{}^0\\&+g_{00}{}^{1/2}(x)\delta^2(x-x_2)\langle 0|T[\psi(x_1)\underline{\gamma}^\mu(x)\gamma^0\overline{\psi}(x_2)]|0\rangle\delta_\nu{}^0 .\end{aligned} \tag{3.9}$$

In deriving this equation I have used

$$\delta(n[x^0-x_1^0])[j^\mu(\vec{x},x^0),\psi(\vec{x}_1,x^0)]=-\gamma^0\underline{\gamma}^\mu(x)\psi(\vec{x}_1,x_0)\frac{\delta^n(x-x_1)}{g_x{}^{1/2}} \tag{3.10}$$

and

$$\delta(n[x^0-x_1^0])[j^\mu(\vec{x},x^0),\overline{\psi}(\vec{x}_1,x^0)]=\overline{\psi}(\vec{x}_1,x_0)\underline{\gamma}^\mu(x)\gamma^0\frac{\delta^n(x-x_1)}{g_x{}^{1/2}} . \tag{3.11}$$

A similar but even longer calculation gives

$$\begin{aligned}\left[\Box_x+\frac{e^2}{\pi}\right]&\langle 0|T[j^\mu(x)\psi(x_1)\overline{\psi}(x_2)]|0\rangle\\&=\Big\{[\underline{\gamma}^\mu(x)\underline{\gamma}^\nu(x)]_1\nabla_\nu^x\delta^2(x-x_1)-[\underline{\gamma}^\nu(x)\underline{\gamma}^\mu(x)]_2\nabla_\nu^x\delta^2(x-x_2)\\&\quad+\frac{e^2}{\pi}[\delta(x^0-x_1^0)\nabla_x^\mu\mathscr{D}(x^1-x_1^1)-\delta(x^0-x_2^0)\nabla_x^\mu\mathscr{D}(x^1-x_2^1)]\Big\}\langle 0|T[\psi(x_1)\overline{\psi}(x_2)]|0\rangle ,\end{aligned} \tag{3.12}$$

where I am using the short-hand notation $\Gamma_1\langle 0|T(\psi\overline{\psi})|0\rangle=\langle 0|T(\Gamma\psi)\overline{\psi}|0\rangle$ and $\Gamma_2\langle 0|T(\psi\overline{\psi})|0\rangle=\langle 0|T(\psi\Gamma\overline{\psi})|0\rangle$ where Γ is a combination of γ matrices. In deriving Eq. (3.12) I have used

$$[\nabla_0 j^\mu(x^1,x^0),\psi(x_1^1,x^0)]=-\epsilon^\mu_\nu \underline{\gamma}^\nu(x)\gamma^0\psi(x_1^1,x^1)\nabla_1\delta(x^1-x_1^1)+\frac{e^2}{\pi}\gamma^0\underline{\gamma}^0(x)\psi(x_1^1,x^0)\nabla^\mu\mathscr{D}(x^1-x_1^1) \quad (3.13)$$

and

$$[\nabla_0 j^\mu(x^1,x^0),\bar\psi(x_2^1,x^0)]=\epsilon^\mu_\nu\bar\psi(x_1^1,x^0)\underline{\gamma}^\nu(x)\gamma^0\nabla_1\delta(x^1-x_2^1)-\frac{e^2}{\pi}\bar\psi(x_2^1,x^0)\underline{\gamma}^0(x)\gamma^0\nabla^\mu\mathscr{D}(x^1-x_2^1)\,. \quad (3.14)$$

