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Riemannian structure of space-time as a consequence of quantum mechanics

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Different axiomatic approaches to general relativity which use light rays and classical test particles as primitive concepts remain incomplete because they end with a Weylian instead of a Riemannian structure of space-time. It is shown that the final step to a Riemann space can be obtained as a necessary consequence if quantum mechanics, as the theory of matter, is made part of the total scheme. Quantum mechanics must contain classical particle mechanics as a limiting case. The self-consistency requirement that in Weyl space this limiting case should agree with the axiomatically introduced classical-test-particle behavior implies the conclusion that the Weyl geometry of space-time must be restricted to the special case of a Riemann geometry. This is shown in detail for massive spin-$\frac{1}{2}$ particles after a general discussion of the theory of unquantized tensor fields and two-spinor fields in Weyl space. The result is independent of the Weyl type chosen for the orthotetrad (Lorentz basis). The same conclusion is obtained from massive Klein-Gordon theory in Weyl space in demanding that the physically reasonable current should be divergence-free.

I. INTRODUCTION

General-relativity theory as a metric theory formulated in Riemann or Riemann-Cartan space is now accepted as the most satisfactory theory of gravitation as far as quantum effects of gravitation may be neglected. During the last centuries there have been many attempts to deduce the Riemannian structure of general-relativity theory from a few axioms. A certain class of approaches seems to be the most important one because the basic tools are not complex physical objects like clocks or gyroscopes, but primitive concepts like light rays and freely falling test particles (see details below). But these approaches have the disadvantage in common that the respective axiomatic constructions of the theory end with the Weyl space as space-time. There seems to be no way to close the remaining gap between Weyl and Riemann space if one is restricted to the use of these primitive concepts only. The axiomatic scheme remains incomplete. *The aim of this paper is to show that the gap can in fact be closed if quantum mechanics as the theory of matter is made part of the total scheme. Quantum mechanics proves that space-time must be a Riemann (or Riemann-Cartan) space.*

This proof is based on two demands a gravitation theory has to fulfill[1]:

(i) Completeness: The theory must mesh with and incorporate all nongravitational laws, in particular the quantum mechanical.

(ii) Self-consistency: if one calculates the prediction for the outcome of an experiment by different methods, one always gets the same result.

The demand (i) forces us to include quantum-mechanically described matter into the scheme of general relativity. Quantum mechanics must contain classical mechanics as a limiting case. The demand (ii) then requires that this classical limit on one hand and the axiomatically introduced classical mechanics on the other agree. It is this demand which will finally lead to the conclusion that gravity as a space-time theory must be described by a Riemannian instead of the more general Weylian structure. Note that our arguments are not based on the outcome of experiments. We are testing the consequences of a theory against its original foundations, which in our case are introduced axiomatically using classical concepts.

A. The gap which is to be closed

Up to 1970 the common axiomatic approach to space-time structure was the one of Synge,[2] which is based on the behavior of standard clocks. The main objection against this chronometric approach[3] is that the real clocks of physicists and astronomers (e.g., atomic clocks) are highly complicated systems which work on the basis of quantum mechanics. Because one can construct ideal clocks showing gravitational time in a more geometric way by means of light rays and freely falling particles, the chronometric axiom reduces to the claim that gravitational and atomic time agree. This, on the other hand, should better be deduced from theory[4] and measured experimentally. Accordingly there have been several efforts after 1970 to describe an alternative constructive approach to general relativity based on more primitive concepts.[5] All these different approaches end up with assigning to space-time a Weyl geometry instead of the further restricted Riemann geometry of general relativity.

A typical axiomatic scheme which results in a Weyl geometry is the one described in Ref. 3. It can be very briefly summarized as follows: primitive concepts are event, light ray, and freely falling particle. The light propagation determines the null cones and therefore a conformal structure, i.e., an equivalence class $e^{\Lambda(x)}g_{\alpha\beta}$ of metrics. The freely falling particles determine the affine geodesics and therefore a projective structure, i.e., an equivalence class of symmetric affine connections. The compatibility requirement that the null geodesics of the conformal structure belong to the geodesics defined by the projective structure then finally results in a Weyl structure W^4.

In a Weyl geometry W^4 (Ref. 6) there is a unique symmetric[7] connection which is semimetric[8]:

$$\partial_\mu g_{\alpha\beta} - \Gamma^{\epsilon}_{\mu\alpha} g_{\epsilon\beta} - \Gamma^{\epsilon}_{\mu\beta} g_{\alpha\epsilon} = -\tilde{a}_\mu g_{\alpha\beta} \ . \tag{1.1}$$

The metric undergoes a conformal transformation (gauge transformation)

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = e^{\Lambda(x)} g_{\alpha\beta} \tag{1.2}$$

with real $\Lambda(x)$. It should be noted that conformal transformations (1.2) appear in physics under several different circumstances. The related mathematical theorems and physical results often differ and cannot necessarily be transcribed to the physics of Weyl space.[9] Related to (1.2) is the transformation

$$\tilde{a}_\mu \rightarrow \tilde{a}'_\mu = \tilde{a}_\mu - \partial_\mu \Lambda \tag{1.3}$$

of the gauge potential $\tilde{a}_\mu$, whereas $\Gamma^{\alpha}_{\beta\gamma}$ remains unaltered. In Weyl space it is possible to introduce gravitational standard clocks in a geometrical way.[10] Accordingly, Weyl spaces may be taken as possible physically reasonable space-times. The question is, therefore, how can we show that Weyl space must be further specialized to a Riemann space (or Riemann-Cartan space) in order to describe the real physical world?

B. How quantum mechanics closes the gap

The usual, non-quantum-mechanical way to do so is to postulate that there is no second-clock effect, i.e., that rates of gravitational standard clocks do not depend on their history. This means that two coinciding clocks of this type of equal rates still have equal rates when they are separated and over different world lines brought together again. This postulate is of course unsatisfactory from the point of view of an axiomatic scheme which has the basic intention to overcome the chronometric arguments. In addition, the physical meaning of such a postulate is unclear because its relation to the behavior of real physical clocks (e.g., atomic clocks) in Weyl space is an open question. No generally valid answer is to be expected. Therefore, the more promising approach seems to be the discussion of *self-consistency of particle physics in Weyl space* in regarding the classical limit of quantum mechanics.

In Riemann space, the trajectory of the free particle can either be described by an affine geodesic or alternatively introduced by means of a variational principle based on proper time. Both definitions agree because the connection is metric. In Weyl space this situation is different. The free motion of classical test particles is by axiomatic construction described by affine geodesics. The Hamilton-Jacobi equation, on the other hand,

$$(\partial_\alpha S)(\partial_\beta S) g^{\alpha\beta} = m^2 \tag{1.4}$$

contains the metric $g^{\alpha\beta}$ with the nontrivial transformation behavior related to (1.2). This induces a corresponding nontrivial transformation behavior of m. If in addition the gradients of the surfaces of equal S

$$p_\alpha = -\partial_\alpha S \tag{1.5}$$

may still be related by some reason to the paths of free test particles, it can be conjectured that the Hamilton-Jacobi approach leads to a different class of particle trajectories. Quantum mechanics is in fact based on the Hamilton formalism. Also, in Weyl space it will therefore imply in an appropriate classical limit a Hamilton-Jacobi equation (1.4) so that a question of self-consistency will arise.

To put it in a different way: While the classical test particle motion is described by trajectories and the respective tangent vectors, the classical particle limit of quantum mechanics is based on the surfaces of equal phase and the related gradients, i.e., on one-forms. In Weyl space, there is because of (1.2) no "natural" isomorphism between vectors and one-forms. It can therefore be supposed that the consistency demand that the classical limit of quantum mechanics should agree with classical mechanics forces one to be restricted to special Weyl spaces with Riemannian structure.[11] This would close the gap described above.

The purpose of this paper is to work out the details of this approach. We restrict ourselves to classical quantum-mechanical fields and introduce only rudiments of the related theory of first quantization. This will already be sufficient. To do so, we first outline the basic structures of vector and spinor fields (which transform nontrivially under Weyl gauge transformations) in a Weyl space W^4. The affine and metric structure of a Weyl space and the gauge transformations of vector fields are discussed in Sec. II. The introduction of orthotetrads will then permit the definition of spinor fields. Section III contains the basic algebraic and differential properties of two-spinor calculus in Weyl space W^4. In Sec. IV Stokes's theorem is given a form which is appropriate for the discussion of physical currents and conservation in W^4. Dirac theory in Weyl space is introduced in Sec. V. In Sec. VI we demonstrate the intended theorem, namely, that the classical WKB limit does not agree with the free fall of classical particles as introduced in the axiomatic scheme above. Section VII contains a discussion of Klein-Gordon theory in Weyl space with the result that again the restriction to a Riemann space is necessary. Our conclusions are given in Sec. VIII. In the Appendix we describe the "standard" choice of the Weyl type of the orthotetrad.

