Thermal gravitational radiation of Fermi gases and Fermi liquids

G. Schäfer and H. Dehnen
Fakultät für Physik der Universität Konstanz, D-7750 Konstanz, Postfach 5560, West Germany
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In view of neutron stars the gravitational radiation power of the thermal "zero-sound" phonons of a Fermi liquid and the gravitational bremsstrahlung of a degenerate Fermi gas is calculated on the basis of a hard-sphere Fermi particle model. We find for the gravitational radiation power per unit volume $P_{(s)} \simeq [(9\pi)^{1/3}/5]GQn^{5/3}(kT)^4/\hbar^2c^5$ and $P_{(g)} \simeq (4^5/5^3)(3/\pi)^{2/3}\,Ga^2n^{5/3}(kT)^4/\hbar^2c^5$ for the cases of "zero sound" and bremsstrahlung, respectively. Here $Q=4\pi a^2$ is the total cross section of the hard-sphere fermions, where a represents the radius of their hard-core potential. The application to very young neutron stars results in a total gravitational luminosity of about 10^{31} erg/sec.

I. INTRODUCTION

In view of the possibility of liquid or solid structures in massive astrophysical objects such as neutron stars, we investigated in a previous paper[1] the gravitational radiation of the thermal phonons of a homogeneous elastic isotropic body. To perform the calculations we restricted ourselves to nonrelativistic elastodynamics and to a special form of the body, namely, to a finite freely vibrating "thin" elastic cylinder. An essential point for the calculation was the fact that the wavelength of the gravitational radiation in the case of thermal excitation is small compared with the length of the cylinder so that the usual quadrupole radiation formula could not be used and a generalized approach was necessary.

For the total gravitational radiation power we found originally a T^4 law, in the case in which the temperature T of the body lies below its Debye temperature T_D. The application to a young neutron star has lead to an extremely high thermal gravitational radiation power. This result, however, is based, unfortunately, on an inconsistent integration procedure, which we already mentioned in a short paper.[2] Taking this into account a consistent calculation yields a T^2 law for the gravitational radiation of the thermal phonons of the body, which leads to a very drastic decrease of the radiation power of a young neutron star.

All these considerations have been performed under the assumption that the body vibrates coherently as a whole. However, this is correct only if no internal damping of the phonons is present.[3] In contrast to this, in the case of non-negligible damping the vibrations of the body fall to coherent pieces of a length comparable to the penetration depth of sound waves, which is for not too low frequencies or—in the case of thermal excitations—too low temperatures, small compared with the linear extension of the body.

Considering that the inner part of a neutron star may have the properties more of a neutron fluid than of a solid neutron lattice, we confine ourselves in this paper to consideration of Fermi liquids. In this case, damping of the vibrations exists, and for high frequencies, caused by thermal excitations, the coherence length of the vibrations is determined by the damping of the "zero-sound" waves of the degenerate Fermi matter.[4] We find in Sec. II that as a consequence of this a T^4 law for the gravitational radiation power of the visco-elastic vibrations is reinstated.

On the other hand, in the case of Fermi liquids gravitational radiation is also emitted by scattering of the gaseous Fermi particles. Therefore, in Sec. III the thermal gravitational radiation power of a degenerate Fermi gas is investigated.

In view of neutron stars we note that the degeneration temperature T_d lies at $\simeq 10^{12}$ K, so that the mean energy of the scattered Fermi particles (neutrons) has a value of approximately 10^2 MeV. For this energy region the collisions of the neutrons can be described by means of a hard-sphere potential with a radius $a \simeq 0.4$ fm.[5] Therefore we perform all our calculations in Secs. II and III on the basis of a hard-sphere Fermi particle model.[6]

Because it is to be expected that the gravitational wavelength is large compared with a, the quadrupole radiation formula for calculations of bremsstrahlung can be used. Furthermore, the de Broglie wavelength of nonrelativistic neutrons is also large compared with the hard-sphere radius α. Consequently, the gravitational bremsstrahlung of the neutrons must be calculated in a quantum-mechanical frame. We have here a situation in which the emission of gravitational radiation of a macroscopic body (neutron star) requires a quantum theory of gravitation, at least in the first-order approximation. In this way we also find for the total gravitational bremsstrahlung a T^4 law according for which, for Fermi liquids, the radiation of the "zero-sound" phonons and the thermal bremsstrahlung of the Fermi particles are of the same order of magnitude.

Finally we give an estimate of the gravitational luminosity of a very hot neutron star for the case in which the assumption of a Fermi liquid for its interior is correct.