Defining $G(x_1,x_2)$ by

$$G(x_1,x_2)=\langle 0\,|\,T[\psi(x_1)\bar\psi(x_2)]\,|\,0\rangle \quad (3.15)$$

and $\Delta_+(x-y;\mu^2)$ by

$$(\Box+\mu^2)\Delta_+(x-y;\mu^2)=\delta^2(x-y) \quad (3.16)$$

[where in the flat–space-time limit $\Delta_+(x-y;\mu^2)\to\int(dk/2\pi)\theta(k^0)\delta(k^2+\mu^2)e^{ik(x-y)}$] gives when used in Eq. (3.12)

$$\begin{aligned}\langle 0\,|\,T[j^\mu(x)\psi(x_1)\bar\psi(x_2)]\,|\,0\rangle\\ =\int d^2y\,\Delta_+\left[x-y;\frac{e^2}{\pi}\right]\Bigg\{&[\underline{\gamma}^\mu(x)\underline{\gamma}^\nu(x)]_1[\nabla^y_\nu\delta^2(y-x_1)]-[\underline{\gamma}^\nu(x)\underline{\gamma}^\mu(x)]_2[\nabla^y_\nu\delta^2(y-x_2)]\\ &+\frac{e^2}{\pi}[\delta(y^0-x_1^0)\nabla^\mu_y\mathscr{D}(y^1-x_1^1)-\delta(y^0-x_2^0)\nabla^\mu_y\mathscr{D}(y^1-x_2^1)]\Bigg\}G(x_1,x_2)\,. \end{aligned}\quad (3.17)$$

Thus

$$\begin{aligned}\langle 0\,|\,T[j^\mu(x)\psi(x_1)\bar\psi(x_2)]\,|\,0\rangle=\Bigg\{&[\underline{\gamma}^\mu(x)\gamma^\nu(x)]\nabla^1_\nu\Delta_+\left[x-x_1;\frac{e^2}{\pi}\right]-[\underline{\gamma}^\nu(x)\underline{\gamma}^\mu(x)]_2\nabla^{(2)}_\nu\Delta_+\left[x-x_2;\frac{e^2}{\pi}\right]\\ &+\frac{e^2}{\pi}\int_0^\infty dy^1\Bigg[\Delta_+\left[x^1-y^1,x^0-x_1^0;\frac{e^2}{\pi}\right]\nabla^\mu_{(1)}\mathscr{D}(y^1-x^1)\\ &-\Delta_+\left[x^1-y^1,x^0-x_2^0;\frac{e^2}{\pi}\right]\nabla^\mu_{(2)}\mathscr{D}(y^1-x_2^1)\Bigg]\Bigg\}G(x_1,x_2)\,.\end{aligned} \quad (3.18)$$

This expression for $\langle 0\,|\,T[j^\mu\psi\bar\psi]\,|\,0\rangle$ can now be used in Eq. (3.8) (with $\mu^2=0$) to get

$$\begin{aligned}\underline{\gamma}^\mu(x_1)\overset{x_1}{\nabla_\mu}G(x_1,x_2)=\delta^2(x_1-x_2)-ie^2\int_0^\infty dy^1\mathscr{D}(x_1^1,y^1)[\underline{\gamma}_0(x_1)]_1\\ \times\Bigg\{&[\underline{\gamma}^0(y)\underline{\gamma}^\nu(y)]_1\nabla^{(1)}_\nu\Delta_+\left[y^1-x_1^1,0;\frac{e^2}{\pi}\right]\\ &-[\underline{\gamma}^\nu(y)\underline{\gamma}^0(y)]_1\nabla^{(2)}_\nu\Delta_+\left[y^1-x_2^1,x_1^0-x_2^0;\frac{e^2}{\pi}\right]\Bigg\}G(x_1,x_2)\,.\end{aligned} \quad (3.19)$$

This is a closed equation for $G(x_1,x_2)$. Before solving this equation I want to look more carefully at the term

$$\int_0^\infty dy^1\mathscr{D}(x_1^1,y^1)[\underline{\gamma}_0(x_1)]_1[\underline{\gamma}^0(y)\underline{\gamma}^\nu(y)]_1\nabla^{(1)}_\nu\Delta_+\left[y^1-x_1^1,0;\frac{e^2}{\pi}\right]. \quad (3.20)$$

In flat space-time this term is equal to zero. The argument is that I can shift variables to $y'=y^1-x_1^1$. The resulting function is odd in y' and is integrated over an even interval. Thus the result is zero. In a two-dimensional Schwarzschild space-time this argument is not correct because the region of integration is not an even interval. The singularity at $r=0$ cuts the space-time in half and the spatial part of the space-time is the half line not the line. This is an artifact of a 1 + 1 space-time but nevertheless the term in Eq. (3.20) cannot be set to zero.

