# Superconformal gravity in Hamiltonian form: Another approach to the renormalization of gravitation

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We reexpress superconformal gravity in Hamiltonian form, explicitly displaying all 24 generators of the group as Dirac constraints on the Hilbert space. From this, we can establish a firm foundation for the canonical quantization of superconformal gravity. The purpose of writing down the Hamiltonian form of the theory is to reexamine the question of renormalization and unitarity. Usually, we start with unitary theories of gravity, such as the Einstein-Hilbert action or supergravity, both of which are probably not renormalizable. In this series of papers, we take the opposite approach and start with a theory which is renormalizable but has problems with unitarity. Conformal and superconformal gravity are both plagued with dipole ghosts when we use perturbation theory to quantize the theories. It is difficult to interpret the results of perturbation theory because the asymptotic states have zero norm and the potential between particles grows linearly with the separation distance. The purpose of writing the Hamiltonian form of these theories is to approach the question of unitarity from a different point of view. For example, a strong-coupling approach to these theories may yield a totally different perturbation expansion. We speculate that canonically quantizing the theory by power expanding in the strong-coupling regime may yield a different set of asymptotic states, somewhat similar to the situation in gauge theories. In this series of papers, we wish to reopen the question of the unitarity of conformal theories. We conjecture that ghosts are "confined."

## I. INTRODUCTION

A unified theory of all four known interactions cannot be complete unless a procedure is found to formulate a quantum theory of gravitation. However, the presence of Newton's constant in the theory, a dimensional coupling constant, precludes a conventional renormalization scheme. Furthermore, the presence of locally symmetric counterterms in the higher-order loop expansion of gravity, supergravity, and extended supergravity casts serious doubt on the conjecture that these theories may be finite to all orders in perturbation theory.

One alternative to the usual theory of gravity and supergravity is to treat these two theories as effective theories, valid only in the long-distance limit. In a previous paper, we (with P. Townsend)[1] proposed that supergravity is only the long-distance, broken version of the more symmetric superconformal gravity[2] theory. Only at distances close to the Planck length will the supergravity theory [based on the group Osp(4/1)] emerge as the superconformal gravity theory [based on the group SU(2,2/1)]. Zee,[3] Smolin,[4] and others[5] have proposed Higgs mechanisms for this kind of scale breaking. Adler[6] even proposes using fermion fields to replace the scalar fields found in the theory, giving a dynamical breaking to scale invariance.

Our knowledge of the dynamical properties of conformal gravity comes from perturbation theory, and there are serious problems before a physical interpretation can be made of this theory. First, in perturbation theory around flat space, the asymptotic states of the theory have positive energy but zero norm. Second, even though the norm of a state is not an observable, and these dipole states may decouple from the $S$ matrix, we encounter problems when we include interactions. Because of the higher-derivative nature of the theory, we find that the potential between particles is linear. Once again, we have difficulty in defining asymptotic states. We do not have confinement in the usual sense because we do not have singlet states under the group, but this indicates some form of bound state. In summary, conformal theories are not unitary in the usual sense when we use perturbation theory.

One way out of this problem is to start with an $R+R^2$ theory and use loop diagrams to push the dipole ghosts off the real axis into the unphysical sheet by the Lee-Wick mechanism.[7] This would apparently preserve unitarity and renormalization, although some questions remain concerning causality and whether or not a consistent prescription can be found to all loop orders in the expansion.

In this paper, we want to explore yet another mechanism to restore unitarity. In QCD, the naive asymptotic states found in weak-coupling perturbation theory are not the actual ones. This is because infrared divergences require a reinterpreting of the naive Hilbert space. In particular, in the strong-coupling Kogut-Susskind approach, we find that a solution to the constraint equation is to postulate color-singlet stringlike states, which cannot be reproduced to any finite order in weak-coupling perturbation theory. These stringlike states naturally emerge when we quantize in the Hamiltonian approach. In the same spirit, we wish to construct the Hamiltonian of superconformal gravity and look for solutions to the Dirac constraints (which are *not,* in general, solved by spin-2 gravitons with two helicities). By canonically quantizing conformal and superconformal gravity and going over to the strong-coupling limit, we hope to find an *entirely new* Hilbert space without the troublesome dipole ghost states found in the weak-coupling limit. We propose that these ghosts are "confined."

We wish to reexamine the question of the unitarity of conformal theories. First, we wish to establish the canonical quantization of conformal and superconformal gravity. There are many different but equivalent quantization schemes, and all of them have subtle problems with ordering, anomalies, etc. We feel that the canonical quantization method is the best understood of these various quantization procedures. Second, we wish to eventually perform a strong-coupling expansion on the theory which will power expand the theory around a new Hilbert space. In this manner, we hope that we can reestablish unitarity from a different point of view.

In a previous paper,[8] we established the canonical formalism for conformal gravity written in metric form. We wrote down the canonical Poisson brackets of the theory and explicitly displayed the five first-class constraints of the theory, four for general coordinate transformations and one for local scale transformations.

In this paper, we complete the discussion of conformal gravity by giving all 10 generators of the conformal group (four general coordinate transformations, six local Lorentz transformations, one scale transformation, and four proper conformal transformations). We go on to reformulate superconformal gravity in canonical form, explicitly displaying all 24 constraints of the theory.

In our next paper, we extend these results by performing a strong-coupling expansion of these theories. One goal is to eventually show that the usual gravitational interactions are actually van der Waals-type effective forces of the conformal theories.

Of course, it is very likely that the problematic behavior of conformal gravity may persist even when we treat the theory nonperturbatively. In this case, we must reject conformal theories as a candidate for a renormalizable theory of gravity at distances smaller than the Planck length. However, if we can somehow eliminate the nonunitary character of conformal gravity found in perturbation theory with methods which employ nonperturbative techniques, then the theory would be a prime candidate for a theory of gravity.

Even if we can establish a positive-definite Hilbert space for these conformal theories, there still are other problems, such as how to retrieve the desired long-range behavior of the Einstein-Hilbert action. One attractive method is to use dynamical symmetry breaking to break down conformal gravity to the usual theory. Since we still want to preserve general covariance, it is likely that dynamical symmetry breaking of scale invariance will reproduce the Einstein action. There are several ways in which the long-range behavior may be retrieved: (a) dynamical symmetry breaking induced by instantons, (b) scale breaking introduced by a cutoff procedure (e.g., such as in the Kogut-Susskind approach), (c) long-range, van der Waals forces between stringlike objects.

## II. SUPERCONFORMAL GRAVITY

The conformal group SU(2,2) is a 15-parameter Lie group which includes the 10-parameter Poincaré group as well as scale invariance and proper conformal transformations. If we enlarge this group to include anticommuting parameters, then the minimal graded Lie group which contains the conformal group is the superconformal group SU(2,2/1). This group is a 24-parameter graded Lie group which includes the six generators of the Lorentz group ($M^{ab}$), the four generators of translations ($P^a$), one generator for scale transformations ($D$), four generators for proper conformal transformations ($K^a$), $4+4$ generators for two supersymmetric transformations ($Q^\alpha, S^\alpha$), and one generator for chiral transformations ($A$).