II. THE GRAVITATIONAL RADIATION OF THERMAL PHONONS

To begin with we repeat the results of our previous paper[1] taking into account explicitly the corrections that follow from consideration of Ref. 2. The energy loss by gravitational radiation of the thin cylinder with mass M and length L vibrating coherently with frequency

$$\omega_j = j\pi c_0/L, \quad j=1,2,3,\ldots \tag{2.1}$$

is given by (G=gravitational constant)

$$-\frac{dE_j}{dt}=\frac{32}{15}\frac{G}{c^5}\frac{Mc_0{}^4}{L^2}E_j \times\left[1-(-1)^j\frac{15}{\omega_j{}^2L^2}j_2(\omega_j L)\right] \tag{2.2}$$

[E_j is the energy of mode j and

$$(2z/\pi)^{1/2}j_2(z)=J_{5/2}(z)$$

is the usual Bessel function], where

$$c_0{}^2=\frac{3\lambda+2\mu}{\lambda+\mu}\frac{\mu}{\rho_0} \tag{2.2a}$$

is the square of the velocity of sound (λ and μ are Lamé's constants and ρ_0 is matter density). Here only one-phonon processes have been taken into account.

In view of the application to thermally excited bodies only knowledge of the high-frequency range of (2.2) is necessary; with the definition

$$P_j=-\frac{1}{2}\frac{d}{dt}(E_j+E_{j+1})$$

one obtains, for $j \gg 1$,

$$P_j=\frac{32}{15}\frac{G}{c^5}Mc_0{}^4L^{-2}E_j\,. \tag{2.3}$$

In the case of thermal excitations of the single vibration modes the energy E_j is to be replaced by the mean thermal energy according to

$$E_j \to \bar{E}_j=\hbar\omega_j/(e^{\hbar\omega_j/kT}-1)\,. \tag{2.3a}$$

Adding up the contributions of all vibration modes we perform, finally, the replacement [cf. (2.1)]

$$\sum_j \to \int \frac{L}{\pi c_0}d\omega\,. \tag{2.3b}$$

Then we get from (2.3) the total thermal radiation power

$$P=\frac{16\pi}{45}\frac{G}{c^5}\frac{M}{L}c_0{}^3(kT)^2/\hbar\,, \tag{2.4}$$

where $T \ll T_D$ (Debye temperature) has been presupposed.[7] This limiting case is of special interest in view of the application to neutron stars, where the Debye temperature is of the order of the degeneration temperature of the neutrons.

With the area of the cover surfaces $F=2\pi R^2$, where R is the radius of the cylinder, we find

$$P=\frac{8\pi}{45}\frac{G}{c^5}\rho_0 c_0{}^3F(kT)^2/\hbar\,. \tag{2.5}$$

Evidently, in case of the coherently vibrating body the emission of gravitational radiation is a surface effect.

On the other hand, when internal damping of matter is present, only pieces of the cylinder with coherence length l radiate coherently. Then Eq. (2.3) is to be applied only to such a piece and subsequently to be multiplied with the number L/l of the pieces; in this way one finds, the gravitational radiation power in the case of vibration with frequency ω_j,

$$P_j=\frac{32}{15}\frac{G}{c^5}\frac{M_l c_0{}^4}{l^2}E_j, \quad M_l=M\frac{l}{L}\,. \tag{2.6}$$

In the case of thermal excitation the insertion of the mean thermal energy according to (2.3a) gives

$$\bar{P}_j=\frac{32}{15}\frac{G}{c^5}Mc_0{}^4\frac{\hbar\omega_j/lL}{e^{\hbar\omega_j/kT}-1}\,. \tag{2.7}$$

Now, the coherence length l is to be specified. For a Fermi liquid the collision time of the fermions is given by ($T \ll T_d$, with T_d the degeneration temperature)[4]

$$\tau=\pi^2\frac{\hbar^3}{mQ}(kT)^{-2}\,, \tag{2.8}$$

where Q is the total cross section of the fermions (m is their mass). On the other hand, the frequency range which contributes most to the radiation power is, according to (2.7), given by $\hat{\omega} \simeq kT/\hbar$. Then using this result together with (2.8) we obtain

$$\hat{\omega}\tau \simeq \pi^2 \frac{\hbar^2/mQ}{kT} \gg 1 \tag{2.8a}$$

for $T \ll T_d$ (note that the numerator is of the order of the Fermi threshold energy ϵ_F). Accordingly we are beyond the hydrodynamic limit and the coherence length is determined by the mean-free-path length of the "zero-sound" waves

$$l = \bar{v}\tau , \tag{2.9}$$

where

$$\bar{v} = \left[\frac{9\pi}{8}\right]^{2/3} \frac{\hbar}{m} n^{1/3} \tag{2.10}$$