C. Empirical background

Although our arguments are based on a discussion of completeness and self-consistency, and therefore remain totally within theory, it may be of interest to inquire about the empirical evidence that freely falling massive elementary particles follow the same trajectories as macroscopic bodies. This can in fact be directly shown for neutrons using a gravity refractometer with an uncertainty of only $\frac{1}{4000}$.[12]

II. WEYL SPACE

A. Gauge structure

We have as a starting point as usual a real, four-dimensional, connected paracompact smooth Hausdorff manifold $\mathscr{M}$. The tangent space at the point x is denoted by $T_x(\mathscr{M})$ and the dual space of one-forms by $\widetilde{T}_x(\mathscr{M})$. Elements of $\widetilde{T}_x(\mathscr{M})$ will be indicated by a tilde. For $\underline{A}\in T_x$ and $\underline{\widetilde{B}}\in \widetilde{T}_x$ there is a linear mapping $\langle \underline{\widetilde{B}},\underline{A}\rangle$ into the real numbers. Tensors are constructed in the usual way.

Apart from the action of the group GL(4,R) of the real four-dimensional linear transformations (represented as local coordinate transformations), all scalar fields $f(x)$ and tensor fields $\underline{T}(x)$ are subject to the following position-dependent real one-parameter gauge transformations:

$$f(x)\rightarrow f(x)'=e^{w(f)\Lambda(x)}f(x)\ , \tag{2.1a}$$

$$\underline{T}(x)\rightarrow \underline{T}(x)'=e^{w(\underline{T})\Lambda(x)}\underline{T}(x) \tag{2.1b}$$

which are called *Weyl transformations.* $\Lambda(x)$ is thereby a real function and $w(f)$ and $w(T)$ are real numbers characterizing the behavior of the fields $f(x)$ and $T(x)$ under Weyl transformations. $w(f)$ and $w(\underline{T})$ are called the *Weyl types* of $f(x)$ and $\underline{T}(x)$, respectively.

Throughout this paper we will assume a coordinate basis $\underline{E}_\alpha=\partial_\alpha$ and $\underline{\widetilde{E}}^\alpha=\underline{d}x^\alpha$ in $T_x(\mathscr{M})$ and $\widetilde{T}_x(\mathscr{M})$ to be of Weyl-type zero:

$$w(\underline{E}_\alpha)=w(\underline{\widetilde{E}}^\alpha)=0\ . \tag{2.2}$$

Components with regard to a coordinate basis $\underline{E}_\alpha=\partial_\alpha$ and $\underline{\widetilde{E}}^\alpha=dx^\alpha$ will be denoted by Greek indices:

$$\underline{T}=T^\alpha{}_\beta\underline{E}_\alpha\otimes\underline{\widetilde{E}}^\beta\ . \tag{2.3}$$

On the manifold is defined exactly one affine connection $\underline{\Gamma}$ called the *Weyl connection* which is the gauge potential of the coordinate transformations. It is of Weyl-type zero: $w(\underline{\Gamma})=0$. We introduce a doubly covariant derivative $\underline{D}$ of scalar and tensor fields, called the *Weyl derivative,* which is covariant under general coordinate transformations and under Weyl transformations. It defines a linear mapping of tensor fields of type (p,q) into tensor fields of type $(p,q+1)$. Apart from the usual property

$$\underline{D}(\underline{S}+\underline{T})=\underline{DS}+\underline{D}\,\underline{T} \tag{2.4}$$

the gauge covariance under Weyl transformations is reflected by

$$\underline{D}(f\underline{T})=[\underline{d}f+w(f)\underline{\widetilde{a}}f]\otimes\underline{T}+f\underline{D}\,\underline{T} \tag{2.5}$$

and by the demand that the Weyl type is preserved,

$$w(\underline{D}\,\underline{T})=w(\underline{T})\ . \tag{2.6}$$

$f(x)$ is thereby any function, $w(f)$ its Weyl type, and $\underline{S}$ and $\underline{T}$ are any vector fields. $\underline{\widetilde{a}}$ is a real-valued one-form called the *Weyl potential.* It is the gauge field of the Weyl transformations (2.1) and transforms inhomogeneously according to

$$\underline{\widetilde{a}}\rightarrow\underline{\widetilde{a}}'=\underline{\widetilde{a}}-\underline{d}\Lambda(x) \tag{2.7}$$

under Weyl transformations.

The components of the Weyl connection $\underline{\Gamma}$ are because of (2.2) given by

$$\underline{D}\,\underline{E}_\nu=\Gamma^\mu_{\nu\lambda}\underline{\widetilde{E}}^\lambda\otimes\underline{E}_\mu\ , \tag{2.8a}$$

$$\underline{D}\,\underline{\widetilde{E}}^\nu=\Gamma^\nu_{\rho\sigma}\underline{\widetilde{E}}^\rho\otimes\underline{\widetilde{E}}^\sigma\ . \tag{2.8b}$$

We restrict the following to the case of vanishing torsion:

$$\Gamma^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\gamma\beta} . \tag{2.9}$$

This will lead us below to a Weyl space instead of a Weyl-Cartan space. The restriction (2.9) is introduced in order to simplify the equations. It can very easily be abandoned. In this case our main physical result, that quantum mechanics implies a Riemann structure of space-time, remains correct if "Riemann" is replaced by "Riemann-Cartan." Because we are discussing quantum mechanics in the extreme classical limit $\hbar \rightarrow 0$ of a WKB approximation, the coupling of torsion with the elementary particle spin can be disregarded.[13]

Introducing components with regard to a coordinate basis

$$\underline{D}\,\underline{T}=(D_\mu T^{\alpha}{}_{\beta})\underline{\tilde{E}}^{\mu}\otimes\underline{\tilde{E}}^{\beta}\otimes\underline{E}_{\alpha} , \tag{2.10}$$

we find with (2.2) and (2.5) for the Weyl derivative

$$D_\mu T^{\alpha}{}_{\beta}=\partial_\mu T^{\alpha}{}_{\beta}+\Gamma^{\alpha}_{\mu\epsilon}T^{\epsilon}{}_{\beta}-\Gamma^{\epsilon}_{\mu\beta}T^{\alpha}{}_{\epsilon}+w(\underline{T})\tilde{a}_\mu T^{\alpha}{}_{\beta} \tag{2.11}$$

and for the transformations of the Weyl potential

$$\tilde{a}_\mu \rightarrow \tilde{a}'_\mu=\tilde{a}_\mu-\partial_\mu\Lambda . \tag{2.12}$$

B. Metrical structure

Apart from the unique Weyl connection $\underline{\Gamma}$ and the Weyl potential $\underline{\tilde{a}}$ we introduce the pseudo-Riemannian metric $\underline{g}(x)$ as an additional geometric field of fundamental importance. $\underline{g}(x)$ is a symmetric (0.2-tensor) with signature -2. The essential point in Weyl space is that $\underline{g}(x)$ is assumed to be of a nontrivial Weyl type:

$$w(\underline{g})=1 . \tag{2.13}$$

Because of the corresponding behavior

$$\underline{g}\rightarrow\underline{g}'=e^{\Lambda(x)}\underline{g} \tag{2.14}$$

under Weyl transformations, the fundamental tensor $\underline{g}$ is only fixed to within an arbitrary positive scale factor. Thus, there is not a unique metric tensor but a whole class.

The structures $\underline{\Gamma}$, $\underline{\tilde{a}}$, and $\underline{g}$ are now correlated in demanding that the Weyl derivative of the metric vanishes:

$$\underline{D}\,\underline{g}=0 . \tag{2.15}$$

This implies for the components $g_{\alpha\beta}$

$$\partial_\mu g_{\alpha\beta}-\Gamma^{\epsilon}_{\alpha\mu}g_{\epsilon\beta}-\Gamma^{\epsilon}_{\beta\mu}g_{\alpha\epsilon}=-\tilde{a}_\mu g_{\alpha\beta} . \tag{2.16}$$

Accordingly, the connection $\underline{\Gamma}$ is not a metric connection as in Riemann space, but a *semimetric connection.*

We introduce in the usual way the symmetric covariant tensor $\underline{\check{g}}$ conjugate to $\underline{g}$ with components $\check{g}^{\alpha\beta}$ defined by

$$g_{\alpha\beta}\check{g}^{\alpha\gamma}=\delta^{\gamma}_{\alpha} . \tag{2.17}$$

The Weyl type of $\underline{\check{g}}$ is $w(\underline{\check{g}})=-1$. Its Weyl derivative vanishes

$$\underline{D}\,\underline{\check{g}}=0 . \tag{2.18}$$

The three quantities $\underline{\Gamma}$, $\underline{\tilde{a}}$, and $\underline{g}$ are not independent. Because of (2.16) the torsion-free connection is determined uniquely by $\underline{g}$ and $\underline{\tilde{a}}$:

$$\Gamma^{\rho}_{\mu\nu}=\left\{{\rho \atop \mu\nu}\right\}+\tfrac{1}{2}(\delta^{\rho}_{\mu}\tilde{a}_\nu+\delta^{\rho}_{\nu}\tilde{a}_\mu-g_{\mu\nu}\check{g}^{\rho\epsilon}\tilde{a}_\epsilon) . \tag{2.19}$$

$\left\{{\rho \atop \mu\nu}\right\}$ is thereby the Christoffel symbol as constructed from $g_{\alpha\beta}$ and $\check{g}^{\alpha\beta}$. Its contraction is $\left\{{\epsilon \atop \mu\epsilon}\right\}=\partial_\mu \ln\sqrt{-g}$, where g is the determinant of $g_{\alpha\beta}$, and so

$$\Gamma^{\epsilon}_{\mu\epsilon}=\partial_\mu\ln\sqrt{-g}+2\tilde{a}_\mu . \tag{2.20}$$

Because of (2.16) or (2.19) not only $\underline{\Gamma}$ and $\underline{g}$ but also the Weyl potential $\underline{\tilde{a}}$ must be interpreted as a geometric field.