In direct analogy with usual gauge theories, we can construct the action by first writing down the curvatures associated with each generator of the theory. The algebra of this graded Lie group is represented by

$$
\begin{aligned}
&[M^{ab},M^{cd}]=\delta^{bc}M^{ad}+\delta^{ad}M^{bc}-\delta^{ac}M^{bd}-\delta^{bd}M^{ac}\ ,\\
&[M^{ab},P^{c}]=\delta^{bc}P^{a}-\delta^{ac}P^{b}\ ,\\
&[M^{ab},K^{c}]=\delta^{bc}K^{a}-\delta^{ac}K^{b}\ ,\\
&[P^{a},D]=P^{a}\ ,\\
&[K^{a},D]=-K^{a}\ ,\\
&[K^{a},P^{b}]=-2(\delta^{ab}D+M^{ab})\ ,\\
&[S,P^{a}]=\gamma^{a}Q,\ \ [S,A]=\tfrac{3}{4}i\gamma_5 S,\ \ [S,D]=-\tfrac{1}{2}S\ ,\\
&[Q,K^{a}]=-\gamma^{a}S,\ \ [Q,A]=-\tfrac{3}{4}i\gamma_5 Q,\ \ [Q,D]=\tfrac{1}{2}Q\ ,\\
&[Q,M^{ab}]=\sigma^{ab}Q,\ \ \{Q^{\alpha},Q^{\beta}\}=-\tfrac{1}{2}(\gamma^{a}C)^{\alpha\beta}P^{a}\ ,\\
&[S,M^{ab}]=\sigma^{ab}S,\ \ \{S^{\alpha},S^{\beta}\}=\tfrac{1}{2}(\gamma^{a}C)^{\alpha\beta}K^{a}\ ,\\
&\{Q^{\alpha},S^{\beta}\}=-\tfrac{1}{2}C^{\alpha\beta}D+\tfrac{1}{2}(\sigma^{ab}C)^{\alpha\beta}M^{ab}+(i\gamma_5 C)^{\alpha\beta}A\ .
\end{aligned}
\tag{2.1}
$$

We now associate a connection field for each of the 24 generators of the theory. We denote the connection field for the Lorentz, translation, scale, proper conformal, supersymmetry, and chiral transformations, respectively, by the following:

$$\omega_\mu^{ab},e^a{}_\mu,b_\mu,f^a{}_\mu,\bar{\psi}^\alpha{}_\mu,\bar{\phi}^\alpha{}_\mu,A_\mu\ .$$

It is now a simple matter to write down all the curvatures associated with each generator of the theory,

$$
\begin{aligned}
&R_{\mu\nu}^{ab}(M)=R_{\mu\nu}^{0ab}-4(e^a{}_\mu f^b{}_\nu-e^b{}_\mu f^a{}_\nu)-2\bar{\psi}_\mu\sigma^{ab}\phi_\nu\ ,\\
&R_{\mu\nu}(D)=-2\partial_\mu b_\nu+4e^a{}_\mu f^a{}_\nu+\bar{\psi}_\mu\phi_\nu\ ,\\
&R_{\mu\nu}(A)=-2\partial_\mu A_\nu-2i\bar{\psi}_\mu\gamma_5\phi_\nu\ ,\\
&R_{\mu\nu}^{\alpha}(Q)=(2D_\nu\bar{\psi}_\mu+2\bar{\phi}_\mu\gamma_\nu+b_\nu\bar{\psi}_\mu\\
&\qquad -\tfrac{3}{2}iA_\nu\bar{\psi}_\mu\gamma_5)^\alpha\ ,\\
&R_{\mu\nu}^{\alpha}(S)=(2D_\nu\bar{\phi}_\mu-2\bar{\phi}_\mu\gamma^a f^a{}_\nu-b_\nu\phi_\mu\\
&\qquad +\tfrac{3}{2}iA_\nu\bar{\phi}_\mu\gamma_5)^\alpha\ ,\\
&R_{\mu\nu}^{a}(P)=-2\partial_\mu e^a{}_\nu+2\omega_\mu^{ab}e^b{}_\nu+\tfrac{1}{2}\bar{\psi}_\mu\gamma^a\psi_\nu\\
&\qquad +2e^a{}_\mu b_\nu\ ,\\
&R_{\mu\nu}^{a}(K)=-2\partial_\mu f^a{}_\nu+2\omega_\mu^{ab}f^b{}_\nu-\tfrac{1}{2}\bar{\phi}_\mu\gamma^a\phi_\nu\\
&\qquad -2f^a{}_\mu b_\nu\ ,
\end{aligned}
\tag{2.2}
$$

where (all curvatures are to be antisymmetrized in $\mu$ and $\nu$)

$$
\begin{aligned}
&R_{\mu\nu}^{0ab}=-\partial_\mu\omega_\nu^{ab}+\partial_\nu\omega_\mu^{ab}+\omega_\mu^{ac}\omega_\nu^{cb}-\omega_\nu^{ac}\omega_\mu^{cb}\ ,\\
&D_\mu\psi_\nu=\partial_\mu\psi_\nu-\tfrac{1}{2}\omega_\mu^{ab}\sigma^{ab}\psi_\nu\ ,\\
&\delta^{ab}=(+,+,+,+)\ ,\\
&\epsilon^{abcd}=1,\ \ \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4,\ \ \gamma_\mu=\gamma_\mu^\dagger,\\
&\gamma_5=\gamma_5^\dagger\ ,\\
&\gamma_\mu{}^2=\gamma_5{}^2=1,\ \ \sigma^{ab}=\tfrac{1}{4}(\gamma^a\gamma^b-\gamma^b\gamma^a),\\
&e=\det(e_{a\mu})\ ,\\
&C=-C^T=-C^{-1},\ \ C\gamma^aC^{-1}=-\gamma^{aT},\\
&g_{\mu\nu}=e^a{}_\mu e^a{}_\nu\ .
\end{aligned}
\tag{2.3}
$$

As in Yang-Mills theory, we can now write down the Lagrangian by taking various products of curvatures,

$$
\begin{aligned}
L=\epsilon^{\mu\nu\rho\sigma}[&\alpha R_{\mu\nu}^{ab}(M)R_{\rho\sigma}^{cd}(M)\epsilon^{abcd}\\
&+\beta R_{\mu\nu}(Q)\gamma_5\bar{R}_{\rho\sigma}(S)\\
&+\gamma R_{\mu\nu}(A)R_{\rho\sigma}(D)]\\
&+\delta eR_{\mu\nu}(A)R^{\mu\nu}(A)\ ,
\end{aligned}
\tag{2.4}
$$

where $\beta=\delta=2i\gamma=-8\alpha$.

It is a tedious, but straightforward task to show that this Lagrangian, indeed, is fully locally gauge invariant under the entire 24-parameter Lie group once we choose the proper constraints on the theory,

$$R_{\mu\nu}^a(P)=0,\ \ R_{\mu\nu}(Q)\gamma^\nu=0\ . \tag{2.5}$$

In the usual theory of gravity and supergravity, the constraint on the $P$ curvature is optional because it is redundant with the equations of motion for the connection field for the local Lorentz transformations. Here, however, we impose this constraint on the theory because it does not yield the naive equations of motion for the connection field. The complete set of constraints is absolutely necessary in proving that the action is locally gauge invariant under the entire group SU(2,2/1).

These constraints, in turn, can be used to eliminate two connection fields in terms of the other fields,

$$
\begin{aligned}
&\omega_\mu^{ab}=-\omega_\mu^{(0)ab}+(e^b{}_\mu b^a-e^a{}_\mu b^b)\\
&\qquad +\tfrac{1}{4}(\bar{\psi}_\mu\gamma^a\psi^b-\bar{\psi}_\mu\gamma^b\psi^a-\bar{\psi}^a\gamma_\mu\psi^b)\ ,\\
&\omega_\mu^{(0)ab}=\tfrac{1}{2}[e^{a\nu}(e^b{}_{\nu,\mu}-e^b{}_{\mu,\nu})\\
&\qquad +e^{a\lambda}e^{b\rho}e^c{}_{\lambda,\rho}e^c{}_\mu]-(a\leftrightarrow b)\ ,\\
&\phi_\mu=\gamma^\nu(S_{\mu\nu}+\tfrac{1}{4}\gamma_5\tilde{S}_{\mu\nu})/3\ ,\\
&S_{\mu\nu}=(D_\nu\psi_\mu+\tfrac{1}{2}b_\nu\psi_\mu-\tfrac{3}{4}iA_\mu\gamma_5\psi_\mu)-(\mu\leftrightarrow\nu)\ ,\\
&\tilde{S}_{\mu\nu}=e^{-1}\epsilon^{\mu\nu\rho\sigma}S_{\rho\sigma}\ .
\end{aligned}
\tag{2.6}
$$

Furthermore, we still have the freedom left in the theory to completely eliminate the connection field $f$ through its equations of motion,

$$f_{\mu\nu}=-\tfrac{1}{4}(\hat{R}_{\nu\mu}-\tfrac{1}{6}g_{\mu\nu}\hat{R})+\tfrac{1}{8}R_{\lambda\mu}(Q)\gamma_\nu\psi^\lambda -(i/16)\tilde{R}_{\mu\nu}(A)\ , \qquad (2.7)$$

where $\hat{R}_{\nu\mu}$ denotes $R_{\nu\rho a\mu}(M)e^{a\rho}$ with $f$ put equal to zero.