is the mean velocity of the degenerate Fermi particles of the fluid (n is the particle number density). From (2.8)–(2.10) it follows immediately that[8]

$$l = (nQ)^{-1}(\hbar^2/2m)^2(3\pi^2 n)^{4/3}(kT)^{-2} . \tag{2.11}$$

On the other hand, in addition to the coherence length knowledge of the velocity c_0 of the zero sound is necessary for calculation of (2.7). Here we use the property that a very viscous fluid behaves in the high-frequency limit as a solid with pure shear waves, where the shear modulus is given by[9]

$$\mu = \eta/\tau . \tag{2.12}$$

Then the velocity of sound reads [compare (2.2a) with $\lambda \to \infty$]

$$c_0 = (3\mu/\rho_0)^{1/2} . \tag{2.13}$$

For the viscosity η we have to use the general definition according to the microscopic transport theory:

$$\eta = \tfrac{1}{3} l n m \bar{v} . \tag{2.14}$$

Using (2.8), (2.10), and (2.11) one finds, for viscosity and shear modulus of the Fermi liquid,

$$\eta = \frac{(3\pi^2)^{5/3}}{16} \frac{\hbar^5}{m^2 Q} n^{5/3}(kT)^{-2} , \qquad \mu = \frac{3^{5/3}\pi^{4/3}}{16} \frac{\hbar^2}{m} n^{5/3} . \tag{2.15}$$

Here we obtain from (2.13) the following relation for the zero-sound velocity [cf. (2.10)]:

$$c_0 = \bar{v} . \tag{2.16}$$

With knowledge of the properties of zero sound we are able to calculate the total radiation power emitted by the thermal phonons of the liquid. Insertion of (2.11) and (2.16) into (2.7) yields

$$\bar{P}_j = \frac{27}{10} \frac{G}{c^5} M \frac{nQ}{m^2 L} (kT)^2 \frac{\hbar\omega_j}{e^{\hbar\omega_j/kT} - 1} . \tag{2.17}$$

Finally we substitute the sum over the contributions of all frequencies ω_j by the integral according to (2.3b). Regarding (2.16) we obtain for the total gravitational luminosity $L_{(s)}$ of the thermal phonons in the limiting case $T \ll T_D$

$$L_{(s)} = \frac{(9\pi)^{1/3}}{5} \frac{G}{c^5} \frac{Q}{\hbar^2} \frac{M}{m} n^{2/3}(kT)^4 . \tag{2.18}$$

In contrast to the "coherent case" [Eq. (2.5)] the result (2.18) represents a volume effect. Furthermore we note that the structure of the relation (2.18) is a quite general one, so that it should be valid not only for the special case of the thin elastic cylinder but for all shapes of bodies.

Considering that between surface F and volume V of spherical bodies the relation

$$F = (36\pi)^{1/3} V^{2/3}$$

is valid, we can transform Eq. (2.18) into

$$L_{(s)} = q\sigma T^4 F \tag{2.19}$$

[$\sigma = (\pi^2/60)k^4/\hbar^3 c^2$ is Stefan-Boltzmann constant], where

$$q = \frac{6}{\pi^2} 2^{1/3} \frac{G}{c^3} Q \hbar n^{5/3} V^{1/3} \tag{2.19a}$$

represents the "opalescence" factor. Its value gives the ratio for which the radiation (2.19) is smaller than that of a blackbody. We note that our results (2.18) and (2.19) are only correct for $q \ll 1$; otherwise the body is nontransparent for gravitational radiation, and absorption and reemission inside the body must be taken into account.

In the case of a Fermi liquid consisting of degenerate neutrons with an energy of about 10^2 MeV we have, for Q,

$$Q = 4\pi a^2, \quad a \simeq 0.4 \text{ fm} \tag{2.20}$$

(hard spheres with diameter a). Then the following gravitational radiation power per unit volume caused by the thermal phonons of the neutron fluid results from (2.18):

$$P_{(s)} = \tfrac{4}{5}(3\pi^2)^{2/3} \frac{G}{c^5} \left[\frac{a}{\hbar}\right]^2 n^{5/3}(kT)^4 . \tag{2.21}$$

For the opalescence factor (2.19a) we get

$$q = \frac{24}{\pi} 2^{1/3} \frac{G}{c^3} a^2 \hbar n^{5/3} V^{1/3} . \tag{2.22}$$

Finally, we point to an interesting feature of our result. According to (2.7) the energy loss of the fluid by gravitational radiation of its thermal phonons increases with decreasing mean path length l, for which the heat conductivity of the thermal phonons decreases also. Consequently, when the energy loss by the electromagnetic radiation emitted from the surface is attenuated, the energy loss by gravitational radiation coming out of the inner parts of the fluid is enhanced.

