

Topological theory of hadrons. I. Mesons

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Spin is incorporated into the hadronic topological expansion scheme. Spin analogs of Chan-Paton factors are introduced in a way that avoids the troubles encountered in earlier attempts. Those troubles, at the meson level, were, first, the occurrence of twice the wanted number of pseudoscalar and vector mesons; second, the occurrence of parity-doublet partners of the pseudoscalar and vector mesons; and third, the occurrence of these parity-doublet partners as particles of negative metric, called ghosts. These troubles are all avoided by introducing a new topological level, called zero entropy, that lies below the ordered level. At the zero-entropy level quarks of opposite chirality are treated as distinct particles. The theory has been extended to all hadrons, and the basic particles are exactly those of the constituent-quark model, which for baryons start with the (56^+) and (70^-) . The theory is formulated in the M -function framework, where the "quarks" are represented by two-component spinors, and it entails $SU(6)_W$ symmetry of the hadronic vertices at a low level of the topological expansion.

I. INTRODUCTION

A scattering amplitude can be represented as a sum of contributions from all ways in which the process can occur. Each contribution has a phase factor, and the scattering amplitude between randomly chosen states tends to be small due to an averaging out of these phase factors. The dominant transitions are between states in which the elements of order characterizing the initial state are carried into the final state in some "direct" way.

This tendency of the dominant transitions to preserve order is particularly important in hadron physics, due to the inherent complexity of the hadrons and their interactions. Indeed, this order-preserving tendency has been made the basis of a successful approximation procedure for meson physics. This procedure is based not on the smallness of any coupling constant but rather on the smallness of contributions that do not preserve order. Order is defined so that it is preserved by contributions to the scattering amplitude that correspond to sequences of scattering events represented by graphs that can be drawn in a plane with no lines crossing. Contributions from nonplanar graphs generally have phase factors that tend to average to zero in high-energy regimes.

This topological approach to hadron dynamics, which originated in some works by Veneziano,¹ and has been pursued by many workers, has been recently reviewed by Chew and Rosenzweig.² They show

how the topological expansion procedure, combined with the requirements of unitarity, analyticity, duality, and Lorentz invariance, organizes and predicts many of the dominant features of meson physics.

The successes achieved in meson physics by this topological approach have motivated efforts to develop it into a comprehensive basis for particle physics. The most obvious deficiency of earlier work is its restriction to mesons. Accordingly, one major aim of the present two-part work is to extend the theory to baryons. Paper II³ is devoted to that task.

But beyond this problem of baryons, there lie other problems of equal importance. To provide a satisfactory basis for particle physics, the theory must, first of all, provide a practical method of determining, through the nonlinear bootstrap conditions, the magnitudes of all coupling constants that occur in particle physics. Phenomenological analyses indicate that the ratios of the hadronic coupling constants satisfy $SU(6)_W$ symmetry to a degree unlikely to be purely accidental.⁴ It is therefore probably essential to the practical viability of the topological approach, considered as a general basis for particle physics, that it be constructed so as to exhibit $SU(6)_W$ symmetry at a low level of topological complexity. Accordingly, a second major aim of this work is to construct a topological bootstrap framework that treats the spin degrees of freedom in a way that ensures $SU(6)_W$ symmetry at the lowest level of the topological expansion.

Historically, this $SU(6)_W$ property emerged, within the present work, as an unexpected by-product of the effort to generalize the Chan-Paton factors⁵ from isospin to ordinary spin. These original Chan-Paton factors enjoy the following important property: every product of amplitudes that contributes at the lowest topological level to any given amplitude has the same Chan-Paton factor. This product property ensures the existence, at the lowest topological level, of a solution to the isotopic-spin part of the dual topological bootstrap dynamical conditions.

It is not obvious that this solution is unique. However, it is a simple solution that is quite possibly unique. For, *a priori*, the infinite number of dual bootstrap conditions need have no solution at all.

To ensure the solubility of the spin part of these dynamical conditions, the following requirement is here imposed: the spin generalization of the Chan-Paton factors are required to satisfy the direct generalization of the product property enjoyed by the original Chan-Paton factors. This demand determines the basic character of the spin formalism described in this paper.

The problem of extending the Chan-Paton factors to ordinary spin was considered by Mandelstam in the late sixties. Mandelstam's work,⁶ like the present work, was based on M functions.⁷ These functions have the combined advantage of possessing simple crossing properties and a minimal number of spin components: crossing is represented by analytic continuation alone, and the redundant components that arise from describing spin- $\frac{1}{2}$ particles by four-component wave functions are avoided. Using these functions, one finds that the required product property (of the spin generalizations of the Chan-Paton factor) cannot be reconciled with the demand of invariance under parity. Thus, Mandelstam, proceeding in the straightforward way, summed two parity-reflected contributions, each of which individually satisfies the product property, in order to obtain a parity-invariant form. However, this procedure of simply summing two separate terms, each of which enjoys also simple factorization properties, led first to a parity-doublet partner for each of the observed mesons π, ρ, η, ω , and then to a second doubling of this set of mesons.

The procedure followed here differs from that of Mandelstam by its strict enforcement of the product-property requirement described above. The present procedure, originally introduced to ensure the solubility of the spin part of the dynamical conditions, generates a number of important further consequences. First, by keeping the treatment of spin closely parallel to the successful Chan-Paton,

treatment of isospin, it leads automatically to $SU(6)_W$ symmetry of the hadronic coupling constants, at the lowest topological level. Second, it automatically produces a basic set of mesons that accords exactly to the phenomenologically observed set $(\pi, \rho, \eta, \omega)$: there is no parity doubling, or any other doubling, of the meson spectrum. Third, when combined with the certain assumptions about the number of flavors, it leads to a supersymmetry connection between the meson-meson-meson coupling constants and the meson-baryon-baryon coupling constants. This connection is in good agreement with experiment. Fourth, when supplemented by a plausible universality requirement, it leads to a value of the ratio of the strong-interaction coupling constants to the electromagnetic coupling constant. This ratio is also in satisfactory agreement with experiment.⁸

The technical basis of these achievements is the fact, recognized and exploited already in the work of Chew and Rosenzweig, that different levels of the topological expansion can act in different Hilbert spaces. In the paper of Chew and Rosenzweig, the lowest level of the topological expansion was the "ordered" level, and each ordered amplitude was associated with a corresponding ordered Hilbert space. These ordered amplitudes were summed to form "planar" amplitudes, which were associated with new "planar" Hilbert spaces. Approximate correspondence to physical amplitudes was possible only at the planar and higher levels of the topological expansion. Yet, the ordered amplitudes were important, for the topological expansion concentrated all nonlinear aspects of the original unitarity equations in the ordered unitary equations satisfied by the ordered amplitudes. These ordered unitarity equations were much simpler than the original unitarity equations because they involved only planar Landau diagrams. Thus, the critical problem of determining the overall strength of the couplings was greatly simplified. Moreover, the representation of physical amplitudes by low-order terms in the topological expansion allowed many aspects of meson physics to be understood even without solving the nonlinear equations.²

The product-property requirement on the spin generalization of the Chan-Paton factors places severe conditions on the theoretical structure. These conditions can be satisfied by introducing a new topological level, called zero entropy, that lies below the ordered level. The individual zero-entropy amplitudes are not invariant under parity, and, like the ordered amplitudes, they cannot be regarded as approximations to the physical amplitudes. It is the planar amplitudes, which do conserve parity, that are again to be considered as the first approximation to the physical amplitudes.

In the work of Chew and Rosenzweig, the planar amplitudes were formed as finite sums of the ordered amplitudes. In the present framework, the planar amplitudes are formed as infinite sums of zero-entropy amplitudes. The infinite sums create new technical problems. But they also create the possibility, and, in fact, the necessity, of calculating, for example, the π - ρ mass difference. For at zero entropy, these two masses are equal.

The present work is part of a long-term collaborative effort with Geoffrey Chew to construct a practical basis for particle physics based on dual topological bootstrap dynamics. Chew's ideas are woven into it in many ways. However, this paper deals only with certain spinor, topological, and group-theoretic aspects of the whole theory; other important aspects are left untouched.

One problem not considered in this paper itself is the extension to all hadrons of the formalism developed herein for mesons. An extension is described in paper II. It incorporates also the group-theoretic properties of the constituent-quark model. The whole work is formulated completely within the S -matrix framework, and hence involves no microscopic description in terms of quark wave functions. Hence, it provides, in principle, the foundation of a Lorentz-covariant approach to particle physics that has a basic set of particles that agrees, as far as spin, parity, and other group-theoretic properties are concerned, with those of the constituent-quark model, yet has no confinement problem. Moreover, it incorporates $SU(6)_W$ symmetry, at the lowest topological level.

The present paper is associated with a recent paper by Chew and Poenaru⁹; it describes technical results that have been used in the development of their ideas. However, the aims of Chew and Poenaru are broader than those of the present work, which simply accepts the group-theoretic structure of the constituent-quark model on the basis of its empirical success. Chew and Poenaru seek to derive the group-theoretic structures from topological considerations and consequently need a richer topological structure than the one used here. Their topological structure contains, in addition to the quark-particle graphs of the present theory and surface upon which these graphs are imbedded, also a second surface, called the quantum surface, in which the group-theoretic relations associated with flavor and other symmetries reside.

In the present work, flavor is an unconstrained variable. The flavor structure may, in fact, be determined by the nonlinear dynamical equations, but it is not determined within the present framework by topological considerations alone.

Proposals for extending the theory to electroweak

interactions have been made recently by Chew, Finkelstein, McMurray and Poenaru.¹⁰⁻¹²

A crucial problem not addressed in any of these works is the development of reliable methods for solving the nonlinear conditions. These conditions should determine the overall strength of the hadronic and electroweak interactions. However, several calculations have been performed,¹³ and they all yield values that differ from the empirically observed overall strength of the hadronic-electroweak interaction by a factor of roughly 2. This result seems significant, particularly because the spinor, topological, and group-theoretic considerations introduce as many as 20 different factors of 2 into this result. These theoretical factors were calculated *prior* to the calculation of the approximate solutions to the nonlinear equations.

A second major problem not addressed in any of the published works is the development of reliable methods for constructing the planar amplitude from zero-entropy amplitudes. This construction must yield, for example, a first approximation to the π - ρ mass difference. Some calculations of this difference have been made, with encouraging results, but the work is still in a developmental stage.

Much of the work contained in this paper was completed several years ago,¹⁴ but was not submitted for publication because of the above-mentioned elements of incompleteness of the whole theory. However, a number of recently published papers⁸⁻¹² are based directly on the spin formalism developed in that earlier work. This fact, in conjunction with the encouraging character of works in progress, makes publication of this expanded version now appropriate.

The theory developed here is based on the M -function formalism. Since the original description of that formalism⁷ was very brief, the key points are described here in Sec. II, with particular emphasis on the results that are important in the context of the present work. The main body of the paper is contained in Sec. III. The results are summarized in Sec. IV. Appendix A shows that discontinuity equations, though usually considered in S -matrix theory as being derived from unitarity, are actually more basic than unitarity. Appendix B explains the failure of unitarity at the zero-entropy level. The planar discontinuity equations nevertheless continue to hold. Appendix C describes the connection of the two-component formalism used in the body of the paper to the four-component formalism based on Dirac matrices.

II. SPIN

A. Lorentz transformations in spin space

Let σ_μ represent the Pauli spin-matrix four-vector

$$\sigma_\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (1, \vec{\sigma}), \quad (2.1)$$

where σ_0 is the two-by-two unit matrix and $\sigma_1, \sigma_2,$ and σ_3 are the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (2.2)$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let A and B be any two-by-two matrices with determinant 1. Then the Lorentz transformation matrix $L^\mu{}_\nu(A, B)$ is defined by

$$A \sigma_\mu B = \sigma_\nu L^\nu{}_\mu(A, B) \equiv (\sigma \cdot L)_\mu. \quad (2.3)$$

(Repeated vector and spinor indices are always to be summed.)

Let $\tilde{\sigma}_\mu$ represent the Pauli spin-matrix four-vector

$$\tilde{\sigma}_\mu = (1, -\vec{\sigma}). \quad (2.4)$$

Then

$$\frac{1}{2} \text{Tr} \sigma_\mu \tilde{\sigma}_\nu = g_{\mu\nu}, \quad (2.5)$$

where $g_{\mu\nu}$ is the Lorentz metric tensor with diagonal elements $(1, -1, -1, -1)$.

Let $C = -i\sigma_2 = -C^{\text{Tr}}$ be the (charge) conjugation matrix, and let M be any two-by-two matrix. Then the Pauli identity

$$C^{-1} M^{\text{Tr}} C M = \det M \quad (2.6)$$

entails that

$$C^{-1} \sigma_\mu^{\text{Tr}} C = \tilde{\sigma}_\mu \quad (2.7)$$

and that

$$B^{-1} \tilde{\sigma}_\mu A^{-1} = \tilde{\sigma}_\nu L^\nu{}_\mu(A, B) \equiv (\tilde{\sigma} \cdot L)_\mu. \quad (2.8)$$

To specify four different ways of applying transforms to spin indices four different types of spinor indices are introduced. The spin transformation $\Lambda \equiv \Lambda(A, B)$ acts on the different types of spinor

$$\frac{1}{2} \text{Tr}[a_1 \cdot \sigma a_2 \cdot \tilde{\sigma} a_3 \cdot \sigma a_4 \cdot \tilde{\sigma}] = (a_1 \cdot a_2)(a_3 \cdot a_4) + (a_1 \cdot a_4)(a_2 \cdot a_3) - (a_1 \cdot a_3)(a_2 \cdot a_4) + i[a_1, a_2, a_3, a_4], \quad (2.14b)$$

where

$$[a_1, a_2, a_3, a_4] = a_1^\mu a_2^\nu a_3^\sigma a_4^\delta \epsilon_{\mu\nu\sigma\delta}. \quad (2.15)$$

Here ϵ is the fully antisymmetric matrix with $\epsilon_{0123} = 1$.

B. Covariant spin-projection operators

Let $p = mv$ be the momentum-energy of a freely moving particle, as measured in some general

indices according to the rules

$$\begin{aligned} \Lambda(\phi_\alpha) &= A_\alpha{}^{\alpha'} \phi_{\alpha'} \equiv (A\phi)_\alpha, \\ \Lambda(\phi_{\dot{\beta}}) &= \phi_{\dot{\beta}} \cdot B^{\dot{\beta}'}{}_{\dot{\beta}} = (\phi B)_{\dot{\beta}}, \\ \Lambda(\phi^{\dot{\beta}}) &= (B^{-1})^{\dot{\beta}'}{}_{\dot{\beta}} \phi^{\dot{\beta}'} = (B^{-1}\phi)^{\dot{\beta}}, \\ \Lambda(\phi^\alpha) &= \phi^\alpha (A^{-1})_{\alpha'}{}^\alpha = (\phi A^{-1})^\alpha. \end{aligned} \quad (2.9)$$

Thus the transformation to be applied is determined by the location of the index (upper or lower) and whether it is dotted or undotted.

The operator Λ acts like the identity on any sum of the form $\phi^\alpha \psi_\alpha$ or $\phi_{\dot{\beta}} \psi^{\dot{\beta}}$. For example,

$$\begin{aligned} \Lambda(\phi^\alpha \psi_\alpha) &= \Lambda(\phi^\alpha) \Lambda(\psi_\alpha) \\ &= (\phi^\alpha A^{-1})_{\alpha'}{}^\alpha (A_{\alpha''}{}^{\alpha'} \psi_{\alpha''}) \\ &= \phi^\alpha \psi_\alpha. \end{aligned} \quad (2.10)$$

Let a_1, a_2, \dots, a_{2n} be any set of $2n$ four-vectors. Then

$$\frac{1}{2} \text{Tr}[a_1 \cdot \sigma a_2 \cdot \tilde{\sigma} a_3 \cdot \sigma \cdots a_{2n} \cdot \tilde{\sigma}] \quad (2.11)$$

is a Lorentz-invariant function of the four-vectors a_1, \dots, a_{2n} . To see this let the indices on σ_μ and $\tilde{\sigma}_\mu$ be specified always in the following way:

$$\sigma_\mu \rightarrow \sigma_{\mu\alpha\dot{\beta}}, \quad \tilde{\sigma}_\mu \rightarrow \sigma_{\dot{\mu}}^{\beta\alpha}. \quad (2.12a)$$

Then (2.3) and (2.8) become

$$\Lambda \sigma_\mu = (\sigma \cdot L)_\mu, \quad \Lambda \tilde{\sigma}_\mu = (\tilde{\sigma} \cdot L)_\mu. \quad (2.12b)$$

Application of the operator Λ leaves invariant the trace (2.11), due to (2.10). It gives, alternatively, by virtue of (2.12),

$$\frac{1}{2} \text{Tr}(\sigma \cdot L a_1)(\tilde{\sigma} \cdot L a_2) \cdots (\tilde{\sigma} \cdot L a_{2n}). \quad (2.13)$$

Thus the trace is invariant under any Lorentz transformation of all the vectors a_i .