I can use a trick to rewrite Eq. (3.19) in a more convenient form. Using $\underline{\gamma}^{\mu}(x)\nabla_{\mu}\underline{\gamma}^{\lambda}(x)\nabla_{\lambda}\Delta_{+}(x,0)=\delta^{2}(x)$ I get

$$ie^{2}\int_{0}^{\infty}dy^{1}\mathscr{D}(x_{1}^{1},y^{1})[\underline{\gamma}_{0}(x_{1})]_{1}\left\{[\underline{\gamma}^{\nu}(y)\underline{\gamma}^{0}(y)]_{2}\nabla_{\nu}^{(2)}\Delta_{+}\left[y^{1}-x_{2}^{1},x_{1}^{0}-x_{2}^{0};\frac{e^{2}}{\pi}\right]\right\}G(x_{1},x_{2})$$

$$=ie^{2}\underline{\gamma}^{\mu}(x_{1})\overset{x_{1}}{\nabla}_{\mu}[\gamma^{\lambda}(x_{1})\underline{\gamma}^{0}(x_{1})]_{1}\overset{x_{1}}{\nabla}_{\lambda}\overset{x_{2}}{\nabla}_{\nu}\left\{\int d^{2}z\,\Delta_{+}(x_{1}-z;0)\int_{0}^{\infty}dy^{1}[\gamma^{\nu}(y)\gamma^{0}(y)]_{2}\right.$$

$$\left.\times\mathscr{D}(x_{1}^{1},y^{1})\Delta_{+}\left[y^{1}-x_{2}^{1};z^{0}-x_{2}^{0};\frac{e^{2}}{\pi}\right]\right\}$$

$$\equiv\underline{\gamma}^{\mu}(x_{1})\overset{x_{1}}{\nabla}_{\mu}F(x_{1},x_{2})\ ,$$

where

$$F(x_{1},x_{2})=ie[\underline{\gamma}^{\lambda}(x_{1})]_{1}\overset{x_{1}}{\nabla}_{\lambda}\overset{x_{2}}{\nabla}_{\nu}\int d^{2}z\,\Delta_{+}(x_{1}-z;0)\underline{\gamma}_{0}(z)$$

$$\times\int_{0}^{\infty}dy^{1}[\underline{\gamma}^{\nu}(y)\underline{\gamma}^{0}(y)]_{2}\mathscr{D}(x_{1}^{1},y^{1})\Delta_{+}\left[y^{1}-x_{2}^{1};z^{0}-x_{2}^{0};\frac{e^{2}}{\pi}\right]$$

$$-\text{ the same term with } x_{1}=x_{2}\ .$$

Since only the derivative of F with respect to x_1 is defined I am free to subtract a function of x_2 and this gives me an $F(x_1,x_2)$ such that $F(x_2,x_2)=0$. After playing the same trick with the term in Eq. (3.20) I can rewrite Eq. (3.19) as

$$\underline{\gamma}^{\mu}(x_{1})\overset{x_{1}}{\nabla}_{\mu}G(x_{1},x_{2})=\delta^{2}(x_{1}-x_{2})-\underline{\gamma}^{\mu}(x_{1})[\overset{x_{1}}{\nabla}_{\mu}H(x_{1},x_{2})]G(x_{1},x_{2})+\underline{\gamma}^{\mu}(x_{1})[\overset{x_{1}}{\nabla}_{\mu}F(x_{1},x_{2})]G(x_{1},x_{2})\ , \tag{3.21}$$