A four-dimensional manifold with a unique symmetric connection $\underline{\Gamma}$, an infinite set of tensor pairs $(\underline{g},\underline{\tilde{a}})$, $(\underline{g}',\underline{\tilde{a}}'),\ldots$ related by Eqs. (2.7) and (2.14), and a doubly covariant derivative $\underline{D}$ with (2.4)—(2.6) and (2.15) is called a *Weyl space* W^4.

To complete the description of a Weyl space, one can introduce *the curvature tensor* of the Weyl connection,

$$R^{\sigma}{}_{\rho\mu\nu}=\partial_\mu\Gamma^{\sigma}_{\rho\nu}-\partial_\nu\Gamma^{\sigma}_{\rho\mu}+\Gamma^{\epsilon}_{\rho\nu}\Gamma^{\sigma}_{\epsilon\mu}-\Gamma^{\epsilon}_{\rho\mu}\Gamma^{\sigma}_{\epsilon\nu} , \tag{2.21}$$

and the *field strength* of the Weyl potential,

$$f_{\mu\nu}=\partial_\mu\tilde{a}_\nu-\partial_\nu\tilde{a}_\mu . \tag{2.22}$$

Both tensors are of Weyl-type zero. Using the lemma (2.20) it can easily be shown that they are related according to

$$f_{\mu\nu}=\tfrac{1}{2}R^{\epsilon}{}_{\epsilon\mu\nu} . \tag{2.23}$$

This demonstrates even more than Eq. (2.19) that the Weyl potential $\tilde{a}_\mu$ is part of the geometry of the Weyl space and should therefore be interpreted as a gravitational field.

A Weyl space with a symmetric connection reduces to a *Riemann space* if and only if the field strength $f_{\mu\nu}$ vanishes. The necessary and sufficient condition for this is that $\tilde{a}_\mu$ is a gradient of a scalar. In this case there exists a gauge transformation (2.12) such that $\tilde{a}'_\mu$ vanishes identically. On the other hand, for vanishing curvature $R^\sigma{}_{\rho\mu\nu}$ we obtain because of (2.23) a *Minkowski space.*

As far as gauge structure of a Weyl space gravitational theory is concerned, the situation is similar to the Riemann case: the theory cannot be called a proper Yang-Mills gauge theory. One of the main differences is that the gauge potentials $\Gamma^\alpha_{\beta\gamma}$ are no primary objects. There is a metrical substructure and the "potentials" $\Gamma^\alpha_{\beta\gamma}$ can be derived according to (2.19) from the Weyl potentials $\tilde{a}_\mu$ and the prepotentials $g_{\alpha\beta}$.

C. Lorentzian structure

In order to be able to define spinor fields in Weyl space W^4 via representations of SL(2,C), we have to introduce the proper orthochronous Lorentz group $\mathscr{L}_+^\uparrow$ first. This can be done as usual by means of tetrad fields. The ratio of the magnitudes of vectors at the same point is invariant under Weyl transformations. The same is the case of the angle between two vectors according to the usual definition. We can therefore introduce in $T_x(\mathscr{M})$ a basis $\{\underline{e}_a\}$ of four orthogonal vectors of equal length. To make our conclusions as general as possible, the Weyl type $w(\underline{e})$ of this Lorentz basis remains unspecified throughout this paper. Without restrictions we may assume that for $\Lambda(x)=0$ the length of the four-vectors is ± 1:

$$g_{ab} = \underline{g}(\underline{e}_a, \underline{e}_b) = \exp\{[2w(\underline{e})+1]\Lambda(x)\}\eta_{ab} \quad (2.24)$$

with the constant matrix

$$\eta_{ab} = \text{diag}(-1,-1,-1,+1) \ . \quad (2.25)$$

Such a basis is called an *orthotetrad.* We demand that the manifold $\mathscr{M}$ be noncompact and that it admit a spinor structure. The latter is a global condition which is fulfilled if $\mathscr{M}$ carries a global field of orthotetrads.[14] For later use we introduce the dual basis $\underline{\tilde{e}}^a$ of $\tilde{T}_X(\mathscr{M})$ with $\langle \underline{\tilde{e}}^b, \underline{e}_a \rangle = \delta_a^b$ and $w(\underline{\tilde{e}}^a) = -w(\underline{e})$. The components of the connection $\underline{\Gamma}$ with regard to an orthotetrad are given by

$$\underline{D}\,\underline{e}_b = \Gamma^a{}_{b\lambda}\underline{\tilde{E}}^\lambda \otimes \underline{e}_a \ , \quad (2.26a)$$

$$\underline{D}\,\underline{\tilde{e}}^b = \Gamma^b{}_{a\lambda}\underline{\tilde{E}}^\lambda \otimes \underline{\tilde{e}}^a \quad (2.26b)$$

so that we have, for the components $T^a{}_b$ of a tensor $\underline{T}$ with regard to $\{\underline{e}_a\}$ and $\{\underline{\tilde{e}}^a\}$,

$$D_\mu T^a{}_b = \partial_\mu T^a{}_b + \Gamma^a{}_{m\mu} T^m{}_b - \Gamma^m{}_{b\mu} T^a{}_m + w(T^a{}_b)\tilde{a}_\mu T^a{}_b \ . \quad (2.27)$$

Because of (2.8) and (2.26) we find for the coordinate components e_a^μ of the orthotetrad ($\underline{e}_a = e_a^\mu \underline{E}_\mu$) the relation

$$D_\lambda e_a^\mu = \partial_\lambda e_a^\mu - \Gamma^b{}_{a\lambda} e_b^\mu + \Gamma^\mu_{\rho\lambda} e_a^\rho + w(\underline{e})\tilde{a}_\lambda e_a^\mu = 0 \ , \quad (2.28)$$

where the covariant derivative is taken with regard to all indices. It follows immediately that

$$\Gamma^b{}_{a\lambda} = [\partial_\lambda e_a^\mu + \Gamma^\mu_{\epsilon\lambda} e_a^\epsilon + w(\underline{e})\tilde{a}_\lambda e_a^\mu]\tilde{e}_\mu^b \ . \quad (2.29)$$

For the introduction of spinors it is important to realize that the change between different orthotetrads,

$$\underline{e}_a \to \underline{e}'_a = l_a{}^b \underline{e}_b \ , \quad (2.30)$$

is, because of (2.24), a *Lorentz transformation,*

$$l_a{}^m l_b{}^n \eta_{mn} = \eta_{ab} \ . \quad (2.31)$$

Introducing the reciprocal matrix $L^a{}_b$ with

$$l_a{}^b L^a{}_c = \delta_c^b, \ \ L^b{}_a l_c{}^a = \delta_a^b \ , \quad (2.32)$$

we obtain from (2.30) the inhomogeneous transformation behavior of the connection $\Gamma^a{}_{b\lambda}$ under position-dependent Lorentz transformations of the orthotetrads,

$$\Gamma^a{}_{b\lambda} \to \Gamma^{a\prime}{}_{b\lambda} = l_b{}^c L^a{}_m \Gamma^m_{c\lambda} + L^a{}_m \partial_\lambda l_b{}^m \ . \quad (2.33)$$

A consequence of this equation is that

$$\Gamma_{(ab)\lambda} = \eta_{e(a}\Gamma^e{}_{b)\lambda} \quad (2.34)$$

transforms with regard to the indices a and b as a tensor under Lorentz transformations.

III. TWO-SPINOR CALCULUS IN WEYL SPACE

A. Spinor algebra

It turns out that with regard to Weyl transformation the two-spinor calculus shows simpler structures than the four-spinor calculus. We therefore restrict ourselves to the first[15] and define at every point x a two-dimensional complex vector space $S_x(\mathscr{M})$ as well as the respective dual $\tilde{S}_x(\mathscr{M})$ and the complex conjugates $\bar{S}_x(\mathscr{M})$ and $\bar{\tilde{S}}_x(\mathscr{M})$. Bases are denoted by $\{\kappa_A\}$, $\{\tilde{\kappa}^A\}$, $\{\kappa_{\dot{A}}\}$, and $\{\tilde{\kappa}^{\dot{A}}\}$, respectively. Components $\xi^{A\cdots\dot{X}\cdots}_{M\cdots\dot{V}\cdots}$ of a spinor ξ are obtained as usual.