Two important special cases are

$$\frac{1}{2} \text{Tr}[a_1 \cdot \sigma a_2 \cdot \tilde{\sigma}] = a_1 \cdot a_2, \quad (2.14a)$$

which follows from (2.5), and

Lorentz frame Σ . Let s be a spin vector that satisfies $s \cdot p = 0$. Let $\Sigma^r(v)$ be the particle rest frame obtained by applying a "boost" to Σ . This boost is a Lorentz transformation that leaves unchanged any space component that is perpendicular to \vec{v} . The vectors v and s as measured in $\Sigma^r(v)$ are

$$v^r = (v^{r\mu}) \equiv (v^{r0}, \vec{v}^r) = (1, 0, 0, 0) \quad (2.16a)$$

and

$$s^r = (s_\mu^r) \equiv (s^{r0}, \vec{s}^r) = (0, s_1^r, s_2^r, s_3^r). \quad (2.16b)$$

The rest-frame projection operator is

$$\begin{aligned} \tilde{P}^r(s) &= \frac{1}{2}(1 + \vec{s}^r \cdot \vec{\sigma}) \\ &= \frac{1}{2}(v^r + s^r)^\mu \tilde{\sigma}_\mu = \frac{1}{2}(v^r + s^r) \cdot \vec{\sigma}. \end{aligned} \quad (2.17)$$

This operator projects onto the spin state in which the spin is directed along $s^r = (0, \vec{s}^r)$ as measured in $\Sigma^r(v)$, and hence along s as measured in Σ .

The operator $\tilde{P}^r(s)$ refers to the rest frame $\Sigma^r(v)$. To eliminate this frame dependence one may apply the boost $\Lambda(A, B)$ that converts \tilde{P}^r from its form in $\Sigma^r(v)$ to its form in the general coordinate frame Σ :

$$\begin{aligned} \tilde{P}^r(s) &\rightarrow \Lambda(v) \tilde{P}^r(s) = B^{-1} \tilde{P}^r(s) A^{-1} \\ &= \frac{1}{2} \vec{\sigma} \cdot (L v^r + L s^r) \\ &= \frac{1}{2} \vec{\sigma} \cdot (v + s) \\ &= \frac{1}{2} (v + s) \cdot \vec{\sigma} \\ &\equiv \tilde{P}(s, v). \end{aligned} \quad (2.18)$$

Real Lorentz transformations are generated by matrices A and B that satisfy $A = B^\dagger$, where a dagger denotes Hermitian conjugation. For rotations A is unitary, but for boosts A is Hermitian. The boost $\Lambda(A, B)$ that converts the rest-frame form \tilde{P}^r into the general coordinate-system form \tilde{P} is

$$A(v) = B^\dagger(v) = A^\dagger(v) = \sqrt{v \cdot \sigma}, \quad (2.19a)$$

$$A^{-1}(v) = B^{-1}(v) = B^{-1}(v) = \sqrt{v \cdot \tilde{\sigma}}, \quad (2.19b)$$

where

$$\begin{aligned} \sqrt{v \cdot \sigma} &= \exp \left[\frac{\theta}{2} (\vec{\sigma} \cdot \vec{n}) \right] \\ &= \cosh \frac{\theta}{2} + \vec{n} \cdot \vec{\sigma} \sinh \frac{\theta}{2} \end{aligned} \quad (2.20)$$

and

$$\begin{aligned} v \cdot \sigma &= \exp[\theta (\vec{\sigma} \cdot \vec{n})] \\ &= \cosh \theta + \vec{n} \cdot \vec{\sigma} \sinh \theta \\ &= v^\mu \sigma_\mu \\ &\equiv v^0 + \vec{v} \cdot \vec{\sigma} \\ &\equiv v^0 + \vec{n} \cdot \vec{\sigma} |\vec{v}|. \end{aligned} \quad (2.21)$$

Note that

$$v \cdot \sigma v \cdot \tilde{\sigma} = 1 \quad (2.22a)$$

and

$$\sqrt{v \cdot \sigma} \sqrt{v \cdot \tilde{\sigma}} = 1. \quad (2.22b)$$

Another useful form is

$$\sqrt{v \cdot \sigma} = \frac{v_0 + 1 + \vec{v} \cdot \vec{\sigma}}{(2v_0 + 2)^{1/2}}. \quad (2.23)$$

The operator

$$\begin{aligned} \tilde{P}(s, v) &= \sqrt{v \cdot \tilde{\sigma}} \frac{1}{2} (1 + \vec{s}^r \cdot \vec{\sigma}) \sqrt{v \cdot \tilde{\sigma}} \\ &= \frac{1}{2} (v \cdot \tilde{\sigma} + s \cdot \tilde{\sigma}) \end{aligned} \quad (2.24)$$

is called a covariant spin operator. The vectors v and s occurring in $\tilde{P}(s, v)$ have components v^μ and s^μ that refer to the general frame of reference Σ .

Because the boost operators A^{-1} and B^{-1} are Hermitian, rather than unitary, the operator $\tilde{P}(s, v)$ is not a true projection operator:

$$\tilde{P}(s, v)^2 \neq \tilde{P}(s, v) \text{ for } \vec{v} \neq 0.$$

The covariant spin operators are Lorentz-invariant spinor functions in the sense that

$$\Lambda \tilde{P}(L^{-1} s, L^{-1} v) = \tilde{P}(s, v). \quad (2.25)$$

Here $\Lambda = \Lambda(A, B)$ and $L = L(A, B)$. This result follows directly from (2.12).

C. M functions

Consider first a scattering process involving one spin- $\frac{1}{2}$ particle in the initial state and one spin- $\frac{1}{2}$ particle in the final state, and an arbitrary number of spinless particles. Let

$$p \equiv (p_a, t_a; p_b, t_b; p_c, t_c; \dots; p_d, t_d),$$

where p_a is the mathematical momentum-energy of the final spin- $\frac{1}{2}$ particle, p_b is the mathematical momentum-energy of the initial spin- $\frac{1}{2}$ particle, and p_c, \dots, p_d are the mathematical momentum-energy vectors of the spinless particles. The mathematical momentum-energy vectors are equal to plus or minus the physical momentum-energy vectors for final and initial particles, respectively. Thus $p_a = m_a v_a$ and $p_b = -m_b v_b$, where $v_a^0 > 0$ and $v_b^0 > 0$.

The t_j are the mathematical-type labels. They are related to the physical-type labels t_j^{phys} by the relation $t_j = t_j^{\text{phys}} / \text{sign } p_j^0$, where t_j and $-t_j$ label relative antiparticles. These type variables are sometimes suppressed.

According to quantum theory the probability for a scattering specified by (p, s_a, s_b) is proportional to

$$\frac{1}{2} \text{Tr} \tilde{P}^r(s_a) S(p) \tilde{P}^r(s_b) S^\dagger(p), \quad (2.26)$$

where $S(p)$ is the S matrix. This can be written equivalently as

$$\frac{1}{2} \text{Tr} \tilde{P}(s_a, v_a) M(p) \tilde{P}(s_b, v_b) M^\dagger(p), \quad (2.27)$$

where, as in (2.24),

$$\tilde{P}(s_a, v_a) = \sqrt{v_a \cdot \tilde{\sigma}} \tilde{P}^r(s_a) \sqrt{v_a \cdot \tilde{\sigma}}, \quad (2.28a)$$

$$\tilde{P}(s_b, v_b) = \sqrt{v_b \cdot \tilde{\sigma}} \tilde{P}^r(s_b) \sqrt{v_b \cdot \tilde{\sigma}}, \quad (2.28b)$$

and

$$M(p) = \sqrt{v_a \cdot \sigma} S(p) \sqrt{v_b \cdot \sigma}, \quad (2.29a)$$

$$M^\dagger(p) = \sqrt{v_b \cdot \sigma} S^\dagger(p) \sqrt{v_a \cdot \sigma}. \quad (2.29b)$$

The physical probability is assumed to be Lorentz invariant. This physical invariance ensures that if the spin indices of $M(p)$ and $M^\dagger(p)$ are assigned spin-index type according to the rules

$$M(p) \rightarrow M_{\alpha\beta}(p) \quad (2.30a)$$

and

$$M^\dagger(p) \rightarrow M^{\dagger\alpha\beta}(p), \quad (2.30b)$$

then the spinor functions $M(p)$ and $M^\dagger(p)$ are Lorentz invariant: for all proper ($\det L = 1$) real Lorentz transformations

$$\Lambda M(L^{-1}(p)) = M(p) \quad (2.31a)$$

and

$$\Lambda M^\dagger(L^{-1}(p)) = M^\dagger(p), \quad (2.31b)$$

with

$$L^{-1}(p) \equiv (L^{-1}p_a, L^{-1}p_b, L^{-1}p_c, \dots, L^{-1}p_d). \quad (2.31c)$$

These invariance properties entail that if $m^\mu(p)$ and

$m^{\dagger\mu}(p)$ are defined by

$$M(p) \equiv m^\mu(p) \sigma_\mu \equiv m(p) \cdot \sigma \quad (2.32a)$$

and

$$M^\dagger(p) \equiv m^{\dagger\mu}(p) \sigma_\mu \equiv m^\dagger(p) \cdot \sigma, \quad (2.32b)$$

then the quantities $m^\mu(p)$ and $m^{\dagger\mu}(p)$ are vector functions of the set of vectors p :

$$m^\mu(L(p)) = L^\mu_\nu m^\nu(p) = (Lm(p))^\mu, \quad (2.33a)$$

$$m^{\dagger\mu}(L(p)) = L^\mu_\nu m^{\dagger\nu}(p) = (Lm^\dagger(p))^\mu. \quad (2.33b)$$

Consequently, by virtue of (2.12), the spinor functions

$$\tilde{M}(p) \equiv C^{-1} M(p) \text{Tr} C = m^\mu(p) \tilde{\sigma}_\mu = m(p) \cdot \tilde{\sigma} \quad (2.34a)$$

and

$$\tilde{M}^\dagger(p) \equiv m^{\dagger\mu}(p) \tilde{\sigma}_\mu = m^\dagger(p) \cdot \tilde{\sigma} \quad (2.34b)$$

are also Lorentz-invariant spinor functions:

$$\Lambda \tilde{M}(L^{-1}(p)) = \tilde{M}(p) \quad (2.35a)$$

and

$$\Lambda \tilde{M}^\dagger(L^{-1}(p)) = \tilde{M}^\dagger(p). \quad (2.35b)$$

These simple transformation properties do not hold for the S matrix $S(p)$.

The foregoing discussion can be immediately extended to processes in which there are n initial spin- $\frac{1}{2}$ particles, n final spin- $\frac{1}{2}$ particles, and n' spinless particles. In this case the M function can be written in the form (with type labels suppressed)

$$M(p_{a_1}, \alpha_1; p_{b_1}, \beta_1; p_{a_2}, \alpha_2; p_{b_2}, \beta_2; \dots; p_{a_n}, \alpha_n; p_{b_n}, \beta_n; p_1, \dots, p_{n'}) = m^{\mu_1 \mu_2 \dots \mu_n}(p) \prod_{i=1}^n \sigma_{\mu_i \alpha_i \beta_i}, \quad (2.36)$$

where $m^{\mu_1 \dots \mu_n}(p)$ is a tensor function of the vectors

$$p = (p_{a_1}, p_{b_1}, \dots, p_{a_n}, p_{b_n}, p_1, \dots, p_{n'}), \quad (2.37)$$

$$m^{\mu_1 \dots \mu_n}(L(p)) = \left[\prod_{i=1}^n L^{\mu_i \nu_i} \right] m^{\nu_1 \dots \nu_n}(p).$$

The way in which the n initial spin- $\frac{1}{2}$ particles are associated with the n final spin- $\frac{1}{2}$ particles is immaterial: (2.37) holds in any case.

D. Parity

Let $S(p)$ be written as

$$S(p) = S_+(p) + S_-(p), \quad (2.38)$$

where

$$S_\pm(p) = \pm S_\pm(\tilde{p}). \quad (2.39)$$

Here

$$\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_N), \quad (2.40a)$$

$$(\tilde{p}_i^\mu) = (p_L^0, -\vec{p}_i),$$

and

$$(p_i^\mu) = (p_i^0, \vec{p}_i). \quad (2.40b)$$

Let an intrinsic parity ϵ_j be assigned to each particle j , and define the parity operator \mathcal{P} by

$$\mathcal{P}(S(p)) = \prod_{j=1}^N \epsilon_j S(\tilde{p}). \quad (2.41)$$

The product of ϵ_i 's for allowed processes must be +1 or -1. Invariance under parity is then expressed by the equation

$$\mathcal{P}(S(p))=S(p). \quad (2.42)$$

If this equation is satisfied, then S_ϵ defined in (2.38) must be zero unless $\epsilon = \prod_{j=1}^N \epsilon_j$.

$$M(p_{a1}, t_{a1}; p_{b1}, t_{b1}; \dots; p_{an}, t_{an}; p_{bn}, t_{bn}; p_1, t_1; \dots; p_n, t_n) = m^{\mu_1 \dots \mu_n}(p, t) \prod_{i=1}^n \sigma_{\mu_i}^{(i)}, \quad (2.43)$$

where the matrix elements of $\sigma_{\mu_i}^{(i)}$ are $\sigma_{\mu_i \alpha_i \beta_i}$. The connection of $M(p)$ to $S(p)$ can be represented by the equation

$$\begin{aligned} M(p) &= \left[\prod_i (v_{ai} \cdot \sigma^{(i)})^{1/2} \right] S(p) \left[\prod_i (v_{bi} \cdot \sigma^{(i)})^{1/2} \right] \\ &\equiv \sqrt{V_a \cdot \sigma} S(p) \sqrt{V_b \cdot \sigma}. \end{aligned} \quad (2.44)$$

Define now

$$M_{\pm}(p) = \left[\prod_i (v_{ai} \cdot \sigma^{(i)})^{1/2} \right] S_{\pm}(p) \left[\prod_i (v_{bi} \cdot \sigma^{(i)})^{1/2} \right]. \quad (2.45)$$

Then

$$\begin{aligned} M_{\pm}(\tilde{p}) &= \left[\prod_i (\tilde{v}_{ai} \cdot \sigma^{(i)})^{1/2} \right] S_{\pm}(\tilde{p}) \left[\prod_i (\tilde{v}_{bi} \cdot \sigma^{(i)})^{1/2} \right] \\ &= \pm \left[\prod_i (\tilde{v}_{ai} \cdot \sigma^{(i)})^{1/2} \right] S_{\pm}(p) \left[\prod_i (\tilde{v}_{bi} \cdot \sigma^{(i)})^{1/2} \right] \\ &= \pm \left[\prod_i v_{ai} \cdot \tilde{\sigma}^{(i)} \right] M_{\pm}(p) \left[\prod_i v_{bi} \cdot \tilde{\sigma}^{(i)} \right]. \end{aligned} \quad (2.46)$$

This equation can be inverted to give

$$M_{\pm}(p) = \pm \left[\prod_i v_{ai} \cdot \sigma^{(i)} \right] M_{\pm}(\tilde{p}) \left[\prod_i v_{bi} \cdot \sigma^{(i)} \right]. \quad (2.47)$$

The parity transformation applied to the M functions is defined to be

$$\begin{aligned} \mathcal{P}(M(p)) &\equiv \left[\prod_{j=1}^N \epsilon_j \right] \left[\prod_i v_{ai} \cdot \sigma^{(i)} \right] \\ &\quad \times M(\tilde{p}) \left[\prod_i v_{bi} \cdot \sigma^{(i)} \right]. \end{aligned} \quad (2.48)$$

Consider a process in which n initial spin- $\frac{1}{2}$ particles, $i=1, \dots, n$, are scattered into n final spin- $\frac{1}{2}$ particles. Let p_{ai} and p_{bi} denote the final and initial mathematical momentum-energies of the i th particle. Let (p_1, \dots, p_n) denote the momenta of n' spinless particles that also participate in the reaction. Then, as already mentioned, the M matrix can be written

Then (2.48) and (2.47) ensure that the condition

$$\mathcal{P}(M(p))=M(p) \quad (2.49)$$

is equivalent to the condition that M_ϵ be zero unless $\epsilon = \prod_{j=1}^N \epsilon_j$, which is equivalent to the parity-invariance condition $\mathcal{P}(S(p))=S(p)$.