where

$$H(x_{1},x_{2})=ie^{2}\underline{\gamma}^{\lambda}(x_{1})\overset{x_{1}}{\nabla}_{\lambda}\overset{x_{1}}{\nabla}_{\nu}\int d^{2}z\,\Delta_{+}(x_{1}-z;0)\int_{0}^{\infty}dy^{1}\mathscr{D}(z^{1},y^{1})$$

$$\times[\gamma_{0}(z)]_{1}[\underline{\gamma}^{0}(y)\gamma^{\nu}(y)]_{1}\Delta_{+}\left[y^{1}-z^{1},0;\frac{e^{2}}{\pi}\right]$$

the same term with $x_1=x_2$. Note that $F(x_1,x_2)$ and $H(x_1,x_2)$ are defined so as to be finite when $x_1=x_2$. These functions are in fact finite everywhere provided the manifold has the singularity at $r=0$ removed. This can be done by defining $\tau=it$ and working in the resulting Riemannian space-time.

In flat space-time the H term is zero. To solve this equation let

$$G(x_{1},x_{2})=e^{F(x_{1},x_{2})-H(x_{1},x_{2})}\mathscr{G}(x_{1},x_{2})\ . \tag{3.22}$$

Plugging this into Eq. (3.21) gives

$$\gamma^{\mu}(x_{1})e^{F(x_{1},x_{2})-H(x_{1},x_{2})}\nabla_{\mu}\mathscr{G}(x_{1},x_{2})=\delta^{2}(x_{1}-x_{2})\ . \tag{3.23}$$

Left multiplying by e^{-F+H} and using $F(x_2,x_2)=H(x_2,x_2)=0$ gives

$$e^{-F(x_{1},x_{2})+H(x_{1},x_{2})}\underline{\gamma}^{\mu}(x_{1})e^{F(x_{1},x_{2})+H(x_{1},x_{2})}\overset{x_{1}}{\nabla}_{\mu}\mathscr{G}(x_{1},x_{2})=\delta(x_{1}-x_{2})\ . \tag{3.24}$$

Since I am going to need to compute $[\gamma^{\mu},e^{F-H}]$ I want to rewrite F and H so as to pull the γ factors out of them. I can do this by writing

$$F(x_1,x_2)=(\gamma^\alpha\gamma^0)_1(\gamma^\beta\gamma^0)_2 f_{\alpha\beta}(x_1,x_2)\ , \tag{3.25}$$

where

$$\begin{aligned} f_{\alpha\beta}(x_1,x_2)=ieb^\lambda_\alpha(x_1)\nabla^{x_1}_\lambda\nabla^{x_2}_\nu \int d^2z\,\Delta_+(x_1-z;0)\eta_{00}g_{00}{}^{1/2}(z) \\ \times\int_0^\infty dy^1 b^\nu_\beta(y)g^{00}(y)^{1/2}\mathscr{D}(x^1_1,y^1)\Delta_+\left[y^1-x^1_2,z^0-x^0_2;\frac{e^2}{\pi}\right] \\ -\text{ same term with } x_1=x_2 \end{aligned} \tag{3.26}$$

and

$$H(x_1,x_2)=(\gamma^\alpha\gamma^0)_1(\gamma^0\gamma^\beta)_1 h_{\alpha\beta}(x_1,x_2)\ , \tag{3.27}$$

where

$$\begin{aligned} h_{\alpha\beta}(x_1,x_2)=ieb^\lambda_\alpha(x_1)\nabla^{x_1}_\lambda\nabla^{x_1}_\nu \int d^2z\,\Delta_+(x_1-z;0)\int_0^\infty dy^1\mathscr{D}(z^1,y^1) \\ \times g_{00}{}^{1/2}(z)\eta_{00}g^{00}(y)^{1/2}b^\nu_\lambda(y)\Delta_+\left[y^1-z^1,0;\frac{e^2}{\pi}\right] \\ -\text{ same term with } x_1=x_2\ . \end{aligned} \tag{3.28}$$