With regard to Weyl transformations it is assumed that correlated with the Weyl transformation of scalar and tensor fields, all spinor fields are simultaneously transformed according to

$$\underline{\xi} \to \underline{\xi}' = e^{w(\underline{\xi})\Lambda(x)}\underline{\xi} \ , \quad (3.1)$$

where the real number $w(\underline{\xi})$ denotes again the Weyl type of $\underline{\xi}(x)$.

We introduce an antisymmetric *spinor metric* γ. For $\lambda=0$ we choose the spinor basis to be normalized in the sense of

$$\gamma(\underline{\kappa}_A,\underline{\kappa}_B)=\gamma_{AB}=\exp\{[w(\underline{\gamma})+2w(\underline{\kappa}_A)]\Lambda(x)\}\epsilon_{AB} \quad (3.2)$$

with the constant matrix

$$\epsilon_{AB}=\epsilon^{AB}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (3.3)$$

The Weyl types $w(\underline{\gamma})$ and $w(\underline{\kappa}_A)=-w(\underline{\tilde{\kappa}}^A)$ of the spinor metric and spinor basis will be specified later. The conjugate metric $\check{\underline{\gamma}}$ is given by

$$\check{\gamma}^{AB}=e^{w(\underline{\gamma})\Lambda(x)}\epsilon^{AB} \quad (3.4)$$

with $w(\check{\underline{\gamma}})=-w(\underline{\gamma})$, and satisfies the relation

$$\check{\gamma}^{AC}\gamma_{BC}=\delta^A_B\ . \quad (3.5)$$

A change of the basis

$$\underline{\kappa}_A\rightarrow\underline{\kappa}'_A=l_A{}^B\underline{\kappa}_B \quad (3.6)$$

which preserves (3.3), fulfills

$$\epsilon_{AB}=l_A{}^M l_B{}^N\epsilon_{MN} \quad (3.7)$$

and is therefore a representation of SL(2,*C*). The matrix reciprocal to $l_A{}^B$ is denoted by $L^A{}_C$.

To summarize, we have the following definition of a spinor ξ in a Weyl space W^4: A *spinor* is a rule which assigns to each orthotetrad $\{\underline{e}_a\}$ at x an array of complex numbers $\xi^{A\cdots\dot{X}\cdots}_{M\cdots\dot{V}\cdots}$ which (i) transforms under a Lorentz transformation $\mathscr{L}^\uparrow_+$ of the orthotetrad according to

$$\xi^{A\cdots\dot{X}\cdots}_{M\cdots\dot{V}\cdots}\rightarrow\xi'^{A\cdots\dot{X}\cdots}_{M\cdots\dot{V}\cdots}=L^A{}_B\bar{L}^{\dot{X}}{}_{\dot{Y}}l_M{}^N\bar{l}_{\dot{V}}{}^{\dot{W}}\cdots\xi^{B\cdots\dot{Y}\cdots}_{N\cdots\dot{W}\cdots}\ , \quad (3.8)$$

where $l_A{}^B$ and $L^A{}_B$ are the corresponding elements of the twofold covering group SL(2,*C*) and which (ii) transforms under a general Weyl transformation according to

$$\xi^{A\cdots\dot{X}\cdots}_{M\cdots\dot{V}\cdots}\rightarrow\xi'^{A\cdots\dot{X}\cdots}_{M\cdots\dot{V}\cdots}=\exp[w(\xi^{A\cdots}_{M\cdots})\Lambda(x)]\xi^{A\cdots\dot{X}\cdots}_{M\cdots\dot{V}\cdots}\ . \quad (3.9)$$

B. Relation to tensors

The isomorphism which maps the spin space $S_x\otimes\bar{S}_x$ onto the vector space T_x can be introduced by

$$\underline{e}_a=\sigma_a{}^{A\dot{X}}\underline{\kappa}_A\otimes\underline{\kappa}_{\dot{X}} \quad (3.10a)$$

with $\sigma_a{}^{A\dot{X}}$ being fixed Hermitian matrices of Weyl-type zero: $w(\sigma_a{}^{A\dot{X}})=0$. The immediate consequence of (3.10a) is the fixation of the Weyl type of the spinor basis

$$w(\underline{\kappa}_A)=\tfrac{1}{2}w(\underline{e})\ . \quad (3.10b)$$

Corresponding to (3.10a) we have

$$\underline{\tilde{e}}^a=\eta^{ab}\epsilon_{AB}\epsilon_{\dot{X}\dot{Y}}\sigma_b{}^{A\dot{X}}\underline{\kappa}^B\otimes\underline{\kappa}^{\dot{Y}}\ . \quad (3.11)$$

A consequence of (3.10a) and (3.11) is the following condition for $\sigma_a{}^{A\dot{X}}$:

$$2\sigma_{(a}{}^{A\dot{X}}\sigma_{b)}{}^{B\dot{Y}}\epsilon_{\dot{X}\dot{Y}}\epsilon_{AC}=\eta_{ab}\delta^B_C\ , \quad (3.12)$$

which has the Pauli matrices[16] as a particular solution.

We complete the coupling between the spinors and tensors in a Weyl space W^4 in demanding, as in Riemann space, the equivalence of the two respective metrics in the sense of (3.10). Because of (3.12) this implies for the Weyl types

$$w(\underline{\gamma})=\tfrac{1}{2}w(\underline{g})=\tfrac{1}{2} \quad (3.13)$$

and therefore

$$2\sigma_{(a}{}^{A\dot{X}}\sigma_{b)}{}^{B\dot{Y}}\gamma_{\dot{X}\dot{Y}}\gamma_{AC}=g_{ab}\delta^B_c\ . \quad (3.14)$$

We note for the later use the consequence

$$\sigma_a{}^{A\dot{X}}\sigma^a{}_{B\dot{Y}}=\delta^A_B\delta^{\dot{X}}_{\dot{Y}}\ , \quad (3.15)$$

where we have introduced the matrix

$$\sigma^a{}_{B\dot{Y}}=\check{g}^{ad}\sigma_d{}^{D\dot{Z}}\gamma_{DB}\gamma_{\dot{Z}\dot{Y}} \quad (3.16)$$

which is as well of Weyl-type zero: $w(\sigma^a{}_{B\dot{Y}})=0$. With reference to the coordinate basis, we finally define

$$\sigma_\mu{}^{A\dot{X}}=e^a_\mu\sigma_a{}^{A\dot{X}}\ , \quad (3.17a)$$

$$w(\sigma_\mu{}^{A\dot{X}})=-w(\underline{e}) \quad (3.17b)$$

which obeys the lemma

$$\sigma_\mu{}^{A\dot{X}}\sigma_\nu{}^{B\dot{Y}}\gamma_{AB}\gamma_{\dot{X}\dot{Y}}=g_{\mu\nu}\ . \quad (3.18)$$

C. Spinor connection

We define a doubly covariant *Weyl derivative* $\underline{D}$ for spinors. The components of the respective *Weyl spinor connection* $\underline{\Gamma}$ are given by

$$\underline{D}\,\underline{\kappa}_A=\Gamma^B{}_{A\lambda}\underline{\tilde{E}}^\lambda\otimes\underline{\kappa}_B\ . \quad (3.19)$$

For the spinor derivative $\underline{D}$ we require, apart from linearity, reality, and Leibnitz's rule, that (i) it reduces for tensors and scalars to the doubly covariant derivative defined in Sec. II, and that (ii) it commutes with $\underline{\gamma}$,

$$\underline{D}\,\gamma=0\ , \qquad (3.20)$$

and $\sigma_a^{A\dot{X}}$ in the sense of

$$D_\lambda \sigma_a^{A\dot{X}} = -\Gamma^b{}_{a\lambda}\sigma_b{}^{A\dot{X}} + \Gamma^A{}_{B\lambda}\sigma_a{}^{B\dot{X}} + \Gamma^{\dot{X}}{}_{\dot{Y}\lambda}\sigma_a{}^{A\dot{Y}} = 0\ . \qquad (3.21)$$

Components of the derivatives are introduced as usual. Taking into account the derivative $\underline{D}f$ of scalars as it can be read off from (2.5), we find with regard to a coordinate basis, for example,

$$D_\lambda \xi^A = \partial_\lambda \xi^A + \Gamma^A{}_{B\lambda}\xi^B + w(\xi^A)\tilde{a}_\lambda \xi^A\ . \qquad (3.22)$$

The Weyl type remains unaltered. With (2.28) and (3.17) we have because of (3.21) also in Weyl space

$$D_\epsilon \sigma_\mu^{A\dot{X}} = 0\ . \qquad (3.23)$$

It can be seen from (3.19) that under a spinor transformation the Weyl spinor connection transforms according to

$$\Gamma^A{}_{B\lambda} \to \Gamma'^A{}_{B\lambda} = l_B{}^C L^A{}_D \Gamma^D{}_{C\lambda} + L^A{}_C \partial_\lambda l_B{}^C\ . \qquad (3.24)$$

Below we will make use of the fact that because of the properties of $L^A{}_C$ and $l_B{}^C$, the trace $\Gamma^A{}_{A\lambda}$ transforms as a scalar under spin transformations. It is therefore a covariant vector.