For n distinguishable spin- $\frac{1}{2}$ particles the no-scattering part of the S matrix has the form

$$S_0(p) = \prod_{i=1}^n [(\sigma_0^{(i)})(2\pi)^3 \delta^3(p_{ai} + p_{bi}) 2\omega_i]. \quad (2.50)$$

The corresponding M function is

$$M_0(p) = \prod_{i=1}^n [v_i \cdot \sigma^{(i)} (2\pi)^3 \delta^3(p_{ai} + p_{bi}) 2\omega_i]. \quad (2.51)$$

In order that this no-scattering part be invariant under parity (for each particle i separately) we must take $\epsilon_{ai} \epsilon_{bi} = 1$ for all i . But then (2.48) gives

$$\mathcal{P}(p_a \cdot \sigma) = (-p_b \cdot \sigma). \quad (2.52)$$

This relationship, which stems from the condition that the no-scattering part be nonzero, is used later.

E. Crossing

Analysis of the pole singularity¹⁵ shows that the analytic continuation of $M(p)$ along an appropriate path from an original region where $p_{ai}^0 > 0$ to a region where $p_{ai}^0 < 0$ gives the function that describes a process in which the final particle of type t_{ai} is replaced by an initial particle of type $-t_{ai}$, i.e., by the antiparticle of the original particle of type t_{ai} . If the final particle t_{ai} carries q units of any conserved quantity out of the reaction then the antiparticle $-t_{ai}$ must carry $-q$ units into the reaction. This holds both for the total momentum energy p_{ai} , for the components of spin, and for any quantity that is conserved by virtue of invariance under a p -independent transformation property. Consequently, the mathematical momentum-energy vector p_{ai} ,

the mathematical spin vector s_{ai} , and the mathematical-type label t_{ai} are equal, after the continuation, to minus their physical values:

$$p_{ai} = p_{ai}^{\text{phys}} / \text{sign} p_{ai}^0, \quad (2.53a)$$

$$s_{ai} = s_{ai}^{\text{phys}} / \text{sign} p_{ai}^0, \quad (2.53b)$$

$$t_{ai} = t_{ai}^{\text{phys}} / \text{sign} p_{ai}^0. \quad (2.53c)$$

A similar argument gives

$$p_{bi} = p_{bi}^{\text{phys}} / \text{sign} p_{bi}^0, \quad (2.53d)$$

$$s_{bi} = -s_{bi}^{\text{phys}} / \text{sign} p_{bi}^0, \quad (2.53e)$$

$$t_{bi} = t_{bi}^{\text{phys}} / \text{sign} p_{bi}^0. \quad (2.53f)$$

The minus sign in (2.53e) arises from the fact that s_{bi} characterizes the physical spin of the initial particle bi , not minus the physical spin. The p_{bi} and t_{bi} were defined originally to be minus the physical momentum-energy vector and minus the physical particle type of the incoming particle (bi), and hence the equations for these are the same as those for p_{ai} and t_{ai} .

The quantities occurring in the transition probability formula

$$\text{Tr} \frac{1}{2} (v_a + s_a) \cdot \tilde{\sigma} M(p_a, t_a; p_b, t_b) \frac{1}{2} (v_b + s_b) \cdot \tilde{\sigma} M^\dagger(p_a, t_a; p_b, t_b) \quad (2.54)$$

are to be interpreted with the aid of (2.53). Thus, for example, if p_a^0 and p_b^0 are both positive, then the s_a and s_b in (2.54) are s_a^{phys} and $-s_b^{\text{phys}}$, respectively, and the particle types t_a and t_b and t_a^{phys} are t_b^{phys} . In this way we can use the same expression (2.54) in all the different channels.

The parity transformation \mathcal{P} was defined to be

$$\mathcal{P}(M(p)) = \left[\prod_{j=1}^N \epsilon_j \right] \left[\prod_{i=1}^n v_{ai} \cdot \sigma \right] M(\tilde{p}) \left[\prod_{L=1}^n v_{bi} \cdot \sigma \right]. \quad (2.55)$$

In the original (direct) channel ($p_{ai}^0 > 0$, $p_{bi}^0 < 0$) the parity invariance equation $\mathcal{P}M((p)) = M(p)$ can be written as

$$M(p) = \left[\prod_{j=1}^N \epsilon_j \right] \left[\prod_{i=1}^n \frac{p_{ai} \cdot \sigma}{m_{ai}} \right] M(\tilde{p}) \left[\prod_{i=1}^n \frac{-p_{bi} \cdot \sigma}{m_{bi}} \right] = \left[\prod_{j=1}^{n'} \epsilon_j \right] \left[\prod_{i=1}^n \frac{p_{ai} \cdot \sigma}{m_{ai}} \right] M(\tilde{p}) \left[\prod_{i=1}^n \frac{-p_{bi} \cdot \sigma}{m_{bi}} \right], \quad (2.56)$$

where use has been made of the direct-channel result $\epsilon_{ai} \epsilon_{bi} = 1$, derived from forward scattering [see (2.51)].

Analytic continuation to the crossed channel avoids all singularities of $M(p)$ and $M(\tilde{p})$. Thus equation (2.56) must hold in all channels, with the factor $\prod_{j=1}^{n'} \epsilon_j$ from the spinless particles defined as in the original direct channel. This equation gives

$$M(p) = \left[\prod_{j=1}^{n'} \epsilon_j \right] \left[\prod_{i=1}^n \text{sign} p_{ai}^0 \right] \left[\prod_{i=1}^n \text{sign} p_{bi}^0 \right] \left[\prod_{i=1}^n v_{ai} \cdot \sigma \right] M(\tilde{p}) \left[\prod_{i=1}^n v_{bi} \cdot \sigma \right]. \quad (2.57)$$

It will be shown presently that the parity transformation is defined in all channels by (2.55). Thus one can conclude that the ϵ_j for the spinless particles is channel independent and that

$$\epsilon_{ai} \epsilon_{bi} = -\text{sign} p_{ai}^0 \text{sign} p_{bi}^0. \quad (2.58)$$

This means, in particular, that the intrinsic parity of each spin- $\frac{1}{2}$ particle must reverse under continuation to a crossed channel and that the intrinsic parity of a particle-antiparticle pair is $-(-1)^l$.

The product of the intrinsic parities of the particles of a parity conserving process is physically well defined: it is equal to the sign ϵ in $S(\tilde{p}) = \epsilon S(p)$, and hence to $(-1)^{\sum l_j}$. The argument leading to the equivalence of $\mathcal{P}(S(p)) = S(p)$ to $\mathcal{P}(M(p)) = M(p)$, with \mathcal{P} as defined in (2.55), was made explicitly in the direct channel. However, it holds equally well in

all channels, provided the same factor $\prod_{j=1}^N \epsilon_j$ occurs in both $\mathcal{P}(S(p))$ and $\mathcal{P}(M(p))$. Any extra sign or phase factor $e^{i\phi}$, that one might introduce into the connection between $S(p)$ and $M(p)$, in any given physical region, would be the same throughout that physical region and would drop out of (2.47), and hence not affect the argument that demonstrates the equivalence between $\mathcal{P}(S(p)) = S(p)$ and $\mathcal{P}(M(p)) = M(p)$, with $\mathcal{P}(M(p))$ defined as in (2.48) or (2.55). Thus this definition is applicable in all channels, and the result (2.58) on the intrinsic parities of spin- $\frac{1}{2}$ particles holds.¹⁶

F. Antiparticle conjugation

Consider a process on which p_{ai}^0 and p_{bi}^0 are both positive, so that the two associated particles are both

final particles. Suppose that $t_{ai} = -t_{bi}$, so that these two final particles are relative antiparticles.

Consider now an original value of (p_{ai}, p_{bi}) and an analytic continuation that stays in the physical region of the process, but interchanges p_{ai} and p_{bi} leaving all other p 's unchanged. Suppose we interchange also s_a^{phys} and s_b^{phys} . Then the original process and the second one are physically the same except for the interchange $t_{ai} \leftrightarrow t_{bi}$, which is just $t_{ai} \leftrightarrow -t_{ai}$.

$$\text{Tr}(v_a + s_a^{\text{phys}}) \cdot \tilde{\sigma} M(p_a, p_b) (v_b - s_b^{\text{phys}}) \cdot \tilde{\sigma} M^\dagger(p_a, p_b) = \text{Tr}(v'_a + s'_a{}^{\text{phys}}) \cdot \tilde{\sigma} M(p'_a, p'_b) (v'_b - s'_b{}^{\text{phys}}) \cdot \tilde{\sigma} M^\dagger(p'_a, p'_b), \quad (2.59)$$

where (2.53) and (2.54) are used, and

$$\begin{aligned} p'_a &= p_b, \quad p'_a{}^0 > 0, \\ p'_b &= p_a, \quad p'_b{}^0 > 0, \\ s'_a{}^{\text{phys}} &= s_b^{\text{phys}}, \\ s'_b{}^{\text{phys}} &= s_a^{\text{phys}}. \end{aligned} \quad (2.60)$$

To see the consequences of this condition define

$$\mathcal{C} M(p_a, t; p_b, -t) = u_a \cdot \sigma \tilde{M}(p_b, t; p_a - t) u_b \cdot \sigma, \quad (2.61)$$

$$\begin{aligned} \text{Tr}(v_a + s_a^{\text{phys}}) \cdot \tilde{\sigma} u_a \cdot \sigma \tilde{M}(p_b, p_a) u_b \cdot \sigma (v_b - s_b^{\text{phys}}) \cdot \tilde{\sigma} u_b \cdot \sigma \tilde{M}^\dagger(p_b, p_a) u_a \cdot \sigma \\ = \text{Tr}(v_a - s_a^{\text{phys}}) \cdot \sigma \tilde{M}(p_b, p_a) (v_b + s_b^{\text{phys}}) \cdot \sigma \tilde{M}^\dagger(p_b, p_a) \\ = \text{Tr}(v_b + s_b^{\text{phys}}) \cdot \tilde{\sigma} M(p_b, p_a) (v_a - s_a^{\text{phys}}) \cdot \tilde{\sigma} M^\dagger(p_b, p_a), \end{aligned} \quad (2.64)$$

where in the second line the relations

$$u_a \cdot \sigma u_a \cdot \tilde{\sigma} = 1 \quad (2.65a)$$

and

$$s_a \cdot \tilde{\sigma} u_a \cdot \sigma = -u_a \cdot \tilde{\sigma} s_a \cdot \sigma \quad \text{for } s_a \cdot u_a = 0 \quad (2.65b)$$

are used, and in the last line the equations

$$\sigma = C^{-1} \tilde{\sigma} \text{Tr} C \quad (2.66a)$$

and

$$\tilde{\sigma} = C^{-1} \sigma \text{Tr} C \quad (2.66b)$$

are used. Comparison of (2.64) to the right-hand side of (2.59), with the substitutions (2.60) made, shows that the condition $M = \pm \mathcal{C} M$ implies antiparticle conjugation invariance.

Notice that

$$\mathcal{C} p_a \cdot \sigma = p_a \cdot \sigma \quad (2.67a)$$

and

Suppose that the transition probabilities for these two processes were the same. Then the process would be invariant under the transformation $t_{ai} \leftrightarrow -t_{ai}$. Antiparticle conjugation invariance is invariance under the analogous change $t_{ai} \leftrightarrow -t_{ai}$ for all i .

If we keep only one particle-antiparticle pair, for notational simplicity, the antiparticle conjugation invariance condition described above is

where $u_a = p_a/m_a$ and $u_b = p_b/m_b$. Define also

$$M_{(\pm)} = \frac{1}{2} (1 \pm \mathcal{C}) M. \quad (2.62)$$

Then $M = M_{(+)} + M_{(-)}$, and the property $(\mathcal{C})^2 = 1$ gives

$$\mathcal{C} M = M_{(+)} - M_{(-)}. \quad (2.63)$$

Hence if $M = M_{(+)}$ or $M_{(-)}$, then $M = \pm \mathcal{C} M$.

Insertion of this condition $M = \pm \mathcal{C} M$ into the left-hand side of (2.59) gives

$$\mathcal{C} p_b \cdot \sigma = p_b \cdot \sigma. \quad (2.67b)$$

Thus both $p_a \cdot \sigma$ and $p_b \cdot \sigma$, and any superposition of them, are invariant under \mathcal{C} .

G. CPT invariance

The physical transformation corresponding to CPT is

$$\begin{aligned} p_j^{\text{phys}} &\rightarrow p_j^{\text{phys}}, \quad s_j^{\text{phys}} \rightarrow -s_j^{\text{phys}}, \\ t_j^{\text{phys}} &\rightarrow -t_j^{\text{phys}}, \quad \text{In} \leftrightarrow \text{Out}. \end{aligned} \quad (2.68a)$$

The corresponding mathematical transformation is, by virtue of (2.53),

$$p_j \rightarrow -p_j, \quad s_j \rightarrow s_j, \quad t_j \rightarrow t_j. \quad (2.68b)$$

Thus CPT invariance is equivalent to invariance of transition probabilities under the transformation $p_j \rightarrow -p_j$ (all j).

Any Lorentz invariant spinor function $M(p)$ is in-

variant, up to a sign, under the transformation $p_j \rightarrow -p_j$ (all j). For the Lorentz invariance condition

$$\Lambda M(L^{-1}(p)) = M(p) \tag{2.69}$$

applied for the case $A=1, B=-1$ gives, by virtue of (2.3) and (2.9),

$$M(-p) = (-1)^{N_d} M(p), \tag{2.70}$$

where N_d is the number of dotted spinor indices. [I mean here dotted two-valued spinor indices: dotted (undotted) spinor indices for spin- $(n/2)$ particles can be constructed trivially by combining $n+2m$ dotted (undotted) two-valued spinor indices by means of the usual Clebsch-Gordan coefficients. Thus a dotted spin- $(n/2)$ spinor index contributes a term n to N_d .]

The matrix $B=-1$ can be continuously connected to $B=1$ by the matrix

$$B(\psi) = \begin{pmatrix} e^{i\pi\psi} & 0 \\ 0 & e^{-i\pi\psi} \end{pmatrix} \tag{2.71}$$

which satisfies $B(0)=1$ and $B(1)=-1$. Since all Lorentz invariants are invariant under all real and complex Lorentz transformations the transformation

$$L(A,B) = L(1,B(\psi)),$$

with $0 \leq \psi \leq 1$, must generate complex values of the p_j , since no real mass-shell vector $p_j(\psi)$ can interpolate $p_j \rightarrow -p_j$.