Now that the preliminaries are over, we can begin a discussion of superconformal gravity in Hamiltonian form.

Because superconformal gravity is a higher-derivative theory, auxiliary fields must be added before we can express the Lagrangian in canonical form,

$$L=p_i\dot{q}_i-H(p,q)+\lambda^a C^a\ , \qquad (2.8)$$

where $C^a$ represents the Dirac constraints of the theory, and $\lambda^a$ represents Lagrange multipliers.

As in the usual Arnowitt-Deser-Misner (ADM) formulation of gravity, we find that the key to rewriting the action lies in a judicious choice of variables. In expressing the theory in terms of the $3+1$ formalism, we must be very careful in selecting variables which separate independent fields from dependent ones and Lagrange multipliers.

We will thus begin our discussion of the $3+1$ formalism by explaining as clearly as possible our choice of variables.

We will first separate out the 16 independent components of the field $e^a{}_\mu$ into two sets, the 12 independent fields represented by $e^a{}_i$ $(i=1,2,3)$, and the 4 remaining fields $e^a{}_0$, which will eventually become Lagrange multipliers for the general coordinate group. (We will use indices $a,b,c,\ldots$ to represent four-dimensional flat space, while $i,j,k,\ldots$ represent curved-space indices.)

When we are manipulating independent fields, we will sometimes use the prefix "3", while the prefix "4" will denote the original fully covariant tensor (which, in general, can be decomposed into Lagrange multipliers, dependent fields, and independent fields).

*From now on, we will use the convention that, if no prefix is given, then all tensors labeled by $i,j,k,\ldots$ are independent fields.*

In general, covariant indices will represent independent fields, while contravariant fields are formed by raising indices by the three-dimensional metric ${}^3g_{ij}$,

$$\begin{aligned}
&g_{ij}={}^3g_{ij}={}^4g_{ij}\ ,\\
&g_{ij}{}^3g^{jk}=\delta_i{}^k\ ,\\
&e^a{}_i={}^3e^a{}_i={}^4e^a{}_i\ ,\\
&e^{ai}={}^3e^{ai}={}^3e^a{}_k{}^3g^{ki}\neq{}^4e^{ai}\ .
\end{aligned} \qquad (2.9)$$

The relationship between the three- and four-dimensional metrics is given by the usual ADM relations

$$\begin{aligned}
&{}^4g_{\mu\nu}=\begin{vmatrix}-N^2+N_kN^k & N_i\\ N_j & {}^3g_{ij}\end{vmatrix}\ ,\\
&{}^4g_{\mu\nu}=\begin{vmatrix}-N^{-2} & N^iN^{-2}\\ N^jN^{-2} & {}^3g_{ij}-N^iN^jN^{-2}\end{vmatrix}\ ,
\end{aligned} \qquad (2.10)$$

where

$$N=(-g^{00})^{-1/2},\ \ N_i=g_{0i}\ ,$$
$$N^i={}^3g^{ij}N_j,\ \ \text{metric}=(-,+,+,+)\ .$$

[Notice that, in curved space, we use the metric $(-,+,+,+)$, while we use the metric $(+,+,+,)$ for flat-space indices $a,b,c,\ldots$. We apologize for using this mixed-metric formalism, but we wish to conform with the metric conventions of ADM and Ref. 2.]

We will find it convenient to introduce a vector field $n^a$, which is a function of only the independent fields $e^a{}_i$, defined by the four relations

$$\begin{aligned}
&n^an^a=-1\ ,\\
&n^ae^a{}_i=n^ae^{ai}=0\ .
\end{aligned} \qquad (2.11)$$

We can also write down a closed form for $n^a$,

$$n^a={}^4e^{a0}/(-g^{00})^{1/2}\ .$$

This second form conceals the fact that $n^a$ is only a function of the independent fields $e^a{}_i$.

One purpose of introducing this new vector is to be able to construct a projection operator for the theory, because inversions in this strange $4\times 3$ space are very tricky. For example, if we start with the 12 independent functions represented by a tensor $A^a{}_i$, then we can form a nine-component antisymmetric tensor as follows:

$$A^{ab}=A^a{}_ie^{bi}-A^b{}_ie^{ai}\ . \qquad (2.12)$$

But if we try to reverse this process, and recreate the original 12-component tensor out of the nine-component antisymmetric tensor, we obviously have problems,

$$A^a{}_i\neq A^{ab}e^b{}_i\ . \qquad (2.13)$$

In other words, the freedom of rapidly contracting back and forth between the flat-space indices $a,b,c,\ldots$ and the curved-space indices $i,j,k,\ldots$ is lost when manipulating tensors in this $4\times 3$-dimensional space.

The problem is solved, however, by introducing a projection operator defined as

$$n^{ab}=e^a{}_i e^{bi}=\delta^{ab}+n^a n^b\ ,$$
$$n^{ab}n^{bc}=n^{ac}\ ,\tag{2.14}$$
$$n^{aa}=3\ .$$

The correct relationship between the 12-component $A^a{}_i$ and the 9-component antisymmetric tensor $A^{ab}$ is therefore given by

$$A^a{}_i n^{ab}=n^{ba}A^{ac}e^c{}_i\tag{2.15}$$

For completeness, we write down the equation which allows us to go back and forth between four- and three-dimensional fields,

$$e^{ai}={}^3e^{ai}={}^4e^{ai}+n^aN^i/N\ .\tag{2.16}$$

Finally, we will find one more notational convention quite useful. Oftentimes, in general relativity, it is more convenient to drop the time direction in favor of a direction denoted by the vector $n$. If we define

$$A_n=-A^n=-NA^0\ ,\tag{2.17}$$

then

$$A_\mu A^\mu=A_nA^n+A_i{}^3g^{ij}A_j$$
$$=-N^2A^0B^0+({}^4A^i+N^iA^0){}^3g_{ij}({}^4B^j+N^jB^0)\ .\tag{2.18}$$

The purpose of this decomposition, as we shall see, is to reduce contractions over four-dimensional space into contractions over three-space, which will prove quite useful later on.

## III. CONFORMAL GRAVITY IN CANONICAL FORM

Before quantizing the complete superconformal theory, we will find it useful to first start by quantizing the smaller conformal group, which is locally invariant only under the 15 local conformal transformations. Many features found in this simpler discussion will carry over into the much more involved superconformal theory.

We will simply set all fermionic fields and the vector fields to zero in order to obtain the conformal action:

$$L=\alpha\epsilon^{\mu\nu\rho\sigma}\epsilon^{abcd}R^{ab}_{\mu\nu}(M)R^{cd}_{\rho\sigma}(M)$$
$$=64(-R^{\mu\nu}f_{\mu\nu}+\tfrac{1}{2}Rf+2f^2-2f_{\mu\nu}f^{\nu\mu})e\alpha\ ,\tag{3.1}$$

where $R_\mu{}^b=R^{0ab}_{\mu\nu}e^{a\nu}$, $R=R_\mu{}^a e^{a\mu}$, and $f=e^{a\mu}f^a{}_\mu$ (we have dropped surface terms).