III. THE GRAVITATIONAL RADIATION OF A DEGENERATE FERMI GAS

As mentioned in the Introduction the gravitational bremsstrahlung emitted by the scattering of two hard-sphere fermions of the Fermi liquid must be calculated quantum mechanically. The radiation intensity produced by the transition from the initial states defined by the momenta $\vec{p}_1,\vec{p}_2$ into the final states $\vec{p}\,'_1,\vec{p}\,'_2$ is given in the quadrupole approximation by[10,11]

$$I_{\vec{p}\,'_1,\vec{p}\,'_2;\vec{p}_1,\vec{p}_2}=\frac{2}{45}\frac{G}{c^5}\,|\langle \vec{p}\,'_1,\vec{p}\,'_2\,|\dddot{Q}_{ab}\,|\,\vec{p}_1,\vec{p}_2\rangle|^2 . \tag{3.1}$$

Going over to center-of-mass coordinates and relative coordinates the quadrupole operator separates into a center-of-mass part and a relative part, $Q_{ab}^{(c)}$ and $Q_{ab}^{(r)}$, respectively. Then in the case of central forces $\dddot{Q}_{ab}^{(c)}\equiv 0$ is valid generally, so that the matrix elements of the relative part of the quadrupole operator remain in Eq. (3.1). For the following it is appropriate to set Eq. (3.1) in the form

$$I_{\vec{p}\,'_1,\vec{p}\,'_2;\vec{p}_1,\vec{p}_2}=\frac{2}{45}\frac{G}{c^5}\omega^2\,|\langle \vec{p}\,'_1,\vec{p}\,'_2\,|\ddot{Q}_{ab}^{(r)}\,|\,\vec{p}_1,\vec{p}_2\rangle|^2 \tag{3.2}$$

with

$$\hbar\omega=\frac{1}{2m}(p_1{}^2+p_2{}^2-p_1'^2-p_2'^2) . \tag{3.2a}$$

In the case of hard-sphere potentials the interaction is taken into account as a boundary condition for the wave functions, whereas for the rest free propagation is valid. Thus $\ddot{Q}_{ab}^{(r)}$ becomes

$$\ddot{Q}_{ab}^{(r)}=\frac{2}{\mu}(3p_ap_b-p^2\delta_{ab}) , \tag{3.3}$$

where

$$\vec{p}=\tfrac{1}{2}(\vec{p}_1-\vec{p}_2)$$

represents the relative momentum of the two particles, and μ denotes their reduced mass in this section. Here it follows, from (3.2), that

$$I_{fi}=\frac{8}{45}\frac{G}{c^5}\frac{\omega^2}{\mu^2}\,|\langle\psi_f\,|\,3p_ap_b-p^2\delta_{ab}\,|\,\psi_i\rangle|^2 . \tag{3.4}$$

The initial and final states ψ_i and ψ_f, respectively, are to be chosen in such a way that the scattering problem has been solved already. This means that ψ_i contains an ingoing spherical and a plane wave and ψ_f contains an outgoing spherical and a plane wave, where the wave functions ψ_i and ψ_f vanish at the radius a of the spherically symmetric hard-sphere potential; in our case a is identical to the diameter of the hard-sphere fermions. Performing this we can restrict ourselves to the s-wave scattering, because the reduced de Broglie wavelength of the particles is large compared with the scattering length a in view of nonrelativistic neutrons. In this way we find for $r>a$, taking into account the indistinguishability of the particles,

$$\begin{aligned}\psi_i&=\frac{1}{\sqrt{2}}\left[e^{i\vec{p}\cdot\vec{x}/\hbar}+e^{-i\vec{p}\cdot\vec{x}/\hbar}-\frac{2a}{r}e^{-ipr/\hbar}\right]e^{i\vec{P}\cdot\vec{X}/\hbar} ,\\ \psi_f&=\frac{1}{\sqrt{2}}\left[e^{i\vec{p}\,'\cdot\vec{x}/\hbar}+e^{-i\vec{p}\,'\cdot\vec{x}/\hbar}-\frac{2a}{r}e^{ip'r/\hbar}\right]e^{i\vec{P}\,'\cdot\vec{X}/\hbar}\end{aligned} \tag{3.4a}$$