The matrices $M(p)$ and $M^\dagger(p)$ have been assigned the transformation properties indicated by the indices $M_{\alpha\beta}(p)$ and $M_{\alpha\beta}^\dagger(p)$. For real p the matrices $M(p)$ and $M^\dagger(p)$ are related by Hermitian conjugation:

$$M_{\alpha\beta}^\dagger(p) = [M_{\beta\alpha}(p)]^* \tag{2.72a}$$

Thus if M is transformed by a real Lorentz transformation to AMB , then M^* is transformed to $A^*M^*B^*$ and M^\dagger is transformed to $B^\dagger M^\dagger A^\dagger = AM^\dagger B$, as indicated by the indices on $M_{\alpha\beta}^\dagger$.

For complex Lorentz transformations the condition $A=B^\dagger$ does not hold. However, (2.7a) is then inappropriate: the appropriate definition is

$$M_{\alpha\beta}^\dagger(p) = M_{\beta\alpha}^*(p) \equiv [M_{\beta\alpha}(p^*)]^* \tag{2.72b}$$

This quantity is an analytic function of p , whereas the function on the right-hand side of (2.72a) is an analytic function of p^* . The function M^\dagger defined in (2.72b) will continue to satisfy the Lorentz-invariance condition

$$\left[\prod_{i=1}^n A^{(i)} \right] M^\dagger(L^{-1}(A,B)(p)) \left[\prod_{i=1}^n B^{(i)} \right] = M^\dagger(p) \tag{2.73}$$

for complex Lorentz transformations.¹³ Thus in the formula for probabilities the factor $(-1)^{N_d}$ from (2.70) will be cancelled by the same factor $(-1)^{N_d}$ from

$$M^\dagger(-p) = (-1)^{N_d} M^\dagger(p). \tag{2.74}$$

Hence probabilities will be invariant under *CPT*.

H. Statistics

The order of writing the variables is important. If the variables in the set of arguments

$$p = (p_1, t_1; p_2, t_2; \dots; p_n, t_n)$$

is such that all variables referring to initial particles stand to the right of all variables referring to final particles, then one may write $p = (p_{\text{fin}}; p_{\text{in}})$. By convention

$$S(p_{\text{fin}}; p_{\text{in}}) \equiv \langle p_{\text{fin}} | S | \bar{p}_{\text{in}} \rangle, \tag{2.75}$$

where \bar{p}_{in} is obtained from p_{in} by reversing the signs of all energy vectors p_j and all type variables t_j , and reversing the order of the variables. Thus if

$$p_{\text{in}} = (p_m, t_m; \dots; p_n, t_n), \tag{2.76a}$$

then

$$\bar{p}_{\text{in}} = (-p_n, -t_n; \dots; -p_m, -t_m). \tag{2.76b}$$

The diagram representing $\langle p_{\text{fin}} | S | \bar{p}_{\text{in}} \rangle$ is generally drawn by ordering the lines from top to bottom in the sequence in which the corresponding arguments of p_{fin} and \bar{p}_{in} appear. The lines corresponding to \bar{p}_{in} are on the right-hand side; those corresponding to p_{fin} are on the left-hand side. The variables in p in (2.50) are in the order

$$(p_{1a}, p_{2a}, \dots, p_{2b}, p_{1b}),$$

so that each particle line goes straight through, without a change in order.

The functions $S(p)$ and $M(p)$ are assumed to be antisymmetric under the interchange of any two spin- $\frac{1}{2}$ particle variables (p_i, t_i) and (p_j, t_j) . Analytic continuation $p_{ai} \rightarrow p_{bi}$ in (2.51) changes the sign of (2.51). This sign change is canceled by the change of the order of variables required to bring the variables back into the form $(p_{\text{fin}}; p_{\text{in}})$. Thus (2.50) and (2.51) hold in all channels, for $p = (p_{\text{fin}}; p_{\text{in}})$, with the corresponding variables of p_{fin} and \bar{p}_{in} occurring in the same order.

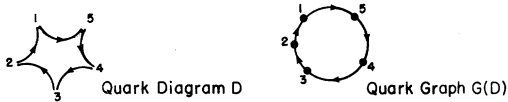


FIG. 1. A zero-entropy quark diagram D and the equivalent quark graph $G = G(D)$.

With these conventions the relationship (2.45) between $M(p)$ and $S(p)$ holds in all channels.

Combinatoric factors $1/n!$ are discussed in Appendix A of Ref. 17.

III. MESONS

A. The zero-entropy amplitudes

The basic building blocks of the topological expansion are the zero-entropy amplitudes. In the meson sector two important zero-entropy amplitudes are represented by the simple quark diagrams D of the kind shown in Fig. 1, or by the equivalent quark graphs G also shown there.

The quark diagram D is converted to the equivalent quark graph $G = G(D)$ by simply connecting to a vertex the ends of the two quark lines at each opening of D . Thus each vertex of a meson quark graph G corresponds, at some level of approximation, to an initial or final particle of a scattering process. The zero-entropy amplitude corresponding to a process with n particles is represented, therefore, by a directed circular graph with n vertices. The n directed edges that connect these vertices all run in the same direction, as illustrated in Fig. 1.

The quark graphs are not abstract graphs, but are graphs placed on an oriented surface. The orienta-

$$F_{(\mu)}(G^0, p) = - \prod_{i=1}^n (2m_i^2)^{-1/2} \text{Tr}[\tilde{\sigma}_{\mu_1} p_1 \cdot \sigma \tilde{\sigma}_{\mu_2} p_2 \cdot \sigma \cdots \tilde{\sigma}_{\mu_n} p_n \cdot \sigma] . \quad (3.2)$$

This factor $F_{(\mu)}(G^0, p)$ is minus the trace of a matrix formed from right to left by following the sense of the quark arrows in G^0 and replacing each vertex i by $\tilde{\sigma}_{\mu_i}/\sqrt{2}$ and each edge by the ortho-quark "propagator"

$$p_{aj} \cdot \sigma / m_{aj} \equiv u_{aj} \cdot \sigma ,$$

where p_{aj} is the mathematical momentum-energy vector associated with the vertex that lies on the leading end of that quark edge.

If G^p is the para-graph obtained from G^0 by reversing the orientation of the disk, then

$$A_{(\mu)}(G^p, p) = \mathcal{P} A_{(\mu)}(G^0, p) . \quad (3.3a)$$

Thus the function $A(G, p)$ is invariant under the parity operation, in the sense that if $\mathcal{P}G^0 = G^p$ and



FIG. 2. Two equivalent ortho-graphs G^0 .

tion of the boundary of the oriented circular disk bounded by the quark line is indicated by a second arrow, as shown in Figs. 2 and 3. The two graphs of Fig. 2 are equivalent to each other, and the two graphs of Fig. 3 are equivalent to each other. But those of Fig. 2 are not equivalent to those of Fig. 3.

The circular graphs in which the directions of all the quark lines agree with the direction of the boundary of the enclosed oriented disk, as in Fig. 2, are called "ortho-graphs." The circular graphs in which the directions of all the quark lines are opposite to the direction of the boundary of the oriented disk, as in Fig. 3, are called "para-graphs."

For each ortho- or para-graph G there is a corresponding amplitude. If G has n vertices, then this amplitude has a set $(\mu) = (\mu_1, \dots, \mu_n)$ of n vector indices. The amplitude corresponding to G has the form

$$A_{(\mu)}(G, p) = F_{(\mu)}(G, p) f(G, p) , \quad (3.1)$$

where $f(G, p)$ is a function of the scalar products of the mathematical momentum-energy vectors p_j appearing in the set of arguments

$$p = (p_1, t_1; \dots; p_n, t_n) .$$

For any ortho-graph $G = G^0$ the function of $F_{(\mu)}(G^0, p)$ is given explicitly by

$$\mathcal{P}G^p = G^0, \text{ then} \\ \mathcal{P} A_{(\mu)}(\mathcal{P}G, p) = A_{(\mu)}(G, p) . \quad (3.3b)$$

The action of \mathcal{P} on any A is given by (2.48). Thus

$$\mathcal{P} f(G, p) = f(G, \bar{p}) , \quad (3.4a)$$

and, by virtue of (2.52),



FIG. 3. Two equivalent para-graphs G^p .

$$F_{(\mu)}(G^p, p) = - \prod_{i=1}^n (2m_i^2)^{-1/2} \text{Tr}(-p_1 \cdot \sigma) \tilde{\sigma}_{\mu_1} (-p_2 \cdot \sigma) \tilde{\sigma}_{\mu_2} \cdots (-p_n \cdot \sigma) \tilde{\sigma}_{\mu_n}. \quad (3.4b)$$

This spinor part of the para-amplitude is minus the trace of a matrix formed from right to left by following the sense of the quark arrows in G^p and replacing each vertex i of G^p by $\tilde{\sigma}_{\mu_i}/\sqrt{2}$ and each edge by the para-quark "propagator"

$$-p_{bj} \cdot \sigma / m_{bj} \equiv -u_{bj} \cdot \sigma,$$

where p_{bj} is the mathematical momentum-energy vector associated with the vertex that lies on the *trailing* end of the quark edge.

Notice that in both the ortho-graph and para-graph cases the orientation of the disk points from each edge to the vertex whose momentum appears in the propagator corresponding to that edge.

Each vertex i is associated with a spin four-vector s_i . For a vector particle $s_i \cdot p_i = 0$ and $s_i^2 = -1$. For a pseudoscalar particle $s_i = u_i$, and $s_i^2 = 1$. The vector (is_i) is the "wave function" of particle i in spin space. The ortho-amplitudes and para-amplitudes themselves are therefore

$$A(G^0, p, s) = - \left[\frac{i}{\sqrt{2}} \right]^n \text{Tr}(s_1 \cdot \tilde{\sigma} u_1 \cdot \sigma \cdots s_n \cdot \tilde{\sigma} \mu_n \cdot \sigma) f(G^0, p) \quad (3.5a)$$

and

$$A(G^p, p, s) = - \left[\frac{-i}{\sqrt{2}} \right]^n \text{Tr}(u_1 \cdot \sigma s_1 \cdot \tilde{\sigma} \cdots u_n \cdot \sigma s_n \cdot \tilde{\sigma}) f(G^p, p). \quad (3.5b)$$

B. Parity

Let G^0 and G^p be ortho-graphs and para-graphs related by disk reversal. Since $A(G^0)$ and $A(G^p)$ are related by $A(G^p) = \mathcal{P}A(G^0)$, the sum $A(G^p) + A(G^0)$ is invariant under parity. To see this explicitly use

$$u_i \cdot \sigma s_i \cdot \tilde{\sigma} = \pm s_i \cdot \sigma u_i \cdot \tilde{\sigma} \begin{cases} + & \text{for spin 0} \\ - & \text{for spin 1,} \end{cases} \quad (3.6)$$

to obtain

$$\begin{aligned} A(G^p, p, s) &= - \left[\frac{-i}{\sqrt{2}} \right]^n (-1)^{\text{number of spin-1 particles}} \text{Tr} s_1 \cdot \sigma u_1 \cdot \tilde{\sigma} \cdots s_n \cdot \sigma u_n \cdot \tilde{\sigma} f^p(p) \\ &= - \left[\frac{i}{\sqrt{2}} \right]^n (-1)^{\text{number of spin-0 particles}} \text{Tr} s_1 \cdot \sigma u_1 \cdot \tilde{\sigma} \cdots s_n \cdot \sigma u_n \cdot \tilde{\sigma} f^p(p), \end{aligned} \quad (3.7)$$

where $f^p(p) \equiv f(G^p, p)$ and $f^0(p) = f(G^0, p)$.

Any trace $\text{Tr} a_1 \cdot \tilde{\sigma} a_2 \cdot \sigma a_3 \cdot \tilde{\sigma} \cdots a_{2n} \cdot \sigma$ is a sum of a scalar part that is unchanged by $a_i \rightarrow \tilde{a}_i$ and a pseudoscalar part that changes sign. Since $f^p(p) = f^0(\tilde{p}) = f^0(p)$ the equations (3.5a) and (3.7) imply [with $A^0 \equiv A(G^0), A^p \equiv A(G^p)$] that

$$A^0 + A^p = 2 \times \text{scalar part of } A^0 \text{ if number of spin-0 particles is even,} \quad (3.8a)$$

$$A^0 + A^p = 2 \times \text{pseudoscalar part of } A^0 \text{ if number of spin-0 particles is odd.} \quad (3.8b)$$

This means that $A^0 + A^p$ conserves parity, provided the spin-0 particles are identified as pseudoscalar particles and the spin-1 particles are identified as vector particles.

C. Antiparticle conjugation

The ortho-propagators and para-propagators are $(p_{aj} \cdot \sigma) / m_{aj}$ and $(-p_{bj} \cdot \sigma) / m_{bj}$, respectively. Ac-

ording to (2.67) these forms are invariant under the antiparticle conjugation operation \mathcal{C} . This result suggests that the ortho-amplitudes and para-amplitudes should be separately invariant under antiparticle conjugation. This invariance would, in fact, be strictly implied if the quarks could be considered separate entities, each with its own initial and final momenta p_{bj} and p_{aj} . It was the analytic continuation $p_{aj} \leftrightarrow p_{bj}$ of these momenta into each

other that was the basis of the discussion of antiparticle conjugation in Sec. II F. In that context antiparticle conjugation was equivalent (up to a sign) to reversing the directions of all the quark arrows. This reversal was accomplished by an equivalent analytic continuation. In that continuation the vector p in the propagator $p \cdot \sigma / m$ continues to be the momentum associated with a fixed end of the quark line. Thus an ortho-propagator is transformed into an ortho-propagator, and a para-propagator goes into a para-propagator.

We therefore define antiparticle conjugation to be the operation of reversing the direction of each quark edge, with the ortho-propagator—para-propagator type left unchanged. Thus antiparticle conjugation interchanges the two graphs (a) and (b) of Fig. 4. The scalar functions $f(p)$ are assumed to be unchanged by antiparticle conjugation. Thus the amplitudes associated with graphs (a) and (b) are

$$A_a^0 = - \left[\frac{i}{\sqrt{2}} \right]^n \times (\text{Tr} s_1 \cdot \tilde{\sigma} u_1 \cdot \sigma \cdots s_n \cdot \tilde{\sigma} u_n \cdot \sigma) f^0(p) \quad (3.9a)$$

and

$$A_b^0 = - \left[\frac{i}{\sqrt{2}} \right]^n \times (\text{Tr} s_n \cdot \tilde{\sigma} u_n \cdot \sigma \cdots s_1 \cdot \tilde{\sigma} u_1 \cdot \sigma) f^0(p) \quad (3.9b)$$

Then use of (2.65) and (2.66) gives

$$A_b^0 = (-1)^{\sum_i (\text{spin})_i} A_a^0 \quad (3.10)$$

D. Isospin

Quark flavors have not yet been discussed. Introduction of the up and down quarks yields the π , ρ , η , and ω mesons. To get the amplitude corresponding to a graph with these mesons as the external particles one includes for each vertex the isotopic-spin factor f_i defined in Fig. 5.

G parity is conserved for the sum of the ortho-amplitudes and the sum of the para-amplitudes

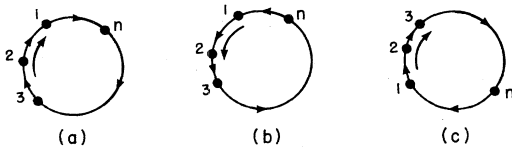


FIG. 4. Graphs (a) and (b) are related by antiparticle conjugation. Graph (c) is graph (b) turned over.