Equation (2.29) shows that $\Gamma^a{}_{b\lambda}$ can be expressed as a function of the orthotetrad field $e_a^\mu(x)$ and the Weyl potential. For practical purposes it may, therefore, be useful to have in addition $\Gamma^A{}_{B\lambda}$ as a function of $\Gamma^a{}_{b\lambda}$. The relation (3.10) causes such a dependence. Taking the derivative, we obtain

$$\Gamma^b{}_{a\lambda}\sigma_b{}^{A\dot{X}} = \sigma_a{}^{B\dot{X}}\Gamma^A{}_{B\lambda} + \sigma_a{}^{A\dot{Y}}\Gamma^{\dot{X}}{}_{\dot{Y}\lambda}\ . \qquad (3.25)$$

Contracting with $\sigma^a{}_{C\dot{Z}}$, using the lemma (3.15), and contracting different pairs of spinor indices, we find

$$\Gamma^A{}_{B\lambda} = \tfrac{1}{2}\sigma_a^{A\dot{X}}\sigma^b{}_{B\dot{X}}\Gamma^a{}_{b\lambda} + \delta_B^A[-\tfrac{1}{8}\Gamma^d{}_{d\lambda} + \tfrac{1}{2}i\,\mathrm{Im}(\Gamma^A{}_{A\lambda})]\ , \qquad (3.26)$$

and with (3.14),

$$\mathrm{Re}(\Gamma^A{}_{A\lambda}) = \tfrac{1}{4}\Gamma^d{}_{d\lambda}\ . \qquad (3.27)$$

The vector $\mathrm{Im}(\Gamma^A{}_{A\lambda})$ remains undetermined by the orthotetrad field. We put it equal to zero to have the correspondence to the usual convention in Riemann space.

IV. STOKES'S THEOREM

In order to perform integrations we introduce quantities which are tensor densities with regard to coordinate transformation. A *Weyl tensor density* $\hat{\underline{T}}$ is characterized by its Weyl type $w(\hat{\underline{T}})$ and its density weight $d(\hat{\underline{T}})$, where $d(\hat{\underline{T}})$ is an integer. To include tensor densities, the Weyl derivative is generalized in the usual way by an additional term [compare (2.11)]:

$$D_\mu \hat{T}^{\cdots}_{\cdots} = \partial_\mu \hat{T}^{\cdots}_{\cdots} + \Gamma_\mu{}^{\cdot}{}_{\cdot}\hat{T}^{\cdots}_{\cdots} + \cdots + w(\hat{\underline{T}})\tilde{a}_\mu \hat{T}^{\cdots}_{\cdots} - d(\hat{T})\Gamma_\mu \hat{T}^{\cdots}_{\cdots} \qquad (4.1)$$

with

$$\Gamma_\mu = \Gamma^\epsilon_{\mu\epsilon}\ . \qquad (4.2)$$

A typical density with a nontrivial behavior under Weyl transformations is $\sqrt{-g}$,

$$w(\sqrt{-g}\,) = 2,\quad d(\sqrt{-g}\,) = 1\ . \qquad (4.3)$$

With (2.20) we have

$$D_\mu \sqrt{-g} = 0\ . \qquad (4.4)$$

The divergence of a vector density $\hat{\underline{S}}$ [i.e., $d(\hat{\underline{S}})=1$] of Weyl-type zero [i.e., $w(\hat{\underline{S}})=0$] reduces in Weyl W^4 as in Riemann space to

$$D_\mu \hat{S}^\mu = \partial_\mu \hat{S}^\mu\ . \qquad (4.5)$$

Applying this to $\sqrt{-g}\,U^\mu$ where U^μ is of the type $w(U^\mu)=-2$ and $d(U^\mu)=0$, we find that (4.4)

$$D_\mu U^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,U^\mu)\ . \qquad (4.6)$$

For a manifold with metric, Stokes's theorem can be given a form which is very close to physical applications:

$$\int_0 A^\mu n^\nu g_{\mu\nu} d^3V = \int_v \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}A^\mu) d^4V\ . \qquad (4.7)$$

d^3V and d^4V are thereby the scalar volume elements built as usual with the help of the line element $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. They are of the types $w(d^3V)=\frac{3}{2}$, $d(d^3V)=0$ and $w(d^4V)=2$, $d(d^4V)=0$. n^ν is the vector orthogonal to the three-surface 0 with

$$n^\nu = \frac{dx^\nu}{ds}\ . \qquad (4.8)$$

It is of Weyl type $w(n^\nu) = -\frac{1}{2}$ and invariantly normalized to unity,

$$n^\mu n^\nu g_{\mu\nu} = \pm 1\ . \qquad (4.9)$$

Integration can only be performed over integrands of Weyl-type zero. The consequence is that Stokes's theorem in the form (4.7) can only be applied to vectors A^μ of type

$$w(A^\mu) = -2,\quad d(A^\mu) = 0\ . \qquad (4.10)$$

Using the lemma (4.6) we then finally obtain from (4.7) the following handy form of *Stokes's theorem in Weyl space:*

$$\int_0 A^\mu n^\nu g_{\mu\nu} d^3V = \int_v D_\mu A^\mu d_4 V , \quad (4.11)$$

where A^μ has to fulfill the condition (4.10).

A consequence of Stokes's theorem (4.11) is that also in Weyl space the existence of a conservation law for a quantity Q is related to a current J^μ of type (4.10) which is divergence-free according to

$$D_\mu J^\mu = 0 . \quad (4.12)$$

In this case, the physical interpretation of the left side of (4.11) which must be invariant under Weyl transformations is the following: $J^\mu n^\nu g_{\mu\nu} d^3V$ is the amount of Q as measured by an observer with tangent vector n^ν in the orthogonal rest space to which d^3V is attributed.

V. DIRAC THEORY IN WEYL SPACE

The scalar Lagrangian L of a field theory in Weyl space with $d(L)=0$ must be of Weyl type

$$w(L) = -2 . \quad (5.1)$$

The scalar density $\hat{L} = \sqrt{-g}L$ is accordingly $d(\hat{L})=1$, $w(\hat{L})=0$. The Lagrangian L of the Dirac theory must be covariantly formulated with regard to coordinate transformations, Weyl transformations, and spinor transformations coupled to the Lorentz transformation of the orthotetrad. We take the two-spinor fields χ^A and ϕ_A as the fundamental fields of Dirac theory and introduce the *convention* that, in the following indices of spinors or tensors will be moved in the usual way with $g_{\alpha\beta}$, $\check{g}^{\alpha\beta}$, γ_{AB}, and $\check{\gamma}^{AB}$. Because we want to describe massive Dirac particles and not neutrinos, a mass term must be included. A real Lagrangian density $\hat{L}$ for the Dirac fields which fulfills all these demands and reduces to the well-known one in the special case of a Riemann space is obtained by minimal coupling (velocity of light $c=1$):

$$\hat{L} = \frac{i\hbar}{\sqrt{2}}\sqrt{-g}\,(\chi^{\dot{X}}\sigma^\alpha{}_{A\dot{X}}D_\alpha\chi^A - \chi^A\sigma^\alpha{}_{A\dot{X}}D_\alpha\chi^{\dot{X}} - \phi_{\dot{X}}\sigma^{\alpha A\dot{X}}D_\alpha\phi_A + \phi_A\sigma^{\alpha A\dot{X}}D_\alpha\phi_{\dot{X}}) - m(\phi_A\chi^A + \varphi_{\dot{X}}\chi^{\dot{X}}) . \quad (5.2)$$

By (5.1) the Weyl types of the spinor fields are uniquely fixed,

$$w(\chi^{\dot{A}}) = -1 - \tfrac{1}{2}w(\underline{e}) , \qquad w(\phi_A) = -\tfrac{1}{2} + \tfrac{1}{2}w(\underline{e}) . \quad (5.3)$$

Another necessary consequence is that *independent of the choice of $w(\underline{e})$ the mass term must be of the nontrivial Weyl type*

$$w(m) = -\tfrac{1}{2} . \quad (5.4)$$

The corresponding space-time behavior of m results from the Weyl transformation factor $e^{-\Lambda(x)/2}$ only. It is the field $m=\text{const}$ which is transformed this way. m in Weyl space is therefore not a new physical field for which field equations would be necessary, it is a parameterlike quantity. The quotient of two such parameters m_1 and m_2 remains constant. Because the measurement of a mass uses in fact a reference quantity (e.g., a reference mass), m may indeed keep its physical meaning, if a local mass measurement is possible in Weyl space.