This theory cannot be put into canonical form because it does not yet contain the desired number of fields. In a previous paper, we were able to quantize conformal gravity (in metric form, only) because we introduced auxiliary variables, like the second fundamental form

$$k_{ij}=(N_{i\,|\,j}-N_{j\,|\,i}-\dot g_{ij})/2N\ .\tag{3.2}$$

In order to introduce the second fundamental form into the action, we simply added a new term into the Lagrangian whose solution gave rise to the previous equation:

$$\pi^{ij}(2Nk_{ij}-N_{i\,|\,j}-N_{j\,|\,i}+\dot g_{ij})\ .\tag{3.3}$$

(The vertical slash, as usual, represents the covariant derivative with the Christoffel symbols using only the three-dimensional metric ${}^3g_{ij}$.)

In conformal theory defined over tetrad fields, rather than simply metric fields, we must define a generalized second fundamental form.

A natural choice for the generalized second fundamental form emerges as we reformulate the theory in 3 + 1 dimensions. Let us define

$$\omega_i^{ab}(e)\equiv\tfrac{1}{2}[e^{aj}(e^b{}_{j,i}-e^b{}_{i,j})$$
$$+\tfrac{1}{2}e^{ak}e^{bj}(e^c{}_{k,j}-e^c{}_{j,k})e^c{}_i$$
$$+n^ae^{bj}n^c(e^c{}_{i,j}-e^c{}_{j,i})-2n^an^b{}_{,i}]$$
$$-(a\leftrightarrow b)\ ,$$
$$\nabla_iA^a\equiv\partial_iA^a+\omega_i^{ab}(e)A^b\ ,\tag{3.4}$$
$$k^a{}_i\equiv[-e^a{}_i+\partial_ie^a{}_0+\omega_i^{ab}(e)e^b{}_0+Ne^a{}_in^cb^c]/N\ .$$

We can add the generalized second fundamental form into the action by simply introducing, as before, the term

$$\pi^{ai}(Nk^a{}_i-\nabla_ie^a{}_0+\dot e^a{}_i-Ne^a{}_in^cb^c)\ .\tag{3.5}$$

Written in this form, we can now decompose the connection fields into the proper 3 + 1 form,

$$\omega_i^{ab}=-\omega_i^{ab}(e)+[e^{aj}k_{ji}n^b+e^b{}_ie^{ak}b_k-(a\leftrightarrow b)]\ ,$$
$$\omega_0^{ab}=\tfrac{1}{2}N({}^4e^{aj}k^b{}_j+e^{aj}e^{b0}e^ck^c{}_j)$$
$$+e^b{}_0e^{ak}b_k-(a\leftrightarrow b)$$
$$=-Nk^{ab}+[e^{ak}n^bk_{ki}N^i-e^{bk}e^a{}_0b_k$$
$$-(a\leftrightarrow b)]\ ,\tag{3.6}$$
$$k_{ij}\equiv\tfrac{1}{2}(e^a{}_ik^a{}_j+e^a{}_jk^a{}_i)\ .$$

Because of the large number of fields present even at the level of conformal gravity, it will be useful to clearly define the 3 + 1 decomposition of all fields.

The generalized second fundamental form $k^a{}_i$, which has 12 independent components, will be decomposed into a six-component symmetric field $k_{ij}$ and a six-component antisymmetric field $k^{ab}$ as follows:

$$k^{ab} \equiv \tfrac{1}{2}(e^{bj}k^a{}_j + n^c k^c{}_j e^{aj} n^b) - (a \leftrightarrow b), \quad k \equiv k_i{}^i ,$$
$$k^a{}_i = k^{ab} e^b{}_i + e^{aj} k_{ji} . \tag{3.7}$$

We will eventually find that the field $k^{ab}$ becomes the Lagrange multiplier for local Lorentz transformations, while the symmetric field $k_{ij}$ splits into two parts: the traceless part becomes an independent field, while the trace of $k_{ij}$ becomes a separate independent field, a scalar under the three-dimensional group.

In addition, we can decompose the dilatation field $b_\mu$ into its components

$$b^a = n^a(-b) + e^{ak} b_k ,$$
$$b \equiv n^a b^a . \tag{3.8}$$

We will find that the $b_k$ become independent fields, while the field $b$ becomes the Lagrange multiplier for local scale transformations.

Finally, we decompose the 16 fields contained within $f^a{}_\mu$ into a three-vector $f_{in}$, another three-vector $f_{ni}$, two scalars $f_{nn}$ and $f_k^k$ and a symmetric traceless tensor $f_{ij}$, while the antisymmetric tensor $a_{ij}$ vanishes from the action:

$$f_{\mu\nu} \equiv {}^4e^a{}_\mu f^a{}_\nu, \quad f_{ij} \equiv e^a{}_i f^a{}_j ,$$
$$f_{in} \equiv -N f_i{}^0 = -N e^a{}_i f^a{}_\mu {}^4g^{\mu 0} ,$$
$$f_{ni} \equiv -N f^0{}_i = -N e^{a0} f^a{}_i ,$$
$$\bar{f}_{ij} \equiv \tfrac{1}{2}(f_{ij} + f_{ji}) - \tfrac{1}{3} g_{ij} f_k{}^k ,$$
$$a_{ij} \equiv \tfrac{1}{2}(f_{ij} - f_{ji}) ,$$
$$f_{nn} \equiv N^2 f^{00} = N^2 e^{a0} f^a{}_\mu {}^4g^{\mu 0} . \tag{3.9}$$

Now that we have decomposed all fields into the proper 3 + 1 formalism, we can categorize all fields, whether they become Lagrange multipliers, or dependent fields, or independent fields.

The Lagrange multipliers for the 15-parameter conformal group are $e^a{}_0$ for general covariance, $k^{ab}$ for local Lorentz transformations, $f_{in}, f_{nn}$ for proper conformal transformations, $b = n^a b^a$ for scale transformations.

The independent fields can now be paired off into canonical-conjugate momenta and independent coordinates,

$$(e^a{}_i, \pi^{ai}) ,$$
$$(\bar{k}_{ij}, \bar{f}^{ij}) ,$$
$$(b_i, f^{ni}) ,$$
$$(k, f_i{}^i) .$$

We are now in a position to decompose the action into its relevant pieces. Let us rewrite the Lagrangian as

$$L = 64\alpha N ({}^3g)^{1/2} [ -f_{ni} R^{ni} + f(\tfrac{1}{2} R_n^n + R/6) - \bar{f}^{ij} \bar{R}_{ij} + \tfrac{4}{3} f_j{}^j - 2\bar{f}^{ij} \bar{f}_{ij} + 2a_{ij} a^{ij} - a_{ij} {}^3R^{ij}$$
$$- f_{in}(R^{in} + 4f^{ni}) + f_n^n(\tfrac{1}{2} R_j{}^j - \tfrac{1}{2} R_n^n + 4 f_i{}^i) ] , \tag{3.10}$$

where the bar over a field means that we must extract out its trace, and where

$$R_{ij} = R_{i\nu}^{ab4} e^{a\nu} e^b{}_j = n^a R_{i0}^{ab} e^b{}_j / N + R_{ik}^{ab} (e^{ak} - n^a N^k / N) e^b{}_j ,$$
$$R_{in} = -n^b R_{i\nu}^{ab4} e^{a\nu} = -R_{ik}^{ab} n^b e^{ak} ,$$
$$R_{ni} = -N R_{\mu\nu}^{ab\,4} e^{a\nu} g^{\mu 0} e^b{}_i = R_{0k}^{ab} e^{ak} e_{bi} / N - N^k R_{kj}^{ab} e^{aj} e^b{}_i ,$$
$$R_{nn} = N^2 R_{\mu\nu}^{ab\,4} e^{a\nu} g^{\mu 0} e^{b0} = -R_{0k}^{ab} e^{ak} n^b / N + N^k R_{kj}^{ab} e^{aj} n^b / N . \tag{3.11}$$