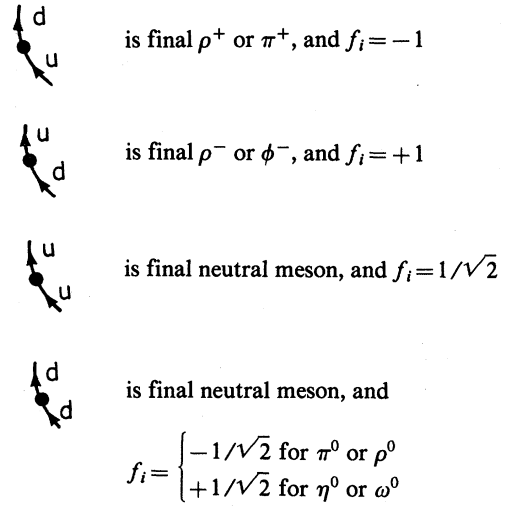


FIG. 5. The isotopic-spin factors. The full zero-entropy amplitude for any process involving a set of n of these mesons is the sum of the amplitudes corresponding to all the ways in which the particles of the reaction can be identified with the vertices of ortho-graphs and para-graphs with n vertices.

separately. To see this note that for each ortho-graph (para-graph) contributing to a process there is another one in which the u and d quarks are interchanged and the cyclic order of the particles is reversed. The two associated ortho-graphs are related as the two graphs (a) and (c) of Fig. 4 are, apart from flavor labels. Since (c) is equivalent to (b) one obtains the factor (3.10) together with the isospin factors f_i shown in Fig. 5. These factors f_i combine to give factors for the graphs (a) and (c) that differ by the factor

$$\prod_i (-1)^{(\text{isospin})_i}$$

Thus the sum of the two contributions is

$$A_a^0 + A_c^0 = A_a^0 [1 + (-1)^g], \quad (3.11a)$$

where

$$g = \sum_i (\text{spin})_i + (\text{isospin})_i \quad (3.11b)$$

The factor $(-1)^g$ is G parity. Hence G parity is conserved for the ortho-graph and para-graph parts separately.

E. Products

The discontinuity formulas involve products of amplitudes represented by graphs of the kind shown in Fig. 6. For each wiggly line there is a sum over

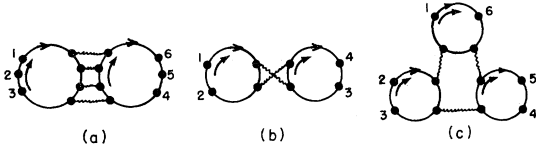


FIG. 6. Diagrams representing products of amplitudes. The wiggly lines represent the intermediate mesons.

the single pseudoscalar meson and the three vector mesons. When this sum is performed the spinor parts of these products are just the spinor parts of the functions associated with the diagrams of Fig. 7. In other words, the spinor parts satisfy the diagrammatic equations of Fig. 8.

To obtain this result, and also a more general one, let the four orthogonal vectors s_i associated with particle i be labeled by $e \in (0,1,2,3)$, with $s_e^u = v_i^u$ for $e=0$. To get the correct normalization we return to the level of the S matrix. Then the four amplitudes S_e are defined by

$$S_e = \text{Tr} \frac{s_e^r \cdot \tilde{\sigma}}{\sqrt{2}} S$$

$$= \frac{1}{\sqrt{2}} s_e^r \cdot \tilde{\sigma}^{\dot{\alpha}\beta} S_{\beta\dot{\alpha}}, \quad (3.12)$$

where the irrelevant indices on S have been suppressed. The arguments of Sec. II C then show that

$$S_e = \text{Tr} \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} M. \quad (3.13)$$

Consider therefore a product of the form

$$\sum_{e=0}^3 S_e S'_e = \sum_{e=0}^3 \left[\text{Tr} \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} M \right] \left[\text{Tr} \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} M' \right]. \quad (3.14)$$

To evaluate it introduce into the second trace the identity

$$s_e \cdot \tilde{\sigma} = v \cdot \tilde{\sigma} s^e \cdot \sigma v \cdot \tilde{\sigma}, \quad (3.15)$$

where v is the velocity $\pm p/m$ of the relevant particle and $s^0 = s_0$, $s^1 = -s_1$, $s^2 = -s_2$, and $s^3 = -s_3$.

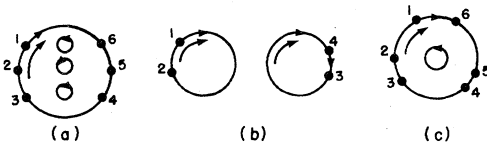


FIG. 7. Alternative representation of the spinor parts of the products represented in Fig. 6. A circle with no vertices represents $-\text{Tr}1 = -2$.

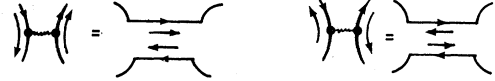


FIG. 8. Spinor identities.

(Each s_e is a four-vector.) Thus

$$\sum_{e=0}^3 S_e S'_e = \frac{1}{2} \sum_{e=0}^3 (\text{Tr} M s_e \cdot \tilde{\sigma}) (\text{Tr} s^e \cdot \sigma v \cdot \tilde{\sigma} M' v \cdot \tilde{\sigma}). \quad (3.16)$$

Use of

$$\sum_{e=0}^3 (s_e)^\mu (s^e)^\nu = g^{\mu\nu} \quad (3.17)$$

gives

$$\sum_{e=0}^3 S_e S'_e = \frac{1}{2} (\text{Tr} M \tilde{\sigma}_\mu) g^{\mu\nu} (\text{Tr} \sigma_{\nu\mu} v \cdot \tilde{\sigma} M' v \cdot \tilde{\sigma}).$$

Use of

$$\frac{1}{2} \tilde{\sigma}_\mu^{\dot{\alpha}\beta} g^{\mu\nu} \sigma_{\nu\gamma\delta} = \delta^{\dot{\alpha}}_{\dot{\delta}} \delta^{\beta}_{\gamma} \quad (3.18)$$

gives

$$\sum_{e=0}^3 S_e S'_e = \sum_{e=0}^3 \left[\text{Tr} M \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} \right] \left[\text{Tr} \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} M' \right]$$

$$= \text{Tr} M v \cdot \tilde{\sigma} M' v \cdot \tilde{\sigma}. \quad (3.19)$$

This result says that summing over all four exchanged particles is equivalent to cutting the two quark lines at the vertices and reconnecting them in the way shown in Fig. 9, with a metric factor $v \cdot \tilde{\sigma}$ placed at each reconnection point.

Consider one of the two reconnection points in Fig. 9 and its associated factor $v \cdot \tilde{\sigma}$. This point connects two line segments that were originally parts of the two loops connected by the meson line shown on the left-hand side of Fig. 9, and also in Fig. 6. Each of these two line segments is associated with a propagator, in accordance with the formulas of (3.5). One of these two propagators is of the form $\pm v \cdot \sigma$, and will cancel the factor $v \cdot \tilde{\sigma}$ associated with the reconnection point, up to a possible sign. Now the meson line connects two vertices. One is associated with an initial particle, the other with a final particle. Thus the signs of the corresponding vectors u_i will be opposite. Consequently, the signs of the two factors $\pm v \cdot \sigma$ that cancel against the two factors, $v \cdot \tilde{\sigma}$ of Fig. 9 will be opposite, and the cancellation of the two factors $v \cdot \tilde{\sigma}$ will leave a residual minus sign. This minus sign cancels one associated with closed quark loops. [This closed-loop sign appears explicit-

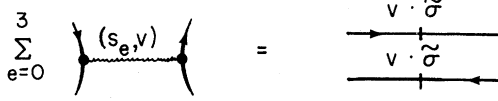


FIG. 9. Diagrammatic representation of (3.19).

ly in front of the trace symbol in (3.5).] For the transformation between the two sides of Fig. 9 changes the number of quark loops by one.

The two factors of i associated with the quark wave functions (is_i) of the two particles that are contracted out compensate in the case of the spin-1 particle for the fact that the physical spin-vector s_e occurring in the derivation of (3.19) is related to the mathematical one occurring in (3.5) by $s^{\text{phys}} = s^{\text{math}} \text{sign} u^0$, and in the case of the spin-0 particle by the fact that the physical vector $s_e = v$ occurring in the derivation of (3.19) is related to the mathematical vector $s_i = u_i$ occurring in (3.5) by this same relationship. Thus the relationship illustrated by Fig. 9 reduces to the simpler one illustrated by Fig. 8, from which follows the equality of the spin factors associated with the corresponding diagrams of Figs. 6 and 7.

An explanation of the signs and factors of i in (3.5) is in order. The minus sign in front of the trace comes from considering the quark wave function to be antisymmetric under the interchange of variables: an odd number of permutations is required to take the quark variables from their normal order (see Ref. 7) in which the propagators are $u_a \cdot \sigma = -u_b \cdot \sigma$ for forward scattering to the cyclic order associated with the closed loop. The imaginary unit is included in the wave function in order to allow the wave function to be folded into the basic unitarity equation for M functions, Eq. (7.1) of Ref. 7, without disrupting either the relative signs of the two terms on the left-hand side or the relationship

$$s^{\text{phys}} = s^{\text{math}} \text{sign} u^0.$$

F. Topological classification

Each circular quark graph G corresponding to a zero-entropy ortho-amplitude or para-amplitude can be transformed by the rule illustrated in Fig. 10 into a particle graph $g = g(G)$ with one internal vertex.

If G is a circular graph with n vertices then $g(G)$ is a tree graph with n edges, n external vertices, and one internal vertex. This internal vertex of $g(G)$ is classified as an ortho-vertex or a para-vertex according to whether G is an ortho-graph or a para-graph. These two kinds of internal vertices can be distinguished in the way illustrated in Fig. 10. The ar-

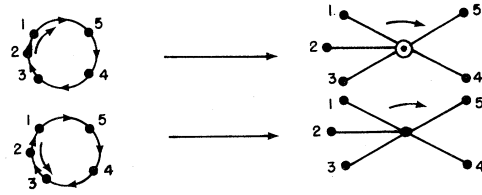


FIG. 10. Transformation of circular ortho-quark and para-quark graphs G into the corresponding basic particle graphs $g(G)$.

row near each internal vertex shows the direction of rotation of the quark line around that vertex. These graphs g are called basic particle graphs.

A product of basic particle graphs g_1, g_2, \dots is formed by identifying certain pairs of external vertices, as illustrated in Fig. 11. Each product graph g has a well-defined genus and boundary structure. These can be calculated by Edmond's rule. One first draws all the orbits of g . An orbit of g is a path in g formed as follows: one picks any point p on any edge of g and a direction $d(p)$ at that point. Then one traces a path in g by a moving point p' that starts from p in the direction $d(p)$. At each non-trivial vertex the moving point p' shifts to the "next" line, with the order of the lines specified by the arrow that indicates the quark-line direction. The orbit is completed when the moving point p' returns to the original point p moving in the original direction $d(p)$.

Some of the orbits may pass through vertices that lie at the ends of single (external) edges. These vertices correspond to the "external particles" associated with the graph. An orbit that passes through at least one external-particle vertex is called a boundary. The boundary structure consists of the collection of boundaries, each identified by the sequence of external-particle vertices through which it passes. Each external-particle vertex appears on exactly one boundary. Graphs with only one boundary are called one-boundary graphs.

The number of different orbits of g —sometimes called faces of g —is denoted by $f(g)$. The numbers of edges and vertices of g are denoted by $e(g)$ and $v(g)$, respectively. Then the genus of g —sometimes called the handle number—is given by the Euler formula

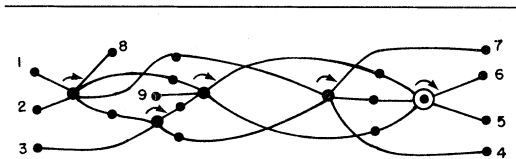


FIG. 11. A product g of five basic particle graphs g_i .

$$h(g) = \frac{e(g) - v(g) - f(g) + 2c(g)}{2}, \quad (3.20)$$

where $c(g)$ is the number of connected components of g . The graph of Fig. 11 has one orbit, which is the boundary (8,6,5,9,7,4,3,2,1), and its genus is two.

The zero-genus one-boundary graphs are the planar graphs. They are the graphs that can be drawn on a plane with no lines crossing and all external vertices identified with a single point at infinity.

An important characteristic of a graph g is its Betti number $\beta(g)$, which is the number of independent closed loops that can be drawn in the graph. Its value is given by

$$\beta(g) = e(g) - v(g) + c(g). \quad (3.21)$$

Let the number of boundaries of g be $b(g)$. The orbits that are not boundaries are called windows, and their number is $w(g) = f(g) - b(g)$. The most important topological characteristic of g is the topological index

$$\begin{aligned} \gamma(g) &= 2h(g) + b(g) - c(g) \\ &= \beta(g) - w(g). \end{aligned} \quad (3.22)$$

This is the number of independent closed loops in g minus the number of windows. For connected graphs with at least one boundary the topological index $\gamma(g)$ is zero if and only if the graph g has zero genus and exactly one boundary, i.e., if and only if g is planar.

This topological index $\gamma(g)$ enjoys the following "entropy" property: if $g_1 g_2$ is some connected product of two connected graphs g_1 and g_2 , then

$$\gamma(g_1 g_2) \geq \gamma(g_1) + \gamma(g_2) - 1. \quad (3.23)$$

To prove this let n be the number of vertices at which g_1 and g_2 are joined. Then (3.21) gives

$$\beta(g_1 g_2) = \beta(g_1) + \beta(g_2) - 1 + n. \quad (3.24)$$

On the other hand,

$$w(g_1 g_2) = w(g_1) + w(g_2) + w'(g_1, g_2), \quad (3.25)$$

where $w'(g_1, g_2)$ is the number of windows of $g_1 g_2$ that lie partly in g_1 and partly in g_2 .

Each of these windows that lies partly in each subgraph must pass at least twice through the n junction points. And each junction point lies exactly twice on the set of orbits. Thus one has the inequality

$$w'(g_1, g_2) \leq n, \quad (3.26)$$

which combines with (3.24) and (3.22) to give (3.23).

The entropy property (3.23) shows that the topological index $\gamma(g_1 g_2)$ of a product graph $g_1 g_2$ is

greater than either component, provided one of them has $\gamma(g_i) > 1$. This means that the topological complexity, as measured by $\gamma(g)$, increases in general. The special case $\gamma(g_i) = 1$ allows the complexity to remain unchanged.

If one of the graphs has $\gamma(g_i) = 0$, then (3.23) would allow for a decrease in complexity. However, if $\gamma(g_1) = 0$ and the product $g_1 g_2$ is such that at least one external vertex of g_1 is an external vertex also of the product graph $g_1 g_2$, then the right-hand side of (3.26) can be replaced by $n - 1$, since then at least one boundary of $g_1 g_2$ must pass twice through the set of junction points, and

$$\gamma(g_1 g_2) \geq \gamma(g_1) + \gamma(g_2).$$

The graphs corresponding to physical-region singularities can always be constructed by taking successive products $g_1, g_2 g_1, g_3 g_2 g_1, \dots$ so that the final external particles of each newly added graph are also final external particles of the new product graph.¹⁸ If the product graphs are built in this way, then the topological index $\gamma(g)$ can never decrease.

The zero-entropy amplitudes discussed in the earlier subsections are not the most general zero-entropy amplitudes. They are those amplitudes represented by circular graphs $G(D)$ like those of Fig. 1 in which all n quark lines represent ortho-propagators, or all n quark lines represent para-propagators.

In the meson sector each zero-entropy amplitude is represented by a circular graph $G(D)$. The complete set of (flavorless) zero-entropy amplitudes associated with any given graph $G(D)$ is obtained by allowing each of the n quark lines of $G(D)$ to represent, independently, either an ortho-propagator or a para-propagator. When flavors are considered each quark line is also assigned a flavor label.

The topological representation of the distinction between ortho-propagators and para-propagators is made by introducing particle-quark graphs $\bar{g}(G)$ of the kind shown in Fig. 12. The particle lines of such a graph divide the disk bounded by the quark lines into sections, each of which can be independently oriented. A quark line represents an ortho-propagator or a para-propagator according to whether the direction of the quark line agrees or disagrees with the direction induced on that line by the orientation of the section of the disk that is adjacent to it.