To clarify additionally the concept of mass in Weyl space, it must be remembered that already in Riemann space, mass cannot be introduced as an eigenvalue of the Casimir operator which is related to the translations of the Poincaré group. Accordingly, in Weyl space, nothing similar can be expected. The reason for this is that in going to curved space-time, the Poincaré group of Minkowski space is not enlarged but restricted. The translations are lost, while the remaining Lorentz symmetry is localized. The latter still permits the usual classification of fields with regard to their spin values. The mass parameter on the other hand can only be introduced in the following way: To obtain the Lagrangian of quantum fields in curved space-time one transcribes the Minkowski-space Lagrangian according to minimal coupling (with regard to the quantum-mechanical interpretation this is the transcription of the spin-position representation in the Schrödinger picture). This results in the Lagrangian (5.2).

Inserting in (5.2) the different terms of the Weyl derivatives according to (3.22), it can be seen that the real Lagrangian density $\hat{L}$ does not contain the Weyl potential $\tilde{a}_\mu$ explicitly due to the "symmetrization" of the kinetic term. Nevertheless, $\tilde{a}_\mu$ reappears explicitly in the Euler-Lagrange equations—e.g.,

$$\frac{\partial\hat{L}}{\partial\chi^A} - \partial_\alpha\left[\frac{\hat{L}}{\partial(\partial_\alpha\chi^A)}\right] = 0$$

(Ref. 17) [see the Appendix for $w(\underline{e}) = -\frac{1}{2}$].

To work out these field equations in detail, one makes use of (2.20) and (2.23) or corresponding relations like $D_\alpha\sigma^\mu{}_{A\dot{X}}=0$. For all values of $w(\underline{e})$ the result is the *Dirac equations*

$$i\sigma^\alpha{}_{A\dot{X}}D_\alpha\chi^A - \frac{m}{\sqrt{2}\hbar}\phi_{\dot{X}} = 0 , \quad (5.5a)$$

$$i\sigma^{\alpha A\dot{X}}D_\alpha\phi_A + \frac{m}{\sqrt{2}\hbar}\chi^{\dot{X}} = 0 . \quad (5.5b)$$

To relate with the Dirac field at least rudiments of a quantum-mechanical interpretation, we need more than a Lagrangian and field equations. Quantities with a physical meaning can be obtained from the vector

$$j^\alpha=\sqrt{2}\sigma^{\alpha A\dot{X}}(\phi_A\phi_{\dot{X}}+\chi_A\chi_{\dot{X}}) . \tag{5.6}$$

It does not contain $\tilde{a}_\mu$ and is independent of $w(\underset{\sim}{e})$. Its type is

$$w(j^\alpha)=-2,\ \ d(j^\alpha)=0 . \tag{5.7}$$

Because of the field equation (5.5) j^α is divergence-free in the sense of

$$D_\alpha j^\alpha=0 . \tag{5.8}$$

According to (4.11) this is related to a global conservation law, which in this case may be interpreted as the conservation of charge or probability. j^α is the physical *Dirac four-current.*

For later use we note a decomposition of the current j^α which can be obtained from (5.6) using some two-spinor algebra:

$$\begin{aligned}j^\alpha=&\frac{\hbar}{2mi}\breve{g}^{\alpha\beta}(\phi_{\dot{X}}D_\beta\chi^{\dot{X}}-\chi^{\dot{X}}D_\beta\phi_{\dot{X}}-\phi_AD_\beta\chi^A\\&+\chi^AD_\beta\phi_A)\\&+\frac{\hbar i}{m}D_\beta(\sigma^{\alpha A(\dot{X}}\sigma^\beta{}_A{}^{\dot{Y})}\chi_{\dot{X}}\phi_{\dot{Y}}\\&-\sigma^{\alpha\dot{X}(A}\sigma^\beta{}_{\dot{X}}{}^{B)}\phi_A\chi_B) .\end{aligned} \tag{5.9}$$

VI. CLASSICAL LIMIT

We obtain the limit of classical particle paths by means of the first step of a *WKB approximation.* The respective ansatz is

$$\chi^A=e^{iS(x)/\hbar}\sum_{n=0}^{\infty}(-i\hbar)^n\chi_n^A(x) , \tag{6.1a}$$

$$\phi_{\dot{A}}=e^{iS(x)/\hbar}\sum_{n=0}^{\infty}(-i\hbar)^n\phi_{n\dot{A}}(x) , \tag{6.1b}$$

where $S(x)$ is of Weyl-type zero,

$$w(S)=0 . \tag{6.2}$$

Inserting (6.1) into the Dirac equations and equating the lowest order of $\hbar$ to zero, we obtain equations which no longer contain the Weyl potential $\underset{\sim}{\tilde{a}}$:

$$\sigma^\alpha{}_{A\dot{X}}\chi_0^A\partial_\alpha S+\frac{m}{\sqrt{2}}\phi_{0\dot{X}}=0 , \tag{6.3a}$$

$$\sigma^{\alpha A\dot{X}}\phi_{0A}\partial_\alpha S+\frac{m}{\sqrt{2}}\chi_0^{\dot{X}}=0 . \tag{6.3b}$$

Combination of these two equations leads—with (3.13)—to the *Hamilton-Jacobi* equation

$$(\partial_\alpha S)(\partial_\beta S)\breve{g}^{\alpha\beta}=m^2 , \tag{6.4}$$

which reflects again $w(m^2)=w(\breve{g}^{\alpha\beta})$. Taking the derivative of (6.4) yields

$$2D_\alpha(\partial_\epsilon S)(\partial_\beta S)\breve{g}^{\alpha\beta}=D_\epsilon(m^2) , \tag{6.5}$$

where we have used (2.18) and

$$D_{[\epsilon}\partial_{\alpha]}S=0 \tag{6.6}$$

which is a consequence of (6.2) and (2.9).

Using (5.9), the four-current j^μ takes in the WKB limit the form

$$j_\mu=j_{0\mu}+O(\hbar)=-\frac{B_0}{m}\partial_\mu S+O(\hbar) \tag{6.7}$$

with

$$B_0=\phi_{0A}\chi_0^A+\phi_{0\dot{X}}\chi_0^{\dot{X}},\ \ w(B_0)=-\tfrac{3}{2} . \tag{6.8}$$

Because of

$$j_{0\mu}j_0^\mu=B_0{}^2 \tag{6.9}$$

the four-current j_0^μ is a timelike vector.

For the directional Weyl derivative of $j_{0\mu}$ we obtain, with (6.5), (6.7), and (6.9),

$$\begin{aligned}(D_\epsilon j_{0\mu})j_0^\epsilon=&D_\epsilon[\ln m^{-1}(j_{0\kappa}j_0^\kappa)^{1/2}]j_0^\epsilon j_{0\mu}\\&+j_{0\kappa}j^{0\kappa}D_\mu(\ln m) .\end{aligned} \tag{6.10}$$

Note that the connection $\Gamma^\alpha_{\beta\gamma}$ is contained in the left side of Eq. (6.10) according to

$$(D_\epsilon j_{0\mu})j_0^\epsilon=(\partial_\epsilon j_{0\mu})j_0^\epsilon-\Gamma^\alpha_{\mu\epsilon}j_{0\alpha}j_0^\epsilon-\tilde{a}_\epsilon j_0^\epsilon j_{0\mu} . \tag{6.11}$$

In a physical situation where a WKB limit (χ_0^A,ϕ_{0A}) defined by the neglect of the terms $O(\hbar)$ is an exact solution of the Dirac equation (or an appropriately good approximation), the corresponding four-current field j_0^μ defines as a tangent vector field a congruence of timelike world lines. They will be called *streamlines* $\mathscr{J}_0$. According to (6.10) and (6.11) parallel propagation of j_0^α by means of $\Gamma^\alpha_{\beta\gamma}$ along a streamline does not result in a vector which is still tangent to the streamline. This leads to the following result: *In a Weyl space W^4, the timelike streamlines $\mathscr{J}_0$ of the extreme classical limit of the Dirac theory (no spin effects) are not geodesic, i.e., they are not autoparallels. Accordingly, the quantum-mechanically defined particle trajectories on one hand and the free-fall trajectories $\mathscr{P}_f$ of classically defined structureless test particles on the other do not agree. This result is independent of the choice of the Weyl type $w(\underset{\sim}{e})$ of the orthotetrad (Lorentz basis).*

On the other hand, the *requirement* that the particle trajectories obtained from quantum mechanics in the classical limit should agree with the geodesic trajectories of freely falling, spherical, nonrotating neutral test particles—introduced as primitive concepts of a space-time theory of gravitation—results with (6.10) in

$$D_\mu m = 0 \ . \tag{6.12}$$

This implies

$$\tilde{a}_\mu = \partial_\mu (\ln m^2) \tag{6.13}$$

so that

$$f_{\mu\nu} = 0 \ . \tag{6.14}$$

The consequence of the requirement is therefore that the Weyl space reduces to a Riemann space and the gap described in Sec. I is closed.[18] Note that the conclusions in this section are essentially based on the nontrivial Weyl type of the mass m (and not on the Weyl potential $\tilde{a}$).