It is now a straightforward, though very tedious, task to rewrite the above curvatures in terms of the 3 + 1 decomposition given earlier. Substituting the expressions for the $\omega$'s in terms of $e$, $k$, and $b$, we now find

$$R_{in} = -\tfrac{2}{3} \partial_i k + \bar{k}^j{}_{i\,|\,j} + \tfrac{2}{3} b_i k + 2b^j \bar{k}_{ji} ,$$
$$R_{ij} = \dot{k}_{ij} / N + N_{|\,i\,|\,j} / N + 2k_{im} k^m{}_j - (N^m{}_{|\,j} k_{mi} + N^m{}_{|\,i} k_{mj}) / N$$
$$- N^m{}_{ij\,|\,m} / N + {}^3R_{ij} - 2b_i b_j + 2b_{j\,|\,i} - k_{ij} b - k k_{ij} + g_{ij} [2b_k b^k + (1/N) b^k N_{,k} + 2b^k{}_{|\,k}] ,$$
$$R_n^n = \dot{k}/N - N^i \partial_i k / N - \bar{k}_{ij} \bar{k}^{ij} + N^{|\,i}{}_{|\,i} / N - k^2/3 + bk + 3N_{|\,k} b^k / N + 2b_k b^k + b^k{}_{|\,k} ,$$
$$R_{ni} = -\tfrac{2}{3} k_{,i} + \bar{k}^j{}_{i\,|\,j} + \tfrac{2}{3} b_i k \;\; 2b_{,i} + 2\dot{b}_i / N + 2b^j k_{ji} - 2(N^j{}_{|\,i} b_j / N + N_{,i} b / N) - 2N^j b_{i\,|\,j} / N . \tag{3.12}$$

We are now in a position to complete the last step of the calculation, which is to rearrange all terms so that we have the desired canonical form. A simple rearrangement of all terms yields

$$L=\pi^{ai}\dot{e}^a{}_i-(2f^{ni}\dot{b}_i+\bar{f}^{ij}\dot{\bar{k}}_{ij}-\tfrac{2}{3}f\dot{k})({}^3g)^{1/2}+e^a{}_0P^a+Nb(D)+f^{an}NK^a+k^{ab}M^{ab}N\ , \tag{3.13}$$

where $f^{a0}\equiv f^a{}_\mu{}^4g^{\mu 0}$, $\alpha=\frac{1}{64}$ (for convenience), $H(p,q)=0$ (actually, the Hamiltonian is a total derivative, which is important depending on the boundary conditions), and

$$\begin{aligned}K^a=&({}^3g)^{1/2}e^{ai}(4f^n{}_i+\tfrac{2}{3}k_{,i}-\nabla^*_j\bar{k}_i{}^j-\tfrac{2}{3}b_ik)\\&+({}^3g)^{1/2}n^a(-\tfrac{1}{2}\bar{k}_{ij}\bar{k}^{ij}+\tfrac{1}{3}k^2+3b_kb^k-2\nabla^*_kb^k-4f_k{}^k-\tfrac{1}{2}{}^3R)\ ,\end{aligned} \tag{3.14}$$

$$\begin{aligned}P^a=&({}^3g)^{1/2}n^a[-2\bar{f}_{ij}\bar{f}^{ij}+f^{ni}(-\nabla^*_j\bar{k}_i{}^j+\tfrac{2}{3}k_{,i}-\tfrac{2}{3}kb_i)\\&\qquad+\bar{f}^{ij}(-2\bar{k}_{im}\bar{k}_j{}^m+\tfrac{1}{3}k\bar{k}_{ij}-{}^3R_{ij}-2\nabla_ib_j+2b_ib_j)\\&\qquad+f_k{}^k(-k^2/3+{}^3R/6+\tfrac{5}{3}b_kb^k+\tfrac{4}{3}\nabla_kb^k-\tfrac{1}{2}\bar{k}_{ij}\bar{k}^{ij})\\&\qquad+\tfrac{2}{3}\nabla^2f_k{}^k-2\nabla_k(b^kf_i{}^i)-\tfrac{1}{8}(b_{j,i}-b_{i,j})^2+\tfrac{4}{3}f_k{}^kf_j{}^j+\pi_{ij}k^{ij}/({}^3g)^{1/2}-\nabla_i\nabla_j\bar{f}^{ij}]\\&+({}^3g)^{1/2}e^{ai}[2b_i\nabla_jf^{nj}-2f^{nj}(b_{j,i}-b_{i,j})+\tfrac{2}{3}\nabla_j(k\bar{f}^j{}_i)+\bar{f}^{rs}\nabla_i\bar{k}_{rs}-\tfrac{2}{3}f_k{}^kk_{,i}]-\nabla_i\pi^{ai}\ ,\end{aligned} \tag{3.15}$$

$$D=({}^3g)^{1/2}(2\nabla_if^{ni}+\bar{k}_{ij}\bar{f}^{ij}+\tfrac{2}{3}kf_k{}^k)-\pi^{ai}e^a{}_i\ , \tag{3.16}$$

$$M^{ab}=\pi^{ab}+\tfrac{1}{2}(\pi^{ac}n^cn^b-\pi^{bc}n^cn^a)\ , \tag{3.17}$$

where we define

$$\nabla^*_iA_j\equiv\nabla_iA_j+g_{ij}b_kA^k-b_jA_i\ . \tag{3.18}$$

Notice that the $b$ field, which usually drops totally out of the action when we eliminate all components of the $f$ field, plays a crucial role in the canonical formalism of the theory. This is because we do not eliminate all components of the $f$ field, which also remains in the theory as a dynamical field.

Now that we have explicitly displayed the action in terms of canonical coordinates, we turn our attention to the more complicated superconformal theory with its 24 generators.

## IV. QUANTIZING SUPERCONFORMAL GRAVITY

Using the conformal theory as a guide, we can now perform the decomposition of superconformal gravity into canonical form. Many of the features of conformal gravity in canonical form carry over directly to the superconformal case.

As before, we want to keep the $f$ field as an independent field in the theory. A straightforward reduction of the original Lagrangian yields

$$\begin{aligned}L=&64\alpha e(-2f_{\mu\nu}f^{\nu\mu}+2f^2+\tfrac{1}{2}Rf-f_{\mu\nu}V^{\mu\nu})\\&+8\epsilon^{\mu\nu\alpha\beta}\alpha[3i\bar{\psi}_\mu\partial_\nu A_\beta\phi_\alpha-4\bar{\phi}_\mu\gamma_\nu\gamma_5D_\beta\phi_\alpha-3i\bar{\phi}_\mu\gamma_\nu A_\beta\phi_\alpha\\&\qquad+\tfrac{1}{2}i\bar{\psi}_\mu\phi_\nu R_{\alpha\beta}(A)-\bar{\psi}_\mu\sigma^{ab}\phi_\nu\bar{\psi}_\alpha\gamma_5\sigma^{ab}\phi_\beta]-8\alpha eR_{\mu\nu}(A)^2\ ,\end{aligned} \tag{4.1}$$

where

$$V_{\mu\nu}=R_{\mu\nu}(M,f=\phi=0)-\tfrac{1}{4}i\tilde{R}_{\mu\nu}(A)-\tfrac{1}{2}R_{\lambda\nu}(Q)\gamma_\mu\psi^\lambda-\bar{\psi}_\mu\sigma^\alpha{}_\nu\phi_\alpha-\bar{\phi}_\mu\sigma^\alpha{}_\nu\psi_\alpha\ . \tag{4.2}$$

[The $R(M)$ curvature in the above equation must have the $f$ and $\phi$ fields set equal to zero.]

Written in this form, it is not at all obvious that the Lagrangian can be put into canonical form. However, our work is made possible by the rather remarkable identity, which reduces out most unnecessary fields:

$$V_{\mu\nu}-V_{\nu\mu}=-\bar{\psi}_\mu\phi_\nu+2b_{\mu,\nu}-\tfrac{1}{4}i\tilde{R}_{\mu\nu}(A)-(\mu\leftrightarrow\nu)\ . \tag{4.3}$$

This identity will prove critical in eliminating out certain fields.