($\vec{x}=\vec{x}_1-\vec{x}_2$ is the relative coordinate, $r=|\vec{x}|$,

$$\vec{X}=\frac{m_1\vec{x}_1+m_2\vec{x}_2}{m_1+m_2}$$

is the center-of-mass coordinate, and $\vec{P}=\vec{p}_1+\vec{p}_2$ is the total momentum). Evidently, only particle pairs with symmetric wave functions (singlet) participate in the scattering. Inserting (3.4a) into (3.4) we find the following intensity of the bremsstrahlung[12]:

$$I_{fi}=\frac{8}{45}\frac{G}{c^5}\left[\frac{4a\hbar}{\mu^2}\right]^2[3(p_ap_b-p_a'p_b')-(p^2-p'^2)\delta_{ab}]^2[(2\pi\hbar)^3\delta^3(\vec{P}-\vec{P}\,')]^2 . \tag{3.5}$$

For calculation of the radiation power $P_{(g)}$ of the bremsstrahlung per unit volume we have to divide (3.5) by $(2\pi\hbar)^3\delta^3(\vec{P}-\vec{P})$ and subsequently multiply by the density of states in the momentum space and the mean occupation numbers according to Fermi statistics. Taking additionally into account that the singlet state represents only $\frac{1}{4}$ of all possible spin states we get

$$d^9P_{(g)}=\frac{64\pi^2}{15}\frac{G}{c^5}\frac{a^2\hbar^2}{\mu^4}[(p^2-p'^2)^2+3p^2p'^2\sin^2\chi]$$
$$\times\left[\exp\left[\frac{-\alpha+(\frac{1}{2}\vec{P}+\vec{p})^2/2m}{kT}\right]+1\right]^{-1}\left[\exp\left[\frac{-\alpha+(\frac{1}{2}\vec{P}-\vec{p})^2/2m}{kT}\right]+1\right]^{-1}$$
$$\times\left\{\exp\left[\frac{\alpha-(\frac{1}{2}\vec{P}+\vec{p}\,')^2/2m}{kT}\right]+1\right\}^{-1}\left\{\exp\left[\frac{\alpha-(\frac{1}{2}\vec{P}-\vec{p}\,')^2/2m}{kT}\right]+1\right\}^{-1}$$
$$\times\frac{d^3p}{(2\pi\hbar)^3}\frac{d^3p'}{(2\pi\hbar)^3}\frac{d^3P}{(2\pi\hbar)^3}\ . \tag{3.6}$$

Here χ is the angle between $\vec{p}$ and $\vec{p}\,'$ and the $\vec{P}\,'$ integration has been evaluated already. The integration over the remaining momenta is to be performed within the following limits:

$$0\le P\le\infty,\ \ 0\le p\le\infty,\ \ 0\le p'\le p\ , \tag{3.6a}$$

whereas the angles in the momenta spaces of $\vec{p}$ and $\vec{p}\,'$ run over half-spheres only because of the identity of the particles. The chemical potential α takes, in the case of high degeneracy, the form

$$\alpha=\frac{1}{2m}(3\pi^2\hbar^3n)^{2/3},\ \ \frac{\alpha}{kT}\gg 1\ . \tag{3.6b}$$

For evaluation of the integrals in (3.6) we consider the products of the brackets explicitly. One finds

$$[\ ][\ ]=1+\exp\left[\left[\frac{P^2}{4}+p^2-2m\alpha\right]\Big/mkT\right]$$
$$+2\cosh(|\vec{P}\cdot\vec{p}|/2mkT)\exp\left[\left[\frac{P^2}{4}+p^2-2m\alpha\right]\Big/2mkT\right], \tag{3.7}$$

$$\{\ \}\{\ \}=1+\exp\left[-\left[\frac{P^2}{4}+p'^2-2m\alpha\right]\Big/mkT\right]$$
$$+2\cosh(|\vec{P}\cdot\vec{p}\,'|/2mkT)\exp\left[-\left[\frac{P^2}{4}+p'^2-2m\alpha\right]\Big/2mkT\right]. \tag{3.8}$$

Evidently, with such denominators the integrals cannot be calculated exactly. For an approximate estimate we use the fact that only particles within an energy range of $\pm kT$ around Fermi's threshold energy participate essentially at the collisions. Therefore, the following holds:

$$\left|\frac{p_1{}^2}{2m}-\frac{p_2{}^2}{2m}\right|\lesssim 2kT,\ \ \left|\frac{p'^2_1}{2m}-\frac{p'^2_2}{2m}\right|\lesssim 2kT\ . \tag{3.9}$$

Because of $2|\vec{p}\cdot\vec{P}|=|p_1{}^2-p_2{}^2|$ and $2|\vec{p}\,'\cdot\vec{P}|=|p'^2_1-p'^2_2|$ it follows immediately that

$$|\vec{p}\cdot\vec{P}|\lesssim 2mkT,\ \ |\vec{p}\,'\cdot\vec{P}|\lesssim 2mkT\ . \tag{3.10}$$