VII. KLEIN-GORDON THEORY IN WEYL SPACE

It is well known that the theory of unquantized Klein-Gordon fields does not allow a probabilistic single-particle interpretation. In Minkowski space, there exists a continuity equation for the four-current but the corresponding density is not positive definite. Anticipating second quantization, it may be interpreted as the charge density of an assembly of positively charged particles and negatively charged antiparticles. It is therefore doubtful from the physical point of view to base a discussion of the particle path in Weyl space (obtained as classical limit from quantum mechanics) on *Klein-Gordon theory.* This is the reason why our discussion of self-consistency and the related closing of the gap has been based on Dirac theory.

On the other hand, if one, nevertheless, rests content with classical Klein-Gordon theory and demands that at least the property of a conserved physical current should be found in Weyl space, Klein-Gordon theory offers a new approach.

The Lagrangian L with $w(L) = -2$ and $d(L) = 0$ for the complex massive scalar fields $\Phi(x)$ is

$$L = \hbar^2 (D_\mu \Phi^*)(D_\nu \Phi) \breve{g}^{\mu\nu} - m^2 \Phi^* \Phi - \lambda \hbar R \Phi^* \Phi \ , \tag{7.1}$$

where R is an appropriate curvature quantity and λ is a real number. A Maxwell potential is neglected but could easily be included. To obtain the values of $w(L)$ and $d(L)$ above, we must again have

$$w(m) = -\tfrac{1}{2} \tag{7.2}$$

as in (5.4) and

$$w(\Phi) = -\tfrac{1}{2}, \quad w(R) = -1 \tag{7.3}$$

as well as $d(m) = d(\Phi) = d(R) = 0$. Possible candidates for R are among others the contractions $R^{\sigma}{}_{\sigma}{}^{\rho}{}_{\rho}$ and $R^{\tau\rho}{}_{\tau\rho}$ of $R^{\tau}{}_{\rho\mu\nu}$ from (2.21). The Klein-Gordon equation is then

$$(\hbar^2 \breve{g}^{\mu\nu} D_\mu D_\nu + m^2 + \lambda \hbar^2 R)\Phi = 0 \ . \tag{7.4}$$

The four-vector

$$j^\mu = \frac{i\hbar}{2} \breve{g}^{\mu\nu} [\Phi^* D_\nu \Phi - (D_\nu \Phi^*)\Phi] \tag{7.5}$$

is of the type $w(j^\mu) = -2$, $d(j^\mu) = 0$ [compare (5.7)]. It is divergence-free

$$D_\mu j^\mu = 0 \ . \tag{7.6}$$

But from the physical point of view, j^μ has the disadvantage that already for dimensional reasons it cannot be taken as a probability current. A possible candidate for this, which shows as well the Schrödinger limit, is instead

$$j^\mu_P = \frac{1}{m} j^\mu \ . \tag{7.7}$$

If one now insists on giving the classical Klein-Gordon field the usual physical interpretation, one has to demand a continuity equation for j^μ_P. The consequence of this *requirement* would be

$$D_\mu m = 0 \tag{7.8}$$

[compare (6.12)] and therefore *the reduction of the Weyl space to the Riemann space.* Note, that this is a way to reach the Riemann space without referring to the classical particle path. *It can be shown on very general physical grounds already that there can be no physically reasonable classical Klein-Gordon theory in the generic Weyl space W^4.*

In order to complete our discussion and to permit a comparison with Dirac theory, we add a WKB treatment of the two currents j^μ and j^μ_p. The WKB expansion

$$\Phi = e^{iS/\hbar} \sum_{n=0}^{\infty} (-i\hbar)^n \Phi_n \tag{7.9}$$

implies with (7.4) to the lowest order in $\hbar$ the Hamilton-Jacobi equation

$$(\partial_\mu S)(\partial_\nu S) \breve{g}^{\mu\nu} = m^2 \ , \tag{7.10}$$

as in (6.4), and

$$j_\mu = -(\partial_\mu S)\Phi_0^* \Phi_0 + O(\hbar^2) \ . \tag{7.11}$$

As far as the geodesic behavior of j_μ and $j_{P\mu}$ is concerned, we obtain, as a consequence of (7.7), (7.10), and (7.11),

$$(D_\epsilon j_\mu)j^\epsilon = D_\epsilon(\ln\Phi^*\Phi)j^\epsilon j_\mu + \tfrac{1}{2}(\Phi^*\Phi)^2 D_\mu(m^2) + O(\hbar) \quad (7.12)$$

and

$$(D_\epsilon j_{P\mu})j_P^\epsilon = D_\epsilon\left[\ln\frac{\Phi^*\Phi}{m}\right]j_P^\epsilon j_{P\mu} + (\Phi^*\Phi)^2 D_\mu(\ln m) + O(\hbar)\ , \quad (7.13)$$

where the latter shows a close analogy to (6.10). In both cases Eq. (7.8) is again the necessary and sufficient condition for a streamline to be geodesic.

VIII. CONCLUSION

The space-time structure of general relativity cannot fully be explored within classical physics using light rays and freely falling structureless test particles. To prove empirically the existence of torsion in Riemann-Cartan space, one already has to make use of the quantum-mechanical spin.[19] An axiomatic scheme based on the two primitive concepts above ends up with a Weyl space and remains incomplete. Only by including quantum-mechanically described matter can one handle torsion axiomatically and make the final step from Weyl to Riemann (or Riemann-Cartan) space. The latter has been shown above.

The following *general structure* can be read off from our calculations, where we have discussed some rudiments of quantum mechanics in first quantization for spin 0 and spin $\frac{1}{2}$ in Weyl space: Whenever the quantum-mechanical field equations containing a mass parameter m allow a WKB limit with

$$(\partial_\alpha S)(\partial_\beta S)\check{g}^{\alpha\beta} = m^2 \quad (8.1)$$

(i.e., whenever they reflect the local energy law), we have

$$w(m) = -\tfrac{1}{2}w(\underline{g}) = -\tfrac{1}{2}\ . \quad (8.2)$$

If in addition $\partial_\alpha S$ is related in a simple way by the tangent vector to the classical particle paths as obtained from the four-current in the WKB limit (this point is to be discussed in detail for every quantum field), then quantum mechanics will contain classical mechanics as a limiting case if and only if the Weyl space is reduced to the Riemann space.

General relativity and quantum theory are paradigms in Kuhn's sense.[20] Our discussion has shown that—already, in a very preliminary stage—the interference between these two paradigms proves to be very useful for a deeper understanding of both of them.

APPENDIX: THE STANDARD CASE

To discuss spin-$\frac{1}{2}$ fields, tetrads are usually taken as the primary fields and the metric is introduced as a derived concept. Coupling according to

$$\underline{g} = \underline{\tilde{e}}^a \otimes \underline{\tilde{e}}^b g_{ab}\ ,\quad w(\underline{g}) = 1 \quad (A1)$$

and demanding that the coupling be mediated by a quantity of trivial Weyl type

$$w(g_{ab}) = 0\ , \quad (A2a)$$

$$g_{ab} = \eta_{ab}\ , \quad (A2b)$$

we obtain

$$w(\underline{e}_a) = -\tfrac{1}{2}\ . \quad (A3)$$

This particular choice is called the *standard case.*

An independent second approach to the standard case (A3) is the demand that the two two-spinors ϕ_A and $\chi^{\dot{A}}$ which combine to the Dirac four-spinor

$$\Psi = \begin{bmatrix}\phi_A\\ \chi^{\dot{A}}\end{bmatrix} \quad (A4)$$

be of equal Weyl type

$$w(\phi_A) = w(\chi^{\dot{A}})\ , \quad (A5)$$

so that it is possible to attribute a unique Weyl type to Ψ. In this case (A3) is an immediate consequence of (5.3).

There is a third possibility of justifying the restriction to (A3). In the general-relativistic theory of measurement in Riemann space, tetrads represent the local Lorentz observer and the respective measured values are obtained by projecting tensors on the related tetrad. If according to (A3) the tetrad components may be written as

$$e_a^\mu = \frac{dx^\mu}{ds_a}\ , \quad (A6)$$

the contraction with tetrad vectors represents the transformation in a local Lorentz system, in which the measured distances are still subject to Weyl transformations (which are in this case conformal transformations). The vector n^μ of (4.8) can be interpreted as a particular tetrad vector. Compare the corresponding physical interpretation of the four-current in Sec. IV.