As in the case of conformal gravity, we begin the 3 + 1 decomposition of the Lagrangian by first decomposing the connection fields for local Lorentz invariance and by defining a generalized second fundamental form, which now are functions of fermionic fields,

$$\omega_i^{ab}=-\omega_i^{ab}(e)+[e^{aj}n^b k_{ij}+e^b{}_i e^{ak}b_k+(\bar{\psi}_i\gamma^b e^{ak}\psi_k-\tfrac{1}{2}\bar{\psi}_k e^{ak}\gamma^c e^c{}_i e^{bm}\psi_m)-(a\leftrightarrow b)]\ ,$$
$$\omega_0^{ab}=\tfrac{1}{2}N({}^4e^{aj}k^b{}_j+e^{ak}e^{b0}e^c{}_0k^c{}_k)+\tfrac{1}{4}N^k\bar{\psi}_k\gamma^b e^{ai}\psi_i-e^a{}_0e^{bk}b_k-\tfrac{1}{4}N\bar{\psi}^n n^a\psi^b-\tfrac{1}{8}\bar{\psi}_a\gamma_0\psi_b-(a\leftrightarrow b)\ , \tag{4.4}$$
$$k^a{}_i=(1/N)[-\dot{e}^a{}_i+\nabla_i e^a{}_0+Ne^a{}_i n^c b^c+\tfrac{1}{2}(\bar{\psi}_0\gamma^a\psi_i+\bar{\psi}_i\gamma^a N^k\psi_k)]\ .$$

Before we begin the final reduction of the Lagrangian, we must add a few new terms to (4.1). First, we must convert (4.6) by adding a Lagrange multiplier $\pi^{ai}$, much the same way that (3.5) was added into the conformal action. Second, we must add a fermionic auxiliary field $\chi$ into the action in order to accommodate the fermionic constraint (2.5). We accomplish this by adding the term

$$e\widetilde{R}_{\mu\nu}(Q)\gamma^\nu\chi^\mu \tag{4.5}$$

into the action. And third, we introduce the field $\pi^k$ into the action, which has the net effect of reducing the term $R_{\mu\nu}(A)^2$ into a term linear in the curvature, in much the same way that Maxwell's equations are put into canonical form by adding in the conjugate field. This conjugate field will, of course, become the conjugate field to the $A$ field. Notice that the net effect of all three additions to the action is to reduce a higher-derivative theory to a theory with lower numbers of time derivatives.

In order to reduce out the term (4.2), we will find the following identity useful:

$$\begin{aligned}R_{\mu\nu}(M,f=\phi=0)=&R_{\mu\nu}(M,f=\phi=\psi=b=0)+\tfrac{1}{2}D^0{}_\mu(\bar{\psi}^\alpha\gamma_\alpha\psi_\nu)-\tfrac{1}{4}D^0{}_\alpha(\Delta_\mu{}^\alpha{}_\nu)-2b_\mu b_\nu+2\nabla_\mu b_\nu+g_{\mu\nu}\nabla_\alpha b^\alpha\\&-\tfrac{1}{16}\Delta_{\mu\alpha\beta}\bar{\psi}^\alpha\gamma_\nu\psi^\beta-\tfrac{1}{8}(\bar{\psi}^\alpha\gamma_\alpha\psi_\beta)\Delta_\mu{}^\beta{}_\nu+\tfrac{1}{2}(-\bar{\psi}_\mu\gamma^\alpha\psi_\nu b_\alpha+\bar{\psi}^\alpha\gamma_\alpha\psi^\beta b_\beta g_{\mu\nu}-\bar{\psi}^\alpha\gamma_\mu\psi_\nu b_\alpha\\&-\bar{\psi}^\alpha\gamma_\alpha\psi_\mu b_\nu)\ ,\end{aligned} \tag{4.6}$$

where

$$\Delta_{\alpha\beta\gamma}=\bar{\psi}_\alpha\gamma_\beta\psi_\gamma-\bar{\psi}_\alpha\gamma_\gamma\psi_\beta+\bar{\psi}_\beta\gamma_\alpha\psi_\gamma\ , \tag{4.7}$$

and where $D^0$ represents the usual covariant derivative with $b=\psi=0$.

Using the above identities, we can now reduce out totally the curvatures given in (4.2) in terms of independent and dependent fields. The calculation is quite tedious and not very illuminating. Instead, we will simply give the final result for the entire reduced action:

$$\begin{aligned}L=&\pi^{ai}\dot{e}^a{}_i+\tfrac{2}{3}\dot{k}f_i{}^i\sqrt{g}-\dot{k}_{ij}\bar{f}^{ij}\sqrt{g}\\&-2f^{ni}\sqrt{g}\,\partial_0(b_i+\tfrac{1}{4}\bar{\psi}^k\gamma_k\psi_i)-\tfrac{3}{8}\dot{A}_k(\sqrt{g}\,\pi^k+i\bar{\psi}_j\phi_i\epsilon^{jki})+\tfrac{1}{2}\bar{\pi}^k_\phi\dot{\phi}_k+\bar{\pi}^k_\psi\dot{\psi}_k+\bar{\pi}^k_\chi\dot{\chi}_k\\&+N\bar{\psi}^nQ+N\bar{\phi}^nS+N\bar{\chi}^nT+NbD+Nf^{an}K^a+e^a{}_0P^a+NK^{ab}M^{ab}+NA^n\theta\ ,\end{aligned} \tag{4.8}$$

where the Hamiltonian is again a total derivative, where $\alpha=\frac{1}{64}$ for convenience, and

$$\begin{aligned}K^a=K^{(0)a}&+e^{ai}\sqrt{g}\,[\tfrac{1}{4}\nabla_j^*(\bar{\psi}_i\gamma\psi_j)-\bar{\psi}_i\sigma^{ka}n^a\phi_k-\bar{\phi}_i\sigma^{ka}n^a\psi_k+(\tfrac{1}{4}i/\sqrt{g})\epsilon_i{}^{jk}R_{jk}(A)-(\tfrac{1}{4}\sqrt{g})\epsilon^{jkl}R_{kl}(Q)\gamma_5\gamma_i\psi_j\\&\qquad-\tfrac{1}{6}k\bar{\psi}^j\gamma_j\psi_i-\tfrac{1}{8}\bar{\psi}_j\gamma\psi_k\bar{\psi}_i\gamma^i\psi^j]\\&+n^a\sqrt{g}\,[-\tfrac{1}{4}\nabla_i^*(\bar{\psi}^k\gamma_k\psi^i)+\tfrac{1}{32}\Delta_{ijk}\bar{\psi}^j\gamma^i\psi^k+\bar{\psi}_k\sigma^{jk}\phi_j+(\tfrac{1}{4}/\sqrt{g})\epsilon^{ijk}R_{jk}(Q)\gamma_5\gamma\psi_i]\ ,\end{aligned} \tag{4.9}$$