Consider any particle-quark graph $\bar{g} = \bar{g}_n \cdots \bar{g}_2 \bar{g}_1$ formed by joining together, in the fashion indicated in Fig. 11, a sequence of graphs $\bar{g}_i \equiv \bar{g}(G_i)$ of the kind shown in Fig. 12. In Fig. 11, each internal line is divided into two parts (edges) by a trivial vertex, which is a vertex that joins less than three edges. Cuts at the trivial internal vertices separate the

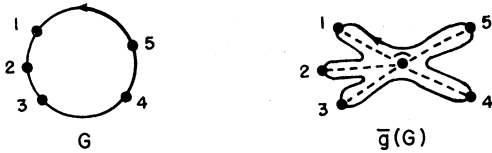


FIG. 12. A graph G and the quark-particle graph $\bar{g}(G)$ formed from G . The particle lines of $g(G)$ are drawn as dashed lines in $\bar{g}(G)$.

graph g of Fig. 11 into its zero-entropy components g_i . To pass from $g = g_n \cdots g_2 g_1$ to the corresponding $\bar{g} = \bar{g}_n \cdots \bar{g}_2 \bar{g}_1$, one replaces each particle graph $g_i = g(G_i)$, of the kind shown in Fig. 10, by its "thickened" version $\bar{g}_i = \bar{g}(G_i)$, of the kind shown in Fig. 12.

A vertex of \bar{g} at which two subgraphs \bar{g}_i meet is called a *junction vertex*. It is also an internal trivial vertex of the subgraph $g = g(\bar{g})$ formed from the vertices and particle lines of \bar{g} .

Each of the subgraphs $\bar{g}_i = \bar{g}(G_i)$ of \bar{g} is placed on the paper so that each quark line runs clockwise around the surface bounded by G_i . Then the surface associated with \bar{g} is, in the immediate neighborhood of any junction vertex V , separated into two sections by the two particle edges incident upon V . The two quark edges lying in each one of these two sections both run in the same direction.

Suppose, for each of these two sections, that both of the two quark lines lying in this section have the same flavor, and both represent ortho-propagators or both represent para-propagators. Then this junction vertex V is said to be a *removable vertex*. The spinor identities represented by Fig. 8, and the two other identities identical to them except that the directions of the two inner arrows agree, rather than disagree, allow the two quark edges lying in each section to be joined together into one: the spinor function will still be correctly represented. The information about the scalar parts of the function represented by the graph is retained in the subgraph g consisting of all the vertices and particle edges of \bar{g} .

The topological classification of a graph $\bar{g} = \bar{g}_n \cdots \bar{g}_2 \bar{g}_1$ is obtained by first disjoining the graphs \bar{g}_i at all junction vertices that are not removable, and then giving for each connected part \bar{g}'_j of this new graph \bar{g}' both its topological index $\gamma(\bar{g}'_j)$ and boundary structure, and giving also a list of pairs of vertices of \bar{g}' that must be equated to form \bar{g} . The boundary structure is a list of orbits. Each orbit is specified by the cyclic order of the vertices that appear on it, together with the specification of the ortho-amplitude or para-amplitude, and flavor, of the quark edge that joins each pair of adjacent vertices on the orbit. The orbits of \bar{g}'_j are identical to the closed quark loops of \bar{g}'_j .

G. Topological expansion

Each physical-region singularity of the S matrix is associated with a Landau graph g_L . A formula for the discontinuity around the singularity associated with graph g_L is obtained by replacing each vertex of g_L by the corresponding scattering function.^{15,16} This scattering function is specified by the set of edges incident upon the vertex to which it corresponds. These edges can be identified with the external edges of the particle graphs g constructed above.

The topological expansion is the assumption that each scattering function can be expressed as a sum of terms, one corresponding to each of the different topological classes specified in the preceding subsection. This expansion is required to be compatible with the discontinuity formulas, in the sense that if the full expansion is introduced into each of the scattering functions that occur in any discontinuity equation, and the full equation is then decomposed into terms of different topological class, then the terms of each class separately satisfy the equation: there is no cancellation among the terms in the equation that have different topological character. This assumption that the contributions to any discontinuity equation corresponding to graphs of the same topological character should cancel among themselves has been discussed extensively before, in connection with the derivation of the discontinuity formulas.¹⁹⁻²¹

H. The zero-entropy functions

Validity of the topological expansion is assumed. Then one can examine the zero-entropy parts of the various discontinuity formulas. Due to the entropy property these equations involve only zero-entropy functions. By virtue of the identity represented in Fig. 8 the spinor factors $F_{(\mu)}(G,p)$ of (3.1) drop completely out, leaving equations that involve only the scalar functions $f(G,p)$. These coupled equations are essentially the same as the set of discontinuity equations for a theory of scalar particles, except that all singularities associated with nonplanar Landau graphs vanish. The functions $f(G,p)$ thus have much simpler analytic properties than the full S matrix, and can be expected to have moving Regge poles but no Regge cuts. They should be similar to the Veneziano dual-resonance model functions,^{22,23} with the addition of a spin-flavor structure, finite widths, and a planar singularity structure of particle poles, normal threshold cuts, and the other singularities associated with planar Landau diagrams. The overall normalization should, however, be fixed by the nonlinear equations. Because of the expected correspondence with dual-resonance model func-

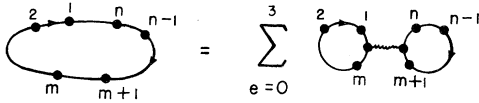


FIG. 13. Pole factorization property.

tions, there is some hope that these equations can be approximately solved, and the full solution then formally reconstructed by an essentially perturbative expansion in which zero-entropy functions replace the usual point coupling functions.

I. Regge recurrences

The property represented in (3.19) and Fig. 9 entails that the spinor factor in the zero-entropy functions A^0 or A^p has the pole-factorization property indicated in Fig. 13. Thus if $f^0(p)$ has a factorizable pole corresponding to a certain value $l \geq 1$ of angular momentum transferred between $(1, \dots, m)$ and $(m+1, \dots, n)$, then the full function A^0 has factorizable poles corresponding to a set of four intermediate states, having total angular momentum values $J=l$ and $l+1, l, l-1$.

If $f^0(p)$ has Regge behavior of the kind exhibited by the Veneziano dual-resonance function,²³ then for each factorizable pole of $f^0(p)$ corresponding to orbital angular momentum $l \geq 1$ there will be a quartet of factorizable poles of $A^0(p)$ corresponding to total angular momentum l and $(l+1, l, l-1)$.

The function $f^0(p)$ is assumed to have a Regge pole with the lowest $l=0$ pole identifiable with our external set of sixteen mesons $(\pi, \rho, \omega, \eta)$, which are assumed to be degenerate in the zero-entropy level. The higher values of l will then generate recurrences of the set $(\pi, \rho, \omega, \eta)$.

If the functions $f^0(p)$ and $f^p(p)$ are now generalized to represent the cases where the external particles are recurrences of the $l=0$ mesons then one must include^{24,25} for each external particle i of angular momentum l_i a set of l_i vector indices, $\mu_1^{(i)} \dots \mu_{l_i}^{(i)}$ that are such that

$$p_i \cdot f^0(p) = p_i \cdot f^p(p) = 0$$

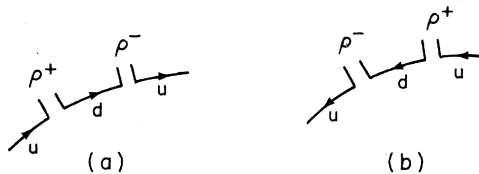


FIG. 14. The normal quark structures of ρ^+ and ρ^- are shown in (a), whereas (b) shows the definition induced by reversing the quark lines. If the quark wave function has angular momentum l_i , then the difference is represented by a factor $(-1)^{l_i}$.

when the inner product is formed with any one of the indices $\mu_{l_i}^{(i)}$. Consequently, the earlier equation

$$f^p(p) = \mathcal{P}(f^0(p)) = f^0(\bar{p}) = f^0(p) \tag{3.27a}$$

becomes replaced by

$$f^p(p) = \mathcal{P}(f^0(p)) = (-1)^{\sum_{i=1}^n l_i} f^0(\bar{p}) \tag{3.27b}$$

When nonzero values of the l_i are allowed there is also an extra factor of

$$(-1)^{\sum l_i}$$

in the charge conjugation equation (3.10). This comes from a consideration of, for example, the two definitions of ρ^+ and ρ^- implicit in Figs. 14(a) and 14(b).

In the discussion of Sec. III D of isospin invariance there was no change in the definitions of ρ^+ and ρ^- of the kind shown in Fig. 14. However, the function $f^0(p)$ was changed due to a reversal of the order of the arguments [see Fig. 4(c)]. In the dual-resonance amplitude⁷ this change induces a change

$$(-1)^{\sum l_i},$$

and we assume that this property holds also for our function $f^0(p)$:

$$f^0(p_n, \dots, p_1) = (-1)^{\sum l_i} f^0(p_1, \dots, p_n) \tag{3.28}$$

The fact that one gets the same factor

$$(-1)^{\sum l_i}$$

by either reversing the direction of the quark arrow, as in Figs. 4(a)–4(b) [or Figs. 14(a)–14(b)], or by reversing the cyclic order of the vertices, as in Figs. 4(a)–4(c), means that the amplitude corresponding to a graph does not depend on how this graph is placed on the paper: the operator of turning over or reflecting a graph, as in Figs. 4(b) and 4(c), does not alter the amplitude corresponding to it. Thus the equivalence of the two graphs of Fig. 2, or of Fig. 3, is maintained also for $l_i > 0$.

IV. SUMMARY AND CONCLUSIONS

Spin can be incorporated into the meson topological expansion if we do the following four things:

(1) Adopt the M function formalism and associate the leading (trailing) end of each quark line with a lower undotted (dotted) index.

(2) Introduce into the topological expansion a “zero-entropy” level that lies below the “ordered” level of Chew and Rosenzweig.²

(3) For each cyclically ordered set of n mesons (of fixed flavor content) form the various possible zero-entropy M functions. Each of these is a product of

a scalar function $f_n(p_1, p_2, \dots, p_n)$ of the scalar products of the momentum-energy vectors p_j of the n mesons, times a spin factor. This spin factor is minus the product of n spin factors, one for each of n quark lines. The spin factor associated with any given quark line is either the ortho-quark factor $u_a \sigma_{\alpha\beta}$ or the para-quark factor $-u_b \sigma_{\alpha\beta}$, where α is the spinor index associated with the leading end of the quark line, β is the spinor index associated with the trailing end of the quark line, u_a is the mathematical covariant velocity p_a/m_a of the physical particle associated with the vertex lying on the leading end of the quark line, and u_b is the mathematical covariant velocity p_b/m_b of the physical particle associated with the vertex lying on the trailing end of the quark line. [The minus sign associated with the spin factor comes from the odd number of permutations of the quark variables required to take them from their normal order associated with the no-scattering part, for which each spin factor is $v_a \cdot \sigma_{\alpha\beta} = v_b \cdot \sigma_{\alpha\beta}$, to the order in which the two indices α_j and β_j associated with physical particle j stand together and in the order (β_j, α_j) .]

(4) Institute invariance under the parity operation by adding together the 2^n zero-entropy functions that arise from the association of each of the n quark lines with, alternatively, either the ortho-function or the para-function.

A quartet of amplitudes corresponding to one pseudoscalar meson and one three-component vector meson is obtained by folding the four wave functions $i2^{-1/2} \tilde{\sigma}^{\beta_j \alpha_j} \cdot u_j$ and $i2^{-1/2} \tilde{\sigma}^{\beta_j \alpha_j} \cdot s_{je}$ ($e=1, 2, 3$) into the M function. Here s_{j1} , s_{j2} , and s_{j3} are three vectors that satisfy $(s_{je})^2 = -1$ and $s_{je} \cdot v_j = 0$.

The theory obtained in this way from the two-component formalism can be directly transcribed into a four-component notation by the methods of Ref. 7. This transcription is carried out in Appendix C, and the results are described here. The use of the four-component notation facilitates comparison to earlier works.

For each cyclic ordering of n mesons (of fixed flavors) there are 2^n different zero-entropy amplitudes, one for each combination of choices of the ortho-function—para-function character of the n quark lines that cyclically join the n vertices corresponding to the n mesons. The basic property of the zero-entropy amplitudes is that the spin factors can be completely factored out of the associated discontinuity equations. Thus the nonlinear integral equations for the scalar factors $f_n(p)$ are the same for all 2^n zero-entropy functions. These equations are relatively simple, and should determine the coupling constants at the zero-entropy level. They also ensure that the same scalar factor $f_n(p)$ occurs in all 2^n zero-entropy functions. Hence the sum \sum_n of the

2^n zero-entropy functions is simply the sum of the spin factors, times the common factor $f_n(p)$.

This sum \sum_n of the 2^n zero-entropy functions is expressed in the four-component formalism as

$$\sum_n = -(\text{Tr} \Gamma_{1\mu_1}(u_1) \Gamma_{2\mu_2}(u_2) \cdots \Gamma_{n\mu_n}(u_n)) \times f_n(p_1, \dots, p_n), \quad (4.1)$$

where the pseudoscalar particle is associated with the factor

$$\Gamma_{j5}(u_j) = i\gamma_5(1 + \gamma \cdot \mu_j) / \sqrt{2}, \quad (4.2a)$$

and the three components of the vector particle are associated with the four factors

$$\Gamma_{j\mu}(u_j) = (\gamma_\mu - i\sigma_{\mu\rho} u_j^\rho) / \sqrt{2}. \quad (4.2b)$$

Here μ and ρ range over the set $(0,1,2,3)$, there is a sum over the repeated vector index ρ , and $u_j^\rho \Gamma_{j\rho}(u_j) = 0$.

The individual zero-entropy functions are obtained by choosing an ortho-function or a para-function character for each quark line, and then inserting after each factor $\Gamma_{j\mu}(u_j)$ of (4.1) a factor of $\frac{1}{2}(1 + \gamma_5)$ or $\frac{1}{2}(1 - \gamma_5)$ according to whether the quark line that leads into vertex j has an ortho-function or a para-function character. These two operators project onto the states of opposite chirality.

Each quintet of factors $\Gamma_{j\mu}(u_j)$ in (4.2) defines a representation of the nonchiral group $(U(2) \times U(\bar{2}))_{u_j}$. That is, in a rest frame of particle j , where u_j is pure timelike and $\Gamma_{j0}(u_j) = 0$, one has

$$A_j^- \Gamma_{j\mu}(u_j) B_j^\dagger = \sum_\nu \Gamma_{j\nu}(u_j) \Lambda_\mu^\nu(A_j^-, B_j^\dagger). \quad (4.3)$$

Here the indices μ and ν range over the set $(0,1,2,3,5)$, and A_j^- and B_j^\dagger are

$$A_j^- = \exp[i(1 - \beta u_j^0) \sigma \cdot a^-] \quad (4.4a)$$

and

$$B_j^\dagger = \exp[i(1 + \beta u_j^0) \sigma \cdot b^+], \quad (4.4b)$$

where a^- and b^+ are two real four-vectors, and the σ_μ are the 4×4 matrices with two 2×2 σ_μ 's in the diagonal corners. The two other operators

$$A_j^+ = \exp[i(1 + \beta u_j^0) \sigma \cdot a^+] \quad (4.5a)$$

and

$$B_j^- = \exp[i(1 - \beta u_j^0) \sigma \cdot b^-] \quad (4.5b)$$

acting on the $\Gamma_{j\mu}(u_j)$'s from the left and right, respectively, act as unit operators [see (C72) and (C48)]. Hence (4.3) holds also if A_j^- and B_j^\dagger are replaced by

$$A(a, a') = \exp(i\sigma \cdot a + i\beta\sigma \cdot a') \tag{4.6a}$$

and

$$B(b, b') = \exp(i\sigma \cdot b + i\beta\sigma \cdot b') , \tag{4.6b}$$

respectively.

The invariance of the $\Gamma_{j\mu}(u_j)$ under the transformations (4.5) is a consequence of the use of the two-component M -function formalism, or, equivalently, of the Dirac equation for the corresponding four-component quantities. It eliminates the possible scalar and axial vector particles, and fixes the couplings of the pseudoscalar and vector particles to be precisely those shown in (4.2).

The transformation properties shown in (4.3) and (4.4) do not in general entail corresponding invariance properties of the S matrix. This is because the velocities u_j of the various particles are generally different, and hence the transformations shown in (4.3) act, for different particles, in different frames of reference, which are related by Lorentz transformations.