Here the cosh functions in (3.7) and (3.8) can be approximated by 1 and (3.7) and (3.8) take the forms

$$[\ ][\ ]=(1+e^x)^2,\ \ \{\ \}\{\ \}=(1+e^{-y})^2 \tag{3.11}$$

with

$$x=\left[\frac{P^2}{4}+p^2\right]\Big/2mkT-\frac{\alpha}{kT},\ \ y=\left[\frac{P^2}{4}+p'^2\right]\Big/2mkT-\frac{\alpha}{kT}\ . \tag{3.11a}$$

Furthermore, we substitute P as follows:

$$z=\frac{P^2}{4}\Big/2mkT\ . \tag{3.11b}$$

Then with x, y, and z as new variables instead of p, p', and P the expression (3.6) results in, with the use of (3.11),

$$d^9P_{(g)}=\frac{64\pi^2}{15}\frac{G}{c^5}\frac{a^2\hbar^2}{\mu^4}\frac{(2mkT)^{13/2}}{(2\pi\hbar)^9}\left[(x-y)^2+3\left[x-z+\frac{\alpha}{kT}\right]\left[y-z+\frac{\alpha}{kT}\right]\sin^2\chi\right]$$
$$\times\left[x-z+\frac{\alpha}{kT}\right]^{1/2}\left[y-z+\frac{\alpha}{kT}\right]^{1/2}\frac{z^{1/2}dx\,dy\,dz}{(1+e^x)^2(1+e^{-y})^2}d^2\Omega_p d^2\Omega_{p'}d^2\Omega_P\ . \tag{3.12}$$

To perform the integrations in (3.12) we have to consider at first the integration limits. With respect to the exponential functions in the denominator of (3.12) the integrand contributes noticeable amounts to the integral only for $x\simeq0$, $y\simeq0$, where $y\leq x$ is valid ($p'\leq p$). Here it follows from (3.11a) immediately that

$$\frac{P^2/4}{2mkT}\simeq\frac{\alpha}{kT}-\frac{p^{(\prime)2}}{2mkT},\quad p^{(\prime)}=p\ \text{or}\ p'\ . \tag{3.13}$$

From (3.10) we obtain, if the azimuthal angles in the momentum spaces of $\vec{p}$ and $\vec{p}\,'$ are measured with respect to the direction of $\vec{P}$,

$$\begin{aligned}&|\cos\Theta|\leq 2mkT/pP,\ \text{ if } pP\geq 2mkT\ ,\\ &|\cos\Theta'|\leq 2mkT/p'P,\ \text{ if } p'P\geq 2mkT\ .\end{aligned} \tag{3.14}$$

In view of (3.13) these domains of P and $p^{(\prime)}$ are the only relevant ones, see Fig. 1. Then the solid-angle integrations in (3.12) can be performed; one finds

$$\begin{aligned}&\int d^2\Omega_P=4\pi\ ,\\ &\int\int d^2\Omega_p d^2\Omega_{p'}=4\pi^2\frac{(2mkT)^2}{pp'P^2}\ ,\\ &\int\int d^2\Omega_p d^2\Omega_{p'}\sin^2\chi\simeq2\pi^2\frac{(2mkT)^2}{pp'P^2}\ ,\end{aligned} \tag{3.15}$$

where the two last integrals run over half-spheres only, as noted above. Taking into account that, according to (3.11a) and (3.11b),

$$\begin{aligned}&pP=4mkT\sqrt{z}\,(x-z+\alpha/kT)^{1/2}\ ,\\ &p'P=4mkT\sqrt{z}\,(y-z+\alpha/kT)^{1/2}\end{aligned} \tag{3.11c}$$

we find, from (3.12) and (3.15),

$$d^3P_{(g)}=\frac{1}{30}\frac{G}{c^5}\frac{a^2}{\pi^4\mu^4\hbar^7}(2mkT)^{13/2}[(x-y)^2+\tfrac{3}{2}(x-z+\alpha/kT)(y-z+\alpha/kT)]$$
$$\times z^{-1/2}(1+e^x)^{-2}(1+e^{-y})^{-2}dx\,dy\,dz\ . \tag{3.16}$$