Now

$$\begin{aligned} e^{-F+H}\underline{\gamma}^\mu(x_1)e^{F-H}&=\underline{\gamma}^\mu(x_1)+e^{F-H}[\gamma^\mu(x_1),e^{F-H}] \\ &=\gamma^\mu(x_1)+e^{-F+H}[\gamma^\mu(x_1),F-H)]+\frac{e^{-F+H}}{2!}[\gamma^\mu(x_1),(F-H)^2]+\frac{e^{-F+H}}{3!}[\gamma^\mu(x_1),(F-H)^3]+\cdots \end{aligned} \tag{3.29}$$

and these commutators can be computed with the help of relations like

$$[\gamma^\nu_1,(\gamma^\alpha\gamma^0)_1(\gamma^\beta\gamma^0)_2]=2\eta^{\alpha 1}(\gamma^\nu\gamma_5)_1(\gamma^\beta\gamma^0)_2 \tag{3.30}$$

and

$$[\gamma^\nu,\gamma^\alpha\gamma^\beta]=2\epsilon^\beta_\delta\eta^{\alpha\delta}\gamma^\nu\gamma_5\ , \tag{3.31}$$

where the ϵ symbol is the flat—space-time ϵ. Doing this gives

$$\begin{aligned} e^{-F(x_1,x_2)+H(x_1,x_2)}\underline{\gamma}^\mu(x_1)e^{F(x_1,x_2)-H(x_1,x_2)}=[\cosh\Phi_1(x_1,x_2)-(\gamma_5)_1\sinh\Phi_1(x_1,x_2)] \\ \times[\cosh\Phi_2(x_1,x_2)-(\gamma_5)_1\sinh\Phi_2(x_1,x_2)]\underline{\gamma}^\mu(x_1)\ , \end{aligned} \tag{3.32}$$

where

$$\Phi_1(x_1,x_2)=-2(\gamma^\beta\gamma^0)_2 f_{1\beta}(x_1,x_2) \tag{3.33}$$

and

$$\Phi_2(x_1,x_2)=-2\epsilon^\beta_\delta\eta^{\alpha\delta}h_{\alpha\beta}(x_1,x_2)\ . \tag{3.34}$$

Using Eq. (3.22) in Eq. (3.24) gives

$$\begin{aligned} [\cosh\Phi_1(x_1,x_2)-(\gamma_5)_1\sinh\Phi_1(x_1,x_2)][\cosh\Phi_2(x_1,x_2)-(\gamma_5)_1\sinh\Phi_2(x_1,x_2)] \\ \times[\underline{\gamma}^\mu(x_1)]_1\nabla^{x_1}_\mu\mathscr{G}(x_1,x_2)=\delta^2(x_1-x_2)\ . \end{aligned} \tag{3.35}$$

Left multiplying by $\cosh\Phi_1+\gamma_5\sinh\Phi_1$ and $\cosh\Phi_2+\gamma_5\sinh\Phi_2$ gives

$$[\gamma^\mu(x_1)]_1\nabla^{x_1}_\mu\mathscr{G}(x_1,x_2)=\delta^2(x_1,x_2)\ . \tag{3.36}$$

This is the equation for a free massless fermion Green's function. Thus

$$\mathscr{G}(x_1,x_2)=G_0(x_1,x_2)\ , \tag{3.37}$$

where G_0 is the free Green's function, and

$$G(x_1,x_2)=e^{F(x_1,x_2)-H(x_1,x_2)}G_0(x_1,x_2) . \tag{3.38}$$

Since one can solve for any other Green's function in just the same way as the two-point Green's functions was solved for, this solves the Schwinger model in a two-dimensional Schwarzschild space-time. The model has been solved in terms of two functions $G_0(x_1,x_2)$ and $\Delta_+(x_1,x_2,\mu^2)$. Since G_0 is the Green's function for a massless particle, G_0 can be obtained from $G_0^{\rm flat}$ by a conformal transformation. Writing the line element as