One subgroup of the group generated by the transformations (4.6) is of particular interest. This is the subgroup $SU(2)_W$ formed by imposing the following restrictions on the coefficients in (4.6):

$$a_3 = -b_3 , \tag{4.7a}$$

$$a'_1 = -b'_1 , \tag{4.7b}$$

$$a'_2 = -b'_2 , \tag{4.7c}$$

$$a_1 = a_2 = b_1 = b_2 = a'_3 = b'_3 = 0 , \tag{4.7d}$$

and

$$\Gamma_{j\mu}(u_j) = \begin{bmatrix} \sqrt{v_j \cdot \sigma} & 0 \\ 0 & \sqrt{v_j \cdot \bar{\sigma}} \end{bmatrix} \Gamma_{j\mu}^0(u_j) \begin{bmatrix} \sqrt{v_j \cdot \bar{\sigma}} & 0 \\ 0 & \sqrt{v_j \cdot \sigma} \end{bmatrix} . \tag{4.11}$$

If v lies in the 0-3 plane then the boost transformations appearing in (4.11) commute with the generators of $SU(2)_W$, since both γ_3 and $\gamma_0 = \beta$ are invariant under $SU(2)_W$. Thus if the three-velocities \vec{v}_j of all n particles are directed along the third coordinate axis, then the Lorentz transformations in (4.11) do not disrupt the invariance of (4.1) under $SU(2)_W$: the function \sum_n remains invariant.

If quarks of three flavors are allowed and the 4×4 matrices $\Gamma_{j\mu}(u_j)$ are expanded to the corresponding 12×12 matrices $\Gamma'_{j\mu, \alpha_j}(u_j)$, then these new 12×12 matrices will define a representation of $SU(6)_W$, and the new function \sum'_n , formed as in (4.1) but with Γ'_j in place of Γ_j , will be invariant under $SU(6)_W$.

A comparison with several earlier works may be

$$a_0 = a'_0 = b_0 = b'_0 , \tag{4.7e}$$

These conditions entail that $B = A^{-1}$, and that the transformations corresponding to $(\delta a'_1, \delta \alpha'_2, \delta \alpha_3)$ are generated by the triad of generators

$$(W_1 = \beta\sigma_1, W_2 = \beta\sigma_2, W_3 = \sigma_3) . \tag{4.8}$$

These three generators enjoy the same commutation relations as the generators $(\sigma_1, \sigma_2, \sigma_3)$ of $SU(2)$.

In the rest frame of particle j the quintet of factors $\Gamma_{j\mu}(u_j)$ occurring in (4.2) reduces to a quartet. If these four factors are identified with four factors X_μ in the following way,

$$\gamma_3(1 + \beta u_j^0) = X_0 , \tag{4.9a}$$

$$i\gamma_5(1 + \beta u_j^0) = X_3 , \tag{4.9b}$$

$$-\gamma_1(1 + \beta u_j^0) = X_2 , \tag{4.9c}$$

$$\gamma_2(1 + \beta u_j^0) = X_1 , \tag{4.9d}$$

then, for all i, j , and k in the set (1,2,3),

$$[W_i, X_0] = 0 \tag{4.10a}$$

and

$$[W_i, X_j] = 2i\epsilon_{ijk}X_k . \tag{4.10b}$$

Thus X_0 transforms as a W -spin singlet, and the set (X_1, X_2, X_3) transforms as a W -spin triplet.

If all particles are at rest, then the function \sum_n defined in (4.1) is invariant under $SU(2)_W$. The Lorentz transformation that boosts $\Gamma_{j\mu}(u_j)$ from its rest frame form $\Gamma_{j\mu}^0(u_j)$ to its form $\Gamma_{j\mu}(u_j)$ in the standard coordinate system is given by

helpful. Bardakci and Halpern²⁶ also introduce spin factors analogous to Chan-Paton factors, but arrive at a 16-particle multiplet in place of our 4-particle multiplet. They find in addition to P and V , the associated parity-doublet partners S and A , which occur, moreover, as ghost (negative-metric) particles. They find also a second set of eight particles (S', P', V', A') that couple differently to the quarks. These 16 particles correspond to the 16 independent matrices needed to span the space of 4×4 (Dirac) matrices.

The present work is based on the two-component formalism and consequently gives in place of the 16 particles of Bardakci and Halpern only 4 particles. Considered from the four-component viewpoint the two Dirac equations (C48) reduce the multiplicity of particles by a factor of 4: they reduce the 16 parti-

cles of Bardakci-Halpern to the 4 independent ones coupled in accordance with (4.2).

Bardakci and Halpern introduce the spinor solutions $U(p_j)$ and $\bar{U}(p_j)$ of the Dirac equation, but their way of using these spinor solutions does not give them the crucial Dirac equations (C48).

The present approach enforces the usual discontinuity equations, including pole factorization, at the zero-entropy level. But parity invariance is not maintained at that level. Parity invariance, and also $SU(6)_W$ invariance for spatially linear processes, is maintained for the sum \sum_n of the 2^n zero-entropy functions, but pole factorization does not hold for these sums.

The terms needed to restore pole factorization at the physical level come from higher-order terms in the topological expansion. To obtain an approximate representation of a physical amplitude near a singularity one can insert into dispersion relations the discontinuity functions obtained from the planar amplitudes of Chew and Rosenzweig. These amplitudes are built out of the zero-entropy functions. The aim of the construction of these zero-entropy amplitudes is not to obtain immediately a good approximation to the physical amplitudes. It is rather to define and determine the basic building blocks of the theory.

The results obtained here are formally similar to those obtained by Delbourgo *et al.*²⁷ from considerations of the group $\tilde{U}(12)$. They use a four-component spin formalism, but impose the Dirac (i.e., Bargmann-Wigner) equations in a way that yields results similar to (C48). The principal difference is that they interpret their analogs of our functions \sum_n as interaction terms of a local field theory, rather than as low-order amplitudes of a topological expansion. Thus the function $f_1(p_1, p_2, \dots, p_n)$ would, in their approach, presumably be an undetermined constant whereas in the present approach it would be constrained (and, it is hoped, determined) by the nonlinear zero-entropy equations. It should also enjoy, for example, Regge asymptotic behavior. The full amplitude would be constructed in their theory by essentially a power-series expansion, but in the present theory by including the remaining terms in the topological expansion. The crucial question is whether the self-consistent structure of the functions $f_n(p_1, \dots, p_n)$ determined by the zero-entropy equations, in conjunction with the topological expansion procedure, will eliminate the divergences of field

theory associated with both renormalization and the divergence of the perturbation series expansion.

Before these questions can be addressed it is necessary to include baryons into the theory: Chew has found that the topological expansion scheme with mesons alone is not soluble, due to the minus sign associated with the closed loop. Inclusion of baryons (and baryonium) leads to a soluble system that gives a predicted ratio of the meson and hadronic coupling constants that is in good agreement with experiment.⁸ Preliminary results indicate that the overall magnitudes of the strong-interaction coupling constants, as determined by the nonlinear integral equations for the zero-entropy functions, agrees with experiment at least in order of magnitude.⁹

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APPENDIX A: ORIGIN OF DISCONTINUITY FORMULAS

Discontinuity formulas are derived in S -matrix theory nominally from macrocausality and unitarity.²⁸ However, it was recognized long ago that unitarity is not essential. What is directly used in the derivations is not unitarity, but rather the property that the *inverse* of the S matrix possess the antinormal analytic structure. This antinormal analytic structure is the same as the normal analytic structure derived for the S matrix from macrocausality, except that the plus $i\epsilon$ rule is replaced by the minus $i\epsilon$ rule. It is the property that the singularities in the real region of definition be confined to the positive- α Landau surfaces, and that the function near these singularities be defined by the minus $i\epsilon$ limiting procedure. That the inverse of S should possess this antinormal analytic structure can be derived from unitarity and the fact that S possesses the normal analytic structure. But the property should hold regardless of whether S is unitary or not.

To see the essential point in the simplest way consider first the formal perturbative solution. Then the S matrix can be written in the form²⁹

$$\langle p' | S | p \rangle = \langle p' | p \rangle - 2\pi i \delta(E_p - E_{p'}) \lim_{\epsilon \downarrow 0} \langle p' | T(E + i\epsilon) | p \rangle, \quad (\text{A1})$$

where

$$T(E) = V + V \frac{1}{E - H_0} V + V \frac{1}{E - H_0} V \frac{1}{E - H_0} V + \dots \quad (\text{A2})$$

If one defines R^\pm by

$$-2\pi i \delta(E_p - E_{p'}) \lim_{\epsilon \rightarrow 0} \langle p' | T(E_p \pm i\epsilon) | p \rangle \\ \equiv \langle p' | R^\pm | p \rangle, \quad (\text{A3})$$

then it is easy to verify, formally, that

$$R^+ - R^- = R^+ R^- = R^- R^+. \quad (\text{A4})$$

Thus the definitions

$$S^\pm = I \pm R^\pm \quad (\text{A5})$$

allow one to write

$$S^+ S^- = S^- S^+ = I. \quad (\text{A6})$$

Hence the operator S^- defined by (A5) is the inverse of $S \equiv S^+$. Consequently, the operators R^+ and R^- defined by the plus $i\epsilon$ and minus $i\epsilon$ limiting procedures on the same function $T(E)$ define formally the operators $(S - I)$ and $-(S^{-1} - I)$, respectively.

These relationships are usually derived from unitarity. But the above derivation does not depend on the Hermitian character of V : it goes through, formally at least, even if S is not unitary.

In S -matrix theory the antinormal analytic structure of S^{-1} is usually derived from unitarity and the normal analytic structure of S . However, it can be derived, alternatively, directly from the antimacrocausality property of S^{-1} . This latter property is the same as the macrocausality property except that the sign of time is reversed, so that physical particles carry positive energy from later times to earlier times, rather than *vice versa*.^{28,30}

If S satisfies macrocausality, then S^{-1} , if it exists, should satisfy antimacrocausality. This will not be proved here, but the following argument makes it very plausible.

Consider a normalized initial state ϕ that represents a system of incoming particles each of which is represented by an incident wave packet with fairly well defined momenta and trajectory region. The action of S takes ϕ to $S\phi = \psi$, and the action of S^{-1} on ψ takes it back to $\phi = S^{-1}\psi$. The two reciprocal processes $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$ are thus closely related. If S is unitary, then ψ is normalized. If S is not unitary, then ψ need not be normalized. But in any case the action of S^{-1} on ψ is closely connected to the action of S on ϕ : S constructs ψ from ϕ , and

S^{-1} reconstructs ϕ from ψ .

One can imagine ψ to be decomposed into components ψ_i corresponding to various combinations of outgoing particles with fairly well defined momenta and trajectory locations. Then the wave functions ϕ and ψ_i can be wave functions of the kind used in the macrocausality arguments of Ref. 30. If S satisfies macrocausality, then in various dilated situations of the kind discussed in that reference the dominant contributions to the process $\phi \rightarrow \psi$ will correspond to physical scattering processes. If S^{-1} should fail to satisfy antimacrocausality, then for some process $\phi \rightarrow \psi$ there would be contributions to $\psi \rightarrow \phi$ that do not correspond to temporally reversed physical processes, yet do not fall off in the way demanded by macrocausality for the corresponding direct process that contributes to $\phi \rightarrow \psi$. But then the dominant contributions to $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$ would not be temporal inverses of each other, and the close reciprocal connection of these two processes would have to be maintained via an intricate interplay of contributions that are not naturally related via temporal inversion.

Although such a situation is perhaps conceivable, it will almost surely not be achievable in situations having the complexity of relativistic particle physics. Thus I think it safe to assume, in the general S -matrix context, that S^{-1} , if it exists, should be antimacrocausal regardless of whether the S matrix is unitary or not: the antimacrocausality property of S^{-1} is a more primitive and basic property than unitarity. From this antimacrocausality property one can deduce immediately from the arguments of Ref. 30 the antinormal analytic structure of S^{-1} needed in the derivation of the discontinuity formulas.

Further insight into the connection between discontinuity equations and the inverse of S can be obtained by considering the S matrix from the point of view of the "in" and "out" parts of the wave functions in radial coordinates. Separating out the center-of-mass motion of an n -particle system, one is left with a function of various relative coordinates (x_1, \dots, x_{n-1}) . An alternative set of coordinates consists of $R = (r_1, \dots, r_{n-1})$ and $\Omega = (\Omega_1, \dots, \Omega_{n-1})$, where (r_i, Ω_i) are the radial and angular coordinates associated with the relative motion of some pair of subsystems. If $\psi(R, \Omega)$ is a steady-state solution and $K = (k_1, \dots, k_{n-1})$ is a set of $n-1$ scalars defining momentum magnitudes, then one may define the asymptotic amplitude

$$A(K, \Omega) = \sum_{i=1}^{n-1} \left[\lim_{\epsilon_i \rightarrow 0^+} \int_0^\infty dr_i \frac{ik_i r_i}{(2\pi)^2} [\theta(r_i - \rho_i) e^{-i(k_i - i\epsilon_i)(r_i - \rho_i)} \right. \\ \left. + \theta(\rho_i - r_i) e^{-i(k_i + i\epsilon_i)(r_i - \rho_i)} \right] \psi(R, \Omega) \quad (\text{A7})$$

where $\rho_i \equiv \epsilon_i^{-2}$, and $R^0 \equiv (r_i - \rho_i, \dots, r_{n-1} - \rho_{n-1})$.

This definition is such that if ψ is a plane wave

$$\sum_{i=1}^{n-1} \exp(i \vec{x}_i \cdot \vec{k}_i^0),$$

and the directions and magnitudes of the $n-1$ three-vectors \vec{k}_i^0 are specified by the sets of coordinates

$$\Omega^0 = (\theta_1^0, \phi_1^0, \dots, \theta_{n-1}^0, \phi_{n-1}^0)$$

and

$$K^0 = (k_1^0, \dots, k_{n-1}^0),$$

where $k_i^0 > 0$ for all i , then the corresponding $A(K, \Omega)$ is

$$A_0(K, \Omega; K^0, \Omega^0) = \delta(K, K^0) \delta(\Omega, \Omega^0) + \delta(K, \bar{K}^0) \delta(\Omega, \bar{\Omega}^0), \quad (\text{A8})$$

where

$$\delta(K, K') = \prod_{i=1}^{n-1} \delta(k_i - k'_i), \quad (\text{A9})$$

$$\delta(\Omega, \Omega') = \prod_{i=1}^{n-1} \delta(\cos \theta_i - \cos \theta'_i) \delta(\phi_i - \phi'_i), \quad (\text{A10})$$

$$\bar{K} = (-k_1, \dots, -k_{n-1}), \quad (\text{A11})$$

and

$$\bar{\Omega} = (\pi - \theta_1, \phi_1 + \pi, \dots, \pi - \theta_{n-1}, \phi_{n-1} + \pi). \quad (\text{A12})$$

If, on the other hand, ψ represents the solution to the scattering problem with the incident or incoming state given by the same plane wave

$$\exp \left[\sum_{i=1}^{n-1} i \vec{k}_i^0 \cdot \vec{r}_i \right]$$

that was used in (A8), then the amplitude $A(\Omega, K)$ can be written as

$$A(K, \Omega; K^0, \Omega^0) = S(K, \Omega; K^0, \Omega^0) + \delta(K, \bar{K}^0) \delta(\Omega; \bar{\Omega}^0), \quad (\text{A13})$$

$$\begin{aligned} A'(K, \Omega) &= \int dK^0 \int d\Omega^0 A(K, \Omega; K^0, \Omega^0) S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0) \\ &= \int dK^0 \int d\Omega^0 S(K, \Omega; K^0, \Omega^0) + \delta(K, \bar{K}^0) \delta(\Omega, \bar{\Omega}^0) S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0) \\ &= \int dK^0 \int d\Omega^0 S(K, \Omega; K^0, \Omega^0) S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0) + S(K, \Omega; \bar{K}_1^0, \bar{\Omega}_1^0). \end{aligned} \quad (\text{A18})$$

The conditions (A14) and (A17) entail that the incoming part of $A'(K, \Omega)$ is the same as the incoming part of the solution whose asymptotic amplitude is given in (A16), with (K^0, Ω^0) set equal to (K_1^0, Ω_1^0) . Thus, by virtue of the causality principle, the asymptotic outgoing parts of these two solutions must also be the same:

where

$$\theta(K) S(K, \Omega; K^0, \Omega^0) = S(K, \Omega; K^0, \Omega^0). \quad (\text{A14})$$

Here

$$\theta(K) = \sum_{i=1}^{n-1} \theta(k_i), \quad (\text{A15})$$

and $\theta(k_i)$ is 0 or 1 according to whether k_i is negative or non-negative. Equation (A14) expresses the condition that the incoming part of the asymptotic wave [i.e., the part having the behavior $\exp(-i \sum k_i r_i)$] is the same as that of the incident plane wave. Equation (A13) defines the S matrix in these variables.