For determination of the lower limits for x and y (the upper limits are ∞ and x, respectively) we conclude, from (3.11a) and (3.14),

$$\left.\begin{matrix}x\\ y\end{matrix}\right\}\geq z+1/4z-\alpha/kT\ , \tag{3.17}$$

where z runs from 0 to ∞. However, because only the values $x\simeq0$, $y\simeq0$ contribute most to the integral of (3.16) the relevant domain for z is, according to (3.17), restricted to

$$kT/4\alpha\lesssim z\lesssim\alpha/kT\ ; \tag{3.18}$$

for most of these z values the lower limits of x and y are, in view of Eq. (3.17), practically $-\infty$. Here, with the fact that because $\alpha/kT\gg1$ the values of x and y in the numerator of (3.16) can be neglected in view of the denominator, we obtain, from (3.16),

$$P_{(g)}=\frac{1}{20}\frac{G}{c^5}\frac{a^2}{\pi^4\mu^4\hbar^7}(2mkT)^{13/2}\int_0^{\alpha/kT}\int\int_{-\infty}^{+\infty}\frac{\Theta(x-y)(\alpha/kT-z)^2dz\,dx\,dy}{(1+e^x)^2(1+e^{-y})^2z^{1/2}}\ , \tag{3.19}$$

where the lower limit of z has been set equal to 0 ($kT/\alpha \ll 1$).

Now we are able to estimate the integrals in (3.19). At first we find

$$\int_0^{\alpha/kT}\iint_{-\infty}^{+\infty}(\;)dz\,dx\,dy=\frac{8}{15}\left[\frac{\alpha}{kT}\right]^{5/2}\iint_{-\infty}^{+\infty}\frac{e^{-|x-y|}dx\,dy}{(1+e^{x})(1+e^{-x})(1+e^{y})(1+e^{-y})}\,, \tag{3.20}$$

where the temperature dependence comes from the z integral and the x,y integrations in (3.20) give a number factor only. Its approximate value lies at $\frac{2}{5}$. Consequently, from (3.19) and (3.20) we obtain after insertion of α according to (3.6b) and of $\mu=m/2$ the following radiation power per unit volume[13]:

$$P_{(g)}=\frac{4^5}{5^3}\left[\frac{3}{\pi}\right]^{2/3}\frac{G}{c^5}\left[\frac{a}{\hbar}\right]^2 n^{5/3}(kT)^4\,. \tag{3.21}$$

The comparison with the expression (2.21) for the thermal gravitational radiation of the phonons shows that both components of a Fermi liquid, the phonon and the gas part, radiate with an intensity of the same order for all values of n and T in question.

IV. APPLICATION TO NEUTRON STARS

For the application of the results (2.21) and (3.21) to neutron stars we test at first if the suppositions of our considerations are fulfilled. Section II is based on equations of motion, in which the gravitational forces are neglected. As one can prove easily this is justified in the high-frequency range of thermal excitations even for neutron stars. Assuming a density of

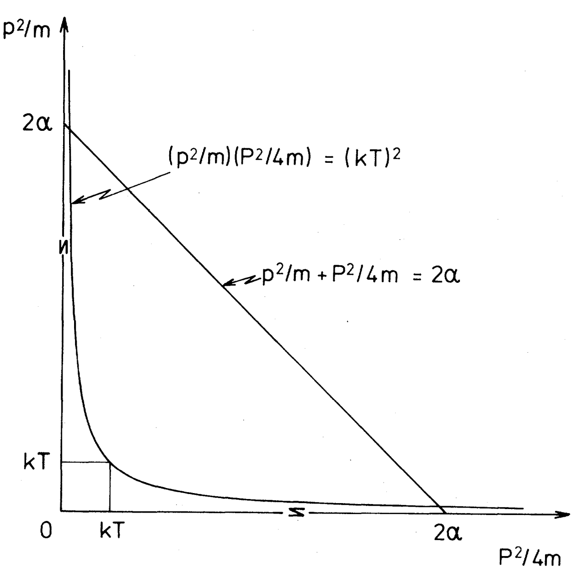

FIG. 1. According to (3.13) most of the contributions to the integrals come from the neighborhood of the straight line $p^2/m+P^2/4m=2\alpha$.