$$ds^2=C(r)du\,dv \tag{3.39}$$

and making a conformal transformation gives

$$G_0(x_1,x_2)=C^{-1/4}(r)C^{-1/4}(r')G_0^{\rm flat}(x,x') . \tag{3.40}$$

Of course one must still specify the vacuum state, which is equivalent to specifying the null coordinates used in Eq. (3.39). The usual choice for an eternal black hole is either the Kruskal or the Schwarzschild vacuum (see Birrell and Davies[22] for a discussion of these vacuums). The function $\Delta_+(x_1,x_2,\mu^2)$ however cannot be explicitly solved for. Because of the mass one cannot use a conformal transformation to calculate it and the equation for the mode functions

$$(\Box+\mu^2)\phi(x)=0 \tag{3.41}$$

is too complicated to solve.

The Ward identity satisfied by $G(x_1,x_2)$ can be easily derived by starting from the generating functional

$$e^{-Z[J_\mu,\chi,\bar\chi]}=N\int \mathscr{D}[A_\mu]\mathscr{D}[\bar\Psi]\mathscr{D}[\Psi]\mathscr{D}[\eta]\mathscr{D}[\eta^*]e^{-S_{\rm eff}-\langle J_\mu A^\mu+i\bar\chi\psi+i\bar\psi\chi\rangle} , \tag{3.42}$$

where χ and $\bar\chi$ are Grassmann sources and

$$S_{\rm eff}=\int \sqrt{g}\,d^nx\left[\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\bar\psi(\underline{\gamma}^\mu\nabla_\mu+ie\,\underline{\gamma}^\mu A_\mu)\psi+\frac{1}{2\alpha}(\nabla_\mu A^\mu)^2-i\nabla_\mu\eta^*\nabla^\mu\eta\right] , \tag{3.43}$$

with η and η^* complex Grassmann ghost fields.

The Ward identity is then derived by making a Becchi-Rouet-Stora (BRS) transformation:

$$\delta A_\mu=\frac{1}{e}\nabla_\mu(\lambda^*\eta+\lambda\eta^*) ,$$

$$\delta\psi=-i(\lambda^*\eta+\lambda\eta^*) ,\quad \delta\bar\psi=i\psi(\lambda^*\eta+\lambda\eta^*) ,$$

$$\delta\eta=\frac{-i}{\alpha e}(\nabla_\mu A^\mu)\lambda ,\quad \delta\eta^*=\frac{i}{\alpha e}(\nabla_\mu A^\mu)\lambda^* ,$$

where λ and λ^* are complex Grassmann constants. Using the fact that under this transformation $\delta S_{\rm eff}=0$ allows one to show that

$$\nabla_\mu\Gamma^\mu(x,y,z)=iS^{-1}(x-z)\delta(z-y) -iS^{-1}(x-z)\delta(x-y) , \tag{3.44}$$

where Γ^μ is the three-point function and S^{-1} is the inverse propagator. Equation (3.44) shows that as expected the Ward identity has the same form as in flat space-time.

IV. CONCLUSIONS AND OUTLOOK

The main point of this paper is that the infrared properties of quantum field theories in curved space-time are important and can (at least in very simple systems) be attacked. However, the subject is very difficult, even with an exact solution to the Schwinger model I have not yet been able to prove that it confines in curved space-time. Since it confines for finite temperatures I expect that it will confine for finite curvatures and I am working on showing this. In a future paper I will use the solution to the Schwinger model to study Hawking radiation from a black hole.

For reasons discussed earlier in this paper, the best place to look for loss of confinement due to curvature would be in a renormalizable but not super-renormalizable model that lost confinement at high temperature. The best model to look at would seem to be CP^{N-1},[23,24] in curved space-time.

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