The k_i^0 in K^0 all satisfy $k_i^0 > 0$. Substitution of the argument $(\bar{K}^0, \bar{\Omega}^0)$ for (K^0, Ω^0) in (A13) gives

$$A(K, \Omega; \bar{K}^0, \bar{\Omega}^0) = S(K, \Omega; \bar{K}^0, \bar{\Omega}^0) + \delta(K, K^0) \delta(\Omega, \Omega^0). \quad (\text{A16})$$

The condition, analogous to (A14), that the two terms in (A16) represent separately the asymptotic incoming and outgoing parts of ψ is

$$\theta(\bar{K}) S(K, \Omega; \bar{K}^0, \bar{\Omega}^0) = S(K, \Omega; \bar{K}^0, \bar{\Omega}^0), \quad (\text{A17})$$

where the k_i^0 in K^0 always satisfy $k_i^0 > 0$.

The wave functions ψ corresponding to the asymptotic amplitudes defined in (A13) and (A16) represent solutions that have incoming and outgoing parts equal, respectively, to the incoming and outgoing parts of the plane wave whose asymptotic amplitude is given in (A8).

We now invoke two general principles. The first is the superposition principle, which asserts that a linear superposition of solutions ψ is a solution ψ' . The second is the causality principle that the incoming parts of a solution should determine uniquely the outgoing parts. Using the superposition principle one can form a solution ψ' by taking a linear superposition over various values of (K^0, Ω^0) in the solutions ψ that correspond to the amplitudes $A(K, \Omega; K^0, \Omega^0)$ of (A13). The weight factor will be chosen to be $S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0)$, for some fixed $(\bar{K}_1^0, \bar{\Omega}_1^0)$. Thus the asymptotic amplitude corresponding to ψ' will be

$$\int dK_1^0 \int d\Omega_1^0 S(K, \Omega; K_1^0, \Omega_1^0) S(\bar{K}_1^0, \bar{\Omega}_1^0; \bar{K}^0, \bar{\Omega}^0) = \delta(K, K_1^0) \delta(\Omega, \Omega_1^0). \quad (\text{A19})$$

But then the introduction of the notation

$$S(\bar{K}^0, \bar{\Omega}^0; \bar{K}_1^0, \bar{\Omega}_1^0) \equiv \bar{S}(K^0, \Omega^0; K_1^0, \Omega_1^0) \quad (\text{A20})$$

allows one to write (A19) in (A20) in the form

$$S\bar{S} = I. \quad (\text{A21})$$

Consequently the transformation of shifting to the barred coordinates takes S to its inverse. The transformation that takes (K, Ω) to $(\bar{K}, \bar{\Omega})$ reverses both the signs of all momentum magnitudes and the directions of all the vectors. Hence the associated sets of vectors \vec{k}_i are transformed into themselves. Thus when the functions S and S^{-1} are expressed in terms of these vector arguments one needs two separate coverings of the physical region, one for S and one for S^{-1} . On the other hand, the points (K, Ω) and $(\bar{K}, \bar{\Omega})$ lie in two separate regions of (k, Ω) space. These regions are joined at the point where all $k_i = 0$.

The formal manipulations given above suggest that there might be some sort of analytic connection between the functions $S(K, \Omega; K', \Omega')$ defined above in the two disjoint regions $k_i, k_i' > 0$ and $k_i, k_i' < 0$. One may examine this question in specific models, and in particular in nonrelativistic models with real or complex (and local or nonlocal) potentials. If the potentials are short ranged, so that the singularities near the transition point where all $k_i = 0$ arise exclusively from the singularities in the propagators, then the singularity structure near the transition point should be correctly represented by perturbation theory.

Near the two-particle threshold in a theory with one kind of particle the propagator $(E - H_0)^{-1}$ becomes, when the overall center-of-mass motion is

$$M_r^+(P'; P'') - M_r^-(P'; P'') = \int M_r^+(P'; P) V \cdot \tilde{\sigma} M_r^-(P; P'') dP, \quad (\text{B2})$$

where $V \cdot \tilde{\sigma}$ stands for the product of factors $v_i \cdot \tilde{\sigma}$. Continuation around the leading threshold is supposed to take the connected part of

$$M_r^+(P'; P'') \equiv M_r(P', P'')$$

into the connected part of $M_r^-(P'; P'')$.

The basic topological assumption is that the separation of $M_r^+(P'; P'')$ and $M_r^-(P'; P'')$ into parts having different topological characters separates the discontinuity equation into parts having different topological characters. This entails that the ortho-function and para-function parts satisfy an equation of the form (B2), but with only planar singularities.

The ortho-function and para-function parts of

factored out, just a one-particle propagator. This propagator, as it occurs in the functions $\langle p' | R^\pm | p \rangle$ defined in (A3), is³¹

$$-\frac{m \exp(\pm i k_p r)}{2\pi r}, \quad (\text{A22})$$

where

$$k_p = |(2mE_p)^{1/2}|. \quad (\text{A23})$$

Consequently the functions $\langle p | R^+ | p' \rangle$ and $\langle p | R^- | p' \rangle$, when expressed in terms of the variables $(K, \Omega; K', \Omega')$, will be analytically connected, regardless of whether the potential is real or not. This analytic connection entails a corresponding connection between $S(K, \Omega; K', \Omega')$ and

$$S^{-1}(K, \Omega; K', \Omega') = S(\bar{K}, \bar{\Omega}; \bar{K}', \bar{\Omega}').$$

These questions can be discussed in greater depth within the context of various special models. However, the point of the above discussion is to note that very general considerations, which lie deeper than particular models, strongly indicate that the familiar analytic connection between S and S^{-1} should be maintained independently of unitarity.

APPENDIX B: FAILURE OF UNITARITY FOR ORTHO-AMPLITUDES AND PARA-AMPLITUDES

Let the M function be decomposed into its "unit" part plus the remainder:

$$M(P'; P'') = M_u(P'; P'') + M_r(P'; P''). \quad (\text{B1})$$

Then the basic discontinuity equation has the form⁷

$M_r^+(P'; P'')$ and $M_r^-(P'; P'')$ have polynomial factors that are specified by the rules given in the text. These polynomial factors have no singularity at the threshold. Thus they are the same for the ortho-function (para-function, respectively) parts of $M_r^+(P', P'')$ and $M_r^-(P', P'')$. Moreover, they are the same also for the ortho-function (para-function, respectively) part of the right-hand side of (B2). This consequence of our rules leads to an important simplification of the discontinuity equations: the polynomial spin factor is the same for each term, and hence can be factored out. Thus the discontinuity formula at the ortho-function or para-function level becomes, essentially, a discontinuity

equation for the residual scalar function.

The polynomial spin factor in the ortho-function part of $M_r^+(P';P'')$ is built of factors $u \cdot \sigma$, one for each quark line. For example, a quark line whose leading end lies at a vertex associated with a variable p_i contributes a factor $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$, where $u_i = p_i/m_i$, and α_i and $\dot{\alpha}_j$ are spinor indices associated with the leading and trailing ends of the quark line, respectively.

The polynomial spin factor associated with $M_r^-(p';P'')$ gets from this same quark line an identical factor $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$. However, the function

$$M_r^+(P';P'') = M_r^*(P'';P')$$

gets a factor $-u_j \cdot \sigma_{\alpha_i \dot{\alpha}_j}$, for real u_j . This is not equal to $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$, nor even proportional to it, since u_i and u_j are generally nonparallel.

On the other hand, the contribution $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$ from any given quark line to the ortho-amplitude $M_r(P';P'')$ is the same as the contribution $u_i \cdot \sigma_{\alpha_i \dot{\alpha}_j}$ from this same quark line to the Hermitian conjugate of the *para-amplitude* contribution to $M_r(P';P'')$. Thus the sum of the ortho-amplitudes and para-amplitudes satisfies

$$M_{r0}^- + M_{rP}^- = -(M_{r0}^+ + M_{rP}^+) . \quad (B3)$$

Hence when the ortho-amplitude and para-amplitude parts are combined one recovers the familiar Hermitian analyticity relation, generally derived from (extended) unitarity.

Since Hermitian analyticity fails for ortho-amplitudes and para-amplitudes one cannot expect unitarity to hold for them.

APPENDIX C: FOUR-COMPONENT FORMALISM

This appendix transcribes the results obtained above in the two-component formalism into the more familiar four-component formalism.

The connection between the two-component and four-component formalisms is most easily expressed

$$\psi_\alpha(x) = \int \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) \theta(p^0) \sum_{\lambda=1,2} [U_\alpha(p, \lambda) e^{-ipx} a(p, \lambda, t) + V_\alpha(p, \lambda) e^{ipx} b^\dagger(p, \lambda, -t)] , \quad (C8)$$

where

$$V(p, \lambda) = EU^*(p, \lambda) \quad (C9)$$

and

$$U(p, \lambda) = EV^*(p, \lambda) . \quad (C10)$$

The charge conjugation operator \mathcal{C} is defined so

by using the Weyl representation for the four-by-four Dirac matrices. In this representation

$$\alpha_\mu = \begin{bmatrix} \tilde{\sigma}_\mu & 0 \\ 0 & \sigma_\mu \end{bmatrix} ,$$

$$\beta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} ,$$

$$\gamma_\mu = \beta \alpha_\mu = \begin{bmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{bmatrix} , \quad (C1)$$

$$\gamma_5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} ,$$

$$\sigma_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \quad (i=1,2,3) .$$

The two-by-two charge conjugation matrix is

$$C = -i\sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (C2)$$

and gives

$$C\sigma_\mu C^{-1} = \tilde{\sigma}_\mu^{\text{Tr}} , \quad (C3)$$

where $\tilde{\sigma} = (\sigma_0, -\vec{\sigma})$. Thus

$$E = \begin{bmatrix} 0 & C \\ C^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix} \quad (C4)$$

satisfies $E = E^{-1} = E^{\text{Tr}} = E^\dagger = E^*$, where Tr, dagger, and star represent transpose, Hermitian conjugation, and complex conjugation, respectively. The important properties of E are

$$E^\dagger \beta E = -\beta^{\text{Tr}} , \quad (C5)$$

$$E^\dagger \alpha_\mu E = \alpha_\mu^{\text{TR}} , \quad (C6)$$

and

$$E^\dagger \sigma_i E = -\sigma_i^{\text{TR}} . \quad (C7)$$

The free-field operator for Dirac particles of type t is

that

$$\mathcal{C}^\dagger \psi(x) \mathcal{C} = E \psi^\dagger(x) \equiv \psi^c(x) . \quad (C11)$$

The interchange $\psi(x) \leftrightarrow \psi^c(x)$ is equivalent to the interchange $a \leftrightarrow b$.

Parity is represented by the operator \mathcal{P} , which satisfies

$$\mathcal{P}^\dagger \psi_i(x) \mathcal{P} = \epsilon_i \beta \psi_i(\bar{x}) \equiv \psi_i^P(x), \quad (\text{C12})$$

where ϵ_i is the intrinsic parity associated with ψ_i .

The time reversal matrix Ω is

$$\Omega = \begin{bmatrix} C^{-1} & 0 \\ 0 & C \end{bmatrix} = \beta E, \quad (\text{C13})$$

and it satisfies

$$\Omega^\dagger \beta \Omega = \beta^{\text{Tr}}, \quad (\text{C14})$$

$$\Omega^\dagger \alpha_i \Omega = -\alpha_i^{\text{Tr}}, \quad (\text{C15})$$

and

$$\Omega^\dagger \sigma_i \Omega = -\sigma_i^{\text{Tr}}. \quad (\text{C16})$$

The Wigner complex conjugation operator K satisfies, for all states Ψ and Φ , and all complex numbers α and β ,

$$\langle K\Psi | K\Phi \rangle = \langle \Psi | \Phi \rangle^* = \langle \Phi | \Psi \rangle, \quad (\text{C17})$$

$$K(\alpha | \Psi \rangle + \beta | \Phi \rangle) = \alpha^* | K\Psi \rangle + \beta^* | K\Phi \rangle, \quad (\text{C18})$$

and

$$K^2 = 1. \quad (\text{C19})$$

Defining the operator A^K by

$$\langle \Psi | A^K | \Psi \rangle \equiv \langle K\Psi | A | K\Psi \rangle \quad (\text{all } \Psi) \quad (\text{C20a})$$

one obtains from application of (C17)–(C19) to

$$\langle \alpha\Psi + \beta\Phi | A^K | \alpha\Psi + \beta\Phi \rangle, \quad (\text{C20b})$$

with $\alpha = 1$ and $\beta = 1$ and i , the result

$$\langle \Phi | A^K | \Psi \rangle = \langle K\Psi | A | K\Phi \rangle \quad (\text{C21})$$

for all Ψ and Φ . The definition of K is completed by taking

$$|Kx\rangle = |x\rangle \quad (\text{C22a})$$

and

$$|Kp\rangle = |-p\rangle \quad (\text{C22b})$$

in first-quantized theory, and by taking

$$K\psi_{\text{op}}(x)K = \psi_{\text{op}}(x) \quad (\text{C23})$$

and

$$K|0\rangle = |0\rangle \quad (\text{C24})$$

in second-quantized theory. That is, the ket $|x\rangle$, the field operator $\psi(x)$, and the vacuum are considered real. Thus in first-quantized theory if

$$\langle x | \psi \rangle = f(x), \quad (\text{C25})$$

then

$$\langle X | K\psi \rangle = \langle Kx | \psi \rangle^* = f^*(x), \quad (\text{C26})$$

whereas in second-quantized theory if

$$\langle 0 | \psi(x) | \Psi \rangle = f(x), \quad (\text{C27})$$

then

$$\begin{aligned} \langle 0 | \psi(x) | K\Psi \rangle &= \langle 0 | \psi(x) K\Psi \rangle \\ &= \langle K0 | K\psi(x) K\Psi \rangle^* \\ &= \langle 0 | \psi(x) \Psi \rangle^* \\ &= f^*(x). \end{aligned} \quad (\text{C28})$$

The time reversal operator is then

$$\mathcal{T} = UKT, \quad (\text{C29})$$

where

$$T^\dagger \psi(x) T = \psi(x'), \quad (\text{C30})$$

with $x' = (-t, x)$, and

$$U^\dagger \psi_\alpha(x) U = [\Omega \psi(x)]_\alpha. \quad (\text{C31})$$

For any operator A one may define A' by

$$\langle \mathcal{T}\psi | A | \mathcal{T}\Psi \rangle = \langle \Psi | A' | \Psi \rangle \quad (\text{C32})$$

for all Ψ . Then arguments similar to those leading to (C21) give

$$A' = T^\dagger K U^\dagger A^\dagger U K T. \quad (\text{C33})$$

The current and spin operators

$$J_\mu(x, t) = \frac{1}{2} [\psi^\dagger(t, x), \alpha_\mu \psi(t, x)] \quad (\text{C34})$$

and

$$\sigma_i(x, t) = \frac{1}{2} [\psi^\dagger(t, x), \sigma_i \psi(t, x)] \quad (\text{C35})$$

then satisfy

$$\begin{aligned} J_\mu^i(t, x) &= \tilde{J}_\mu(-t, x) \\ &\equiv (J_0(-t, x), -\vec