$$\rho_0\simeq 5\times 10^{14}\ \mathrm{g/cm^3}$$

for a neutron star[14] the degeneration temperature of the neutrons amounts to $T_d\simeq 10^{12}$ K, whereas the Debye temperature for the phonons resulting from the hard-sphere fermion model is about $T_D\simeq 10^{11}$ K. Therefore even the temperature of $T\simeq 5\times 10^{10}$ K of a very young neutron star[15] lies below T_d and T_D, respectively. Furthermore with these values for ρ_0 and T the coherence length (2.11) is in view of (2.20) of the order of

$$l\simeq 7\times 10^{-11}\ \mathrm{cm}\,.$$

On the other hand the wavelengths corresponding to $\omega\simeq kT/\hbar$ have the values

$$\lambdabar_G\simeq 5\times 10^{-12}\ \mathrm{cm}$$

and

$$\lambdabar_S\simeq 10^{-12}\ \mathrm{cm}$$

for the gravitational and the sound wave, respectively, whereby the sound velocity is, according to (2.16), $c_0\simeq 0.3c$. Consequently, the conditions $\lambdabar_S<\lambdabar_G\ll l\ll R$ (R is the radius of the star) are satisfied for a large temperature range of

$$T\lesssim 5\times 10^{10}\ \mathrm{K}\,.$$

Furthermore $\lambdabar_S$ is large compared with the mean distance of the neutrons. For the opalescence factor (2.22), valid for the radiation of the phonons as well as for the bremsstrahlung, one finds $q\simeq 10^{-21}$, so that even neutron stars are very transparent for gravitational radiation. Finally we note that in case of the bremsstrahlung of Sec. III the use of the quadrupole formula is satisfied because of

$$\lambdabar_G\gg a\simeq 4\times 10^{-14}\ \mathrm{cm}\,.$$

Consequently, all suppositions of our calculations are fulfilled.

With the approximate supposition of constant density and temperature inside the neutron star we find, by combination of (2.21) and (3.21), the following thermal gravitational luminosity of the star:

$$L_{\mathrm{th}}\simeq\frac{8}{5}(3\pi^2)^{2/3}\frac{G}{c^5}\left[\frac{a}{\hbar}\right]^2\frac{M}{m}n^{2/3}(kT)^4\,. \tag{4.1}$$

For a very young object with typical properties, i.e.,

$$T=5\times10^{10}\ \mathrm{K}\ ,$$

$$\rho_0=5\times10^{14}\ \mathrm{g/cm^3},\ \ M=M_\odot\ ,$$

one finds a thermal gravitational radiation power of about 10^{31} erg/sec, if $a=0.4$ fm is chosen. In contrast to this, in the case of pure coherent vibrations, the radiation power of the same object would be, according to (2.5), approximately 10^{14} erg/sec only. In any case the frequencies of the gravitons lie in the very-high-frequency range of 10^{21} Hz.

V. FINAL REMARK

The foregoing result is based on the assumption of a nonsuperfluid Fermi liquid for the main part of neutron stars. Up to now it is not clear if this assumption is fulfilled within the range of the numerical values for T and ρ_0 used by us. It could be that a description of the inner parts by a superfluid neutron liquid or a neutron solid would be more appropriate.[16,17] In these cases new damping mechanisms will take place which, of course, imply new calculations of the temperature dependence and of the power of the thermal gravitational radiation.

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[3]For two-phonon gravitational radiation processes phonon damping has been discussed by L. Halpern and R. Desbrandes, Ann. Inst. Henri Poincaré 11A, 309 (1969).

[4]G. Baym and C. Pethick, in *The Physics of Liquid and Solid Helium, Part II,* edited by K. Bennemann and J. Ketterson (Wiley, New York, 1978), pp. 1 ff.

[5]G. Eder, *Kernkräfte* (Braun, Karlsruhe, 1965), pp. 29 ff.

[6]A. Isihara, *Statistical Physics* (Academic, New York, 1971), pp. 332 ff.

[7]Concerning the lower limit of the frequency integral there exists a second condition for the temperature, namely, $T\gg\pi(\hbar/k)(c_0/L)$, which is fulfilled in general, of course.

[8]Compare Ref. 6, p. 99.

[9]L. Landau and E. Lifshitz, *Elastizitätstheorie* (Akademie, Berlin, 1975), Sec. 36.

[10]Compare L. Landau and E. Lifshitz, *Quantum Electrodynamics* (Pergamon, Oxford, 1982), Sec. 92.

[11]G. Schäfer and H. Dehnen, J. Phys. A 13, 2703 (1980).

[12]For evaluation of the integrals in (3.4) the lower limit of the radial part has been set equal to zero in a good approximation.

[13]For a different estimation of the integrals resulting from (3.6) see, e.g., J. Ziman, *Electrons and Phonons* (Oxford University, Oxford, 1960), pp. 412 ff.

[14]V. Canuto and R. Bowers, *IAU Symposium No. 95,* Pulsars, edited by W. Sieber and R. Wielebinski (Reidel, Dordrecht, 1981), pp. 321 ff.

[15]E. Flowers and N. Itoh, Astrophys. J. 230, 847 (1979).

[16]F. Lamb, *IAU Symposium No. 95,* Pulsars, Ref. 14, pp. 303 ff.

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