$$\begin{aligned}P^a=P^{(0)a}+n^a\sqrt{g}\,\{&f^{ni}[-\tfrac{1}{4}\nabla_j(\bar{\psi}^j\gamma\psi_i)+\tfrac{8}{9}k\bar{\psi}^j\gamma_j\psi_i-\tfrac{1}{2}\bar{\psi}^jb_j\gamma\psi_i-\tfrac{1}{2}R_{ji}(Q)\gamma\psi^j-(\tfrac{1}{4}i/\sqrt{g})\epsilon_i{}^{jk}R_{jk}(A)]\\&+\bar{f}^{ij}[\tfrac{1}{2}\bar{\psi}^kb_k\gamma_i\psi_j+\tfrac{1}{16}\Delta_{ilk}\bar{\psi}^l\gamma_j\psi^k-\tfrac{1}{2}\nabla_i^*(\bar{\psi}^k\gamma_k\psi_j)\\&\qquad+\tfrac{1}{4}\nabla_k(\Delta_i{}^k{}_j)-\tfrac{1}{2}\bar{\psi}_i\gamma\psi^m\bar{k}_{mj}+\bar{\psi}_i\sigma^k{}_j\phi_k+\bar{\phi}_i\sigma^k{}_j\psi_k+\tfrac{1}{2}R_{kj}(Q)\gamma_i\psi^k]\\&-\tfrac{1}{4}\nabla_k(\Delta_i{}^k{}_j\bar{f}^{ij})+f_k{}^k[\tfrac{1}{12}\nabla_i(\bar{\psi}^k\gamma_k\psi^i)+\tfrac{1}{32}\bar{\psi}_i\gamma\psi_j\bar{\psi}^i\gamma\psi^j-\tfrac{1}{96}\Delta_{ijk}\bar{\psi}^j\gamma^i\psi^k\\&\qquad+\tfrac{1}{6}\epsilon^{ijk}R_{ij}(Q)\gamma_5\gamma\psi_k+\bar{\psi}^i\sigma^k{}_i\phi_k+\tfrac{1}{3}\bar{\psi}^k\gamma_k\psi^ib_i]\\&+(\tfrac{1}{2}/\sqrt{g})\bar{\phi}_i\gamma\gamma_5D_j\phi_k\epsilon^{ijk}+(3i/8\sqrt{g})\bar{\phi}_i\gamma A_j\phi_k\epsilon^{ijk}-\tfrac{1}{8}R_{ij}(A)^2-\tfrac{3}{16}\pi_k\pi^k-\tfrac{1}{48}(\bar{\psi}_i\phi_j-\bar{\psi}_j\phi_i)^2\\&+\tfrac{1}{12}(\bar{\psi}^i\phi^j-\bar{\psi}^j\phi^i)(b_{i,j}-b_{j,i})-(i/4)\epsilon^{ijk}\pi_k(b_{i,j}-\tfrac{1}{2}\bar{\psi}_i\phi_j)\end{aligned}$$

$$
\begin{aligned}
&\quad -\tfrac{1}{12}(b_{i,j}-b_{j,i})^2+\epsilon^{ijk}[-R_{ij}(Q)-D_i\bar{\psi}_j]\gamma\chi_k/\sqrt{g}\\
&\quad +(2/\sqrt{g})\bar{\phi}_i\gamma\gamma_j\chi_k\epsilon^{ijk}-\bar{\psi}_i\gamma D_j\chi_k\epsilon_{ijk}/\sqrt{g}+\bar{\psi}_i\gamma\chi_k\epsilon^{ijk}b_j\sqrt{g}\\
&\quad -\bar{\psi}^a\gamma\psi^b\tfrac{1}{4}[-\tfrac{1}{2}\epsilon^{ijk}\bar{\phi}_i\gamma_j\gamma_5\sigma^{ab}\phi_k-\epsilon^{ijk}\bar{\psi}_i(\gamma_j,\sigma^{ab})_-\chi_k]\}\\
&+e^{am}\sqrt{g}\,\{\tfrac{1}{2}f^{nk}\nabla_m(\bar{\psi}^j\gamma_j\psi_k)-\tfrac{1}{2}\nabla_k(f^{nk}\bar{\psi}^j\gamma_j\psi_m)\\
&\quad -\tfrac{3}{8}\epsilon^{ljk}[A_m\nabla_j(\bar{\psi}_l\phi_k)-\bar{\psi}_l\phi_k(A_{j,m}-A_{m,j})]-\tfrac{3}{8}\sqrt{g}A_m\nabla_k\pi^k-\tfrac{3}{8}\sqrt{g}\,\pi^j(A_{j,m}-A_{m,j})\\
&\quad +\epsilon^{ijk}\nabla_k(\tfrac{1}{2}\bar{\phi}_i\gamma_j\gamma_5\phi_m+\bar{\psi}_i\gamma_j\chi_m+\bar{\psi}_m\gamma_j\chi_k)-\tfrac{1}{2}\epsilon^{ijk}\bar{\phi}_i\gamma_j\gamma_5D_m\phi_k\\
&\quad -\epsilon^{ijk}\bar{\psi}_i\gamma_jD_m\chi_k+\epsilon^{ijk}(D_m\bar{\psi}_i)\gamma_j\chi_k+(2k_{mj}e^{aj}n^b+\tfrac{1}{2}\bar{\psi}_m\gamma^bn^a-\tfrac{1}{4}\bar{\psi}_a\gamma_m\psi_b)\\
&\quad \times[-\tfrac{1}{2}\epsilon^{ijk}\bar{\phi}_i\gamma_j\gamma_5\sigma^{ab}\phi_k-\epsilon^{ijk}\bar{\psi}_i(\gamma_j,\sigma^{ab})_-\chi_k]\}\;,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
Q/\sqrt{g}&=\tfrac{1}{2}\gamma^j\psi_k\nabla^*_jf^{nk}-\gamma^k\psi_k\nabla_if^{ni}+f^{ni}[\tfrac{1}{3}\gamma\psi_ik-\tfrac{1}{2}b^k\gamma_k\psi_i-\tfrac{1}{4}\gamma_k\psi_i\bar{\psi}^j\gamma_j\psi^k-\tfrac{1}{2}\gamma\psi_j\bar{k}^j{}_i\\
&\qquad +\tfrac{1}{8}\gamma\psi^j\bar{\psi}_j\gamma\psi_i-\tfrac{1}{4}\sqrt{g}\,\gamma_5\gamma R_{jk}(Q)\epsilon_i{}^{jk}-\sigma_{ji}\phi^j]\\
&\quad +\bar{f}^{ij}[\tfrac{1}{3}\gamma_i\psi_j+\tfrac{1}{8}\gamma_k\psi_j\bar{\psi}_i\gamma\psi^k+\tfrac{1}{2}\gamma^m\psi_jk_{im}+n^a\sigma^a{}_j\phi_i+(\tfrac{1}{4}g)\epsilon_j{}^{km}\gamma_5\gamma_i\bar{R}_{km}(Q)]\\
&\quad +\tfrac{1}{2}\gamma\psi_j\nabla^*_i\bar{f}^{ij}-b_i\gamma_j\chi_k\epsilon^{ijk}/\sqrt{g}+\gamma_iD_j\chi_k\epsilon^{ijk}\sqrt{g}+D_i(\gamma_j\psi_k\epsilon^{ijk})/\sqrt{g}\\
&\quad +f_k{}^k[-\tfrac{2}{3}\gamma\psi^ib_i-\tfrac{1}{6}\gamma\psi_i\bar{\psi}^j\gamma_j\psi^i+\tfrac{2}{3}\sigma^{ia}n^a\phi_i-\tfrac{1}{4}\gamma^j\psi^ik_{ij}+\tfrac{1}{48}\gamma^i\psi^k\bar{\psi}_i\gamma\psi_k]\\
&\quad +\tfrac{1}{3}\gamma\psi^k\nabla_k(f_i{}^i)+\tfrac{1}{4}\sigma^{ab}\phi_i\bar{\psi}_j\gamma_5\sigma^{ab}\phi_k\epsilon^{ijk}/\sqrt{g}-(i/16\sqrt{g})\phi_iR_{jk}(A)\epsilon^{ijk}+\tfrac{3}{8}i\nabla_iA_j\phi_k\epsilon^{ijk}/\sqrt{g}\\
&\quad -\tfrac{3}{16}\gamma_5\phi_k\pi^k+(3i/4)\gamma_5A_i\gamma_j\chi_k\epsilon^{ijk}/\sqrt{g}+\tfrac{1}{2}\gamma n_b\psi_a[-\tfrac{1}{2}\epsilon^{ijk}\bar{\phi}_i\gamma_j\gamma_5\sigma^{ab}\phi_k-\epsilon^{ijk}\bar{\psi}_i(\gamma_j,\sigma^{ab})_-\psi_k]/\sqrt{g}\\
&\quad +\gamma^a\chi_i\epsilon^{ijk}(\tfrac{1}{4}\psi_i\gamma^a\psi_j+e^a{}_ib_j)/\sqrt{g}-\tfrac{1}{2}\gamma^a\psi_j\epsilon^{ijk}\bar{\psi}_i\gamma^a\chi_k/\sqrt{g}\;,
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
S&=-\bar{f}^{ij}n^a\sigma^a{}_j\psi_i\sqrt{g}+(3i/8)\psi_i\nabla_jA_k\epsilon^{ijk}-\tfrac{1}{2}\gamma_i\gamma_5D_j\phi_k\epsilon^{ijk}\\
&\quad -(3i/4)\gamma_iA_j\phi_k\epsilon^{ijk}+\tfrac{1}{8}\sigma^{ab}\psi_i\bar{\psi}_j\gamma_5\sigma^{ab}\psi_k\epsilon^{ijk}+(i/8)\psi_iR_{jk}(A)\epsilon^{ijk}+\tfrac{2}{3}\sqrt{g}\,\sigma^{ia}n^a\psi_if_k{}^k\\
&\quad +(3i/16)\gamma_5\phi_k\pi^k\sqrt{g}-2\epsilon^{ijk}\gamma_i\gamma_j\chi_k-\tfrac{1}{2}\epsilon^{ijk}D_k(\gamma_j\gamma_5\phi_i)\;,
\end{aligned}
\tag{4.12}
$$