Assuming (A3) we have

$$w(g_{ab}) = 0,\quad g_{ab} = \eta_{ab} \quad (A7)$$

which imples, with $D_\lambda g_{ab} = 0$,

$$\Gamma_{(ab)\lambda} = 0\ . \quad (A8)$$

Using this to simplify (3.26) we obtain, with (2.29) and (2.19),

$$\sigma^{\lambda}{}_{A\dot{X}}\Gamma^{A}{}_{B\lambda}=\sigma^{\lambda}{}_{A\dot{X}}\Gamma^{\{\}A}{}_{B\lambda}+\tfrac{3}{4}\tilde{a}_{\lambda}\sigma^{\lambda}{}_{B\dot{X}}\ , \qquad \text{(A9)}$$

where $\Gamma^{\{\}A}{}_{B\lambda}$ denotes the two-spinor connection with the Weyl connection $\Gamma^{\alpha}_{\beta\gamma}$ replaced by the Christoffel connection $\{{}^{\alpha}_{\beta\gamma}\}$. Taking into account that for the choice (A3) the two-spinors $\chi^{\dot{A}}$ and ϕ_A are of Weyl type $-\frac{3}{4}$, we find that the differential operator part of the Dirac equation can be written with the help of the Christoffel connection only:

$$\sigma^{\alpha A\dot{X}}D_{\alpha}\phi_A=\sigma^{\alpha AX}(\partial_{\alpha}\phi_A-\Gamma^{\{\}B}{}_{A\alpha}\phi_B)\ , \qquad \text{(A10a)}$$

$$\sigma^{\alpha}{}_{\dot{A}X}D_{\alpha}\chi^{\dot{A}}=\sigma^{\alpha}{}_{\dot{A}X}(\partial_{\alpha}\chi^{\dot{A}}+\Gamma^{\{\}\dot{A}}{}_{\dot{C}\alpha}\chi^{\dot{C}})\ . \qquad \text{(A10b)}$$

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[1]K. S. Thorne, D. L. Lee, and A. P. Lightman, Phys. Rev. D 7, 3563 (1973).

[2]J. L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1956); *Relativity: The General Theory* (North-Holland, Amsterdam, 1964).

[3]J. Ehlers, F. A. E. Pirani, and A. Schild, in *General Relativity*, edited by L. O'Raifeartaigh (Oxford University, Oxford, 1972).

[4]J. Audretsch and G. Schäfer, Gen. Relativ. Gravit. 9, 243 (1978); 9, 489 (1978); L. Parker, Phys. Rev. D 22, 1922 (1980); E. Fischbach, B. S. Freeman, and W.-K. Cheng, Phys. Rev. D 23, 2157 (1981).

[5]Apart from Ref. 3, see also M. Enosh and A. Kovetz, Ann. Phys. (N.Y.) 69, 279 (1971); M. A. Castagnino, J. Math. Phys. 12, 2203 (1971); M. Enosh and A. Kovetz, *ibid.* 14, 572 (1973).

[6]H. Weyl, *Raum, Zeit, Materie,* 5th ed. (Springer, Berlin, 1923); J. A. Schouten, *Ricci Calculus,* 2nd ed. (Springer, Berlin, 1954).

[7]In the axiomatic approaches sketched above, torsion is usually excluded by postulate without any reference to physical arguments. It is in fact an additional disadvantage of an axiomatic scheme based on classical test matter only that torsion cannot be handled (compare Sec. VIII).

[8]We use the following conventions: $\alpha,\beta,\ldots=1,\ldots,4$ are tensor indices; $a,b,\ldots=1,\ldots,4$ are tetrad indices; $A,\ldots,\dot{A},\ldots,=1,2$ are two-spinor indices. The signature of the metric tensors is $e^{\Lambda(x)}g_{\alpha\beta}=(---+)$. The partial derivative is denoted by ∂_{α}. Symmetrization is denoted by $A_{(\alpha\beta)}=\frac{1}{2}(A_{\alpha\beta}+A_{\beta\alpha})$. Antisymmetrization is denoted by $A_{[\alpha\beta]}=\frac{1}{2}(A_{\alpha\beta}-A_{\beta\alpha})$.

[9]It is important to distinguish clearly the different contexts in which conformal transformations appear. See for this and for further literature T. Fulton, F. Rohrlich, and L. Witten, Rev. Mod. Phys. 34, 442 (1962); H. A. Kastrup, Ann. Phys. (Leipzig) 9, 388 (1967). The different topics are (i) discussion of Weyl space as an extension of Riemann space, (ii) discussion of conformally related Riemann spaces (one-to-one mappings between Riemann spaces with related change of connection), (iii) discussion of the 15-parameter conformal group replacing the Poincaré group in Minkowski space [see, e.g., S. Ferrara, R. Gatto, and A. F. Grillo, in *Springer Tracts in Modern Physics 67,* edited by G. Höhler (Springer, Berlin, 1973) for a survey], (iv) conformal supergravity as a gauge theory of the superconformal algebra generalizing Poincaré supergravity [see, e.g., M. Kaku, P. K. Townsend, and P. van Nieuwenhuizen, Phys. Rev. D 17, 3179 (1978)]. We are concerned with point (i) only.

[10]See M. A. Castagnino of Ref. 5.

[11]K. Hayashi and T. Shirafuji, Prog. Theor. Phys. 57, 302 (1977), have made the observation that in the real Lagrangian of a scale-invariant Dirac theory the corresponding gauge potential does not couple minimally. Based on this they have concluded that, accordingly, (i) the corresponding gauge-covariant Dirac equations do not contain the gauge potential, and (ii) that therefore the gauge field is devoid of physical significance for these particles and fields and can be completely disregarded (i.e., put globally equal to zero). This justifies—according to these authors—the step from Weyl space to Riemann space. These arguments are wrong already for the following reasons. In K. Hayashi and T. Kugo, Prog. Theor. Phys. 61, 334 (1979), the authors stress the fact that a scalar matter field does indeed couple to the Weyl gauge field, so that the conclusion (ii) above, that the gauge field is devoid of physical significance, cannot be drawn and it remains an open question how the specialization to Riemann space can be justified. In addition, conclusion (i) is valid only if $w(\underline{e})=-\frac{1}{2}$, as will be discussed in more detail in Sec. V and Ref. 17.

[12]L. Koester, Phys. Rev. D 14, 907 (1976). Earlier experiments with neutrons are reported by A. W. McReynolds, Phys. Rev. 83, 172 (1951); 83, 233 (1951); J. W. T. Dabbs *et al.*, *ibid.* 139, B756 (1965); L. Koester, Z. Phys. 198, 187 (1967).

[13]J. Audretsch, Phys. Rev. D 24, 1470 (1981); 25, 605 (1982).

[14]R. Geroch, J. Math. Phys. 9, 1739 (1968). Compare, in addition, R. Geroch, J. Math. Phys. 11, 343 (1970); K. Bichteler, *ibid.* 9, 813 (1968).

[15]We assume the reader to be familiar with two-spinor calculus. For a review see F. A. E. Pirani, in *Lectures on General Relativity* edited by A. Trautman, F. A. E. Pirani, and H. Bondi (Prentice-Hall, Englewood Cliffs, 1965).

[16]

$$\sigma_1^{A\dot{X}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2^{A\dot{X}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$\sigma_3^{A\dot{X}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_4^{A\dot{X}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

[17]That the Dirac equations (5.5) are the correct Euler-Lagrange equations can as well be directly demonstrated in starting from the complex Lagrangian density

$$\hat{L}_2 = \sqrt{-g}\,[i\hbar\sqrt{2}(\chi^{\dot{X}}\sigma^{\alpha}{}_{A\dot{X}}D_\alpha\chi^A - \phi_{\dot{X}}\sigma^{\alpha A\dot{X}}D_\alpha\phi_A) - m(\phi_A\chi^A + \varphi_{\dot{X}}\chi^{\dot{X}})]$$

which differs from $\hat{L}$ of (5.2) by the total divergence

$$\frac{i\hbar}{\sqrt{2}}\partial_\alpha(\sqrt{-g}\,\chi^{\dot{X}}\sigma^{\alpha}{}_{A\dot{X}}\chi^A - \sqrt{-g}\,\phi_{\dot{X}}\sigma^{\alpha A\dot{X}}\phi_A)$$

[use (4.6) to show this], and which therefore leads to the same field equations. $\hat{L}_2$ contains $\tilde{a}^\mu$ explicitly if $w(\underline{e}) \neq -\frac{1}{2}$ (compare the Appendix). The conclusion of K. Hayashi and T. Kugo [Prog. Theor. Phys. 61, 334 (1979)] that the Weyl gauge field $\tilde{a}^\mu$ disappears from the Weyl gauge-covariant field equations for massive Dirac spin-$\frac{1}{2}$ fields holds only for a particular weight. That $\tilde{a}_\mu$ disappears in the case of $w(\underline{e}) = -\frac{1}{2}$ does by no means justify choosing $\tilde{a}_\mu = 0$ globally, because this cannot be obtained by a gauge transformation. The mass term in the Dirac equation still transforms under Weyl transformations. See Sec. VI for the consequences.

[18]To demonstrate that the corresponding calulation in a Weyl-Cartan space (nonvanishing torsion) results in a Riemann-Cartan space, transcribe the WKB procedure in the Riemann-Cartan space of Ref. 13.

[19]Compare Ref. 13.

[20]J. Audretsch, Zeitschr. f. allg. Wissenschaftstheorie/J. for General Philosophy of Science 12, 323 (1981).