$$M^{ab}=M^{(0)ab}+\tfrac{1}{2}\epsilon^{ijk}\bar{\phi}_i\gamma_j\gamma_5\sigma^{ab}\phi_k-\bar{\psi}_i(\sigma^{ab},\gamma_j)_-\chi_k\epsilon^{ijk}\;, \tag{4.13}$$

$$\theta=(3i/8)\nabla_j(\bar{\psi}_i\phi_k)\epsilon^{ijk}-(3i/8)\bar{\phi}_i\gamma_j\phi_k\epsilon^{ijk}+\tfrac{3}{16}\sqrt{g}\,\nabla_k\pi^k-(3i/4)\bar{\psi}_i\gamma_5\gamma_j\chi_k\epsilon^{ijk}\;, \tag{4.14}$$

$$D=D^{(0)}+2\bar{\psi}_i\gamma_j\chi_k\epsilon^{ijk}\;, \tag{4.15}$$

$$\bar{\pi}^k_\phi\equiv\bar{\phi}_i\gamma_j\gamma_5\epsilon^{ijk},\;\;\bar{\pi}^k_\psi\equiv-\bar{\chi}_i\gamma_j\epsilon^{ijk},\;\;\bar{\pi}^k_\chi\equiv\bar{\psi}_i\gamma_j\epsilon^{ijk}\;, \tag{4.16}$$

$$T=-2\epsilon^{ijk}\gamma_k[\gamma_i\phi_j+\tfrac{1}{2}b_i\psi_j-(3i/4)A_i\gamma_5\gamma_j-\tfrac{1}{2}D_i\psi_j]+\gamma^a\psi_i\epsilon^{ijk}(\tfrac{1}{4}\bar{\psi}_j\gamma^a\psi_k+e^a{}_jb_k)-\epsilon^{ijk}D_k(\gamma_i\psi_j)\;, \tag{4.17}$$

and where we have used the following definitions:

$$n^a\gamma^a=\gamma,\;\;\nabla^*_iA_j=\nabla_iA_j+A^k\Delta_{ijk}+g_{ij}b^kA_k-b_jA_i\;,$$

$$g={}^3g,\;\;\psi^n=N^4g^{0\alpha}\psi_\alpha,\;\;(\gamma_i,\sigma^{ab})_-=\gamma_i\sigma^{ab}-\sigma^{ab}\gamma_i,\;\;\epsilon^{ijk}=\epsilon^{0ijk}\;.$$

The symbol (0) over the generators of the algebra represent the original generators found in (3.18), with the exception that the asterisk in (3.18) is to be replaced by the generalized star symbol given above, and that all $k_{ij}$'s now are functions of the fermion fields as well. We also use the notation that the $D$ derivative on a fermion field requires a connection field given by (2.6), while the $\nabla$ derivative on a fermion field requires the connection given by (3.8).

This completes our discussion of the construction of the generators of the algebra. Presumably, these generators have Poisson brackets which close on themselves, generating a representation of the superconformal group.

## V. CONCLUSION

We have constructed the canonical formalism for superconformal gravity, which is required before we can rigorously quantize the model. The correct Hilbert space of the theory must satisfy

$$C^a|\psi\rangle=0\ ,$$

where the $C^a$ represent the constraints on the theory. Our conjecture is that solutions of the above equation using nonperturbative methods may bear little, if any, resemblance to the Hilbert space generated by perturbation theory. If this new Hilbert space is positive definite, then we have a new candidate for a renormalizable theory of gravity which is unitary.

In the next paper, we will complete the straightforward but tedious steps necessary to calculate in the strong-coupling limit.

First, we will construct the "reduced Hamiltonian." Notice that the Hamiltonian in (4.8) is formally *zero*. This is because we have neglected surface terms. When we carefully calculate variations of (4.8) under changes in the variables, we see surface terms are necessary in order to complete Hamilton's equations. In particular, we will see that the Hamiltonian is actually a reduced version of $n^aP^a$, just as in the usual gravity theory.

Second, we will make a redefinition of fields to bring out the strong-coupling limit. In particular, we will find that the term $\bar{f}_{ij}\bar{f}^{ij}$ in (3.15) naturally emerges as the dominant term. As expected, we also find that the usual spin-2 gravitons with two helicities are not solutions to the constraint equations.

Third, we must go to the lattice approximation in order to construct a Hilbert space which is quantized (rather than continuous). We will construct some solutions to the Dirac constraints, and we will see that stringlike states (with zero energy to lowest order) emerge as solutions.

We will also ask the difficult question: is the energy spectrum in this strong-coupling limit positive? In the canonical approach, unitarity is probably restored, but the price one may have to pay is the positivity of the energy. Unfortunately, this must remain an open question because the positivity of the energy probably cannot be established to any finite order in the strong-coupling expansion.

And finally, we will make the conjecture that the solutions to the Dirac constraints represent the "real" graviton, i.e., the real graviton actually has structure in the same sense as glueballs having structure. Thus, at large distances we see the graviton as a point particle, while at distances close to the Planck length the graviton begins to exhibit structure (which is the reason why the theory is renormalizable while the usual theory of gravity is not). Furthermore, if we single out these composite graviton states and neglect all others, we find that conformal invariance is necessarily broken. In other words, if we only keep these composite graviton states and discard the rest, the net effect is to dynamically break local scale invariance.

Of course, the next step, then, is to actually show that graviton-graviton interactions actually reproduce the usual Einstein-Hilbert action in the long-distance approximation.

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[1]M. Kaku and P. Townsend, Phys. Lett. 76B, 54 (1978).

[2]M. Kaku, P. Townsend, and P. van Nieuwenhuizen, Phys. Rev. D 17, 3179 (1978); Phys. Lett. 69B, 304 (1977); Phys. Rev. Lett. 39, 1109 (1977).

[3]A. Zee, Phys. Rev. Lett. 42, 417 (1979); 44, 703 (1980).

[4]L. Smolin, Nucl. Phys. B160, 253 (1979).

[5]Y. Fujii, Phys. Rev. D 9, 874 (1974); P. Minkowski, Phys. Lett. 71B, 419 (1977); T. Matsuki, Prog. Theor. Phys. 59, 235 (1978); A. D. Linde, Pis'ma Zh. Eksp. Teor. Fiz. 30, 479 (1979) [JETP Lett. 30, 447 (1979)].

[6]S. Adler, Phys. Rev. Lett. 44, 1567 (1980); K. Akama, Y. Chikashige, T. Matsuki, and H. Terazawa, Prog. Theor. Phys. 60, 1900 (1980).

[7]E. Tomboulis, Phys. Lett. 70B, 361 (1977); 97B, 77 (1980). See also B. Hasslacher and E. Mottola, *ibid.* 95B, 237 (1980); 99B, 221 (1981); S. L. Adler, *ibid.* 95B, 241 (1980); Rev. Mod. Phys. 54, 729 (1982).

[8]M. Kaku, Nucl. Phys. B203, 285 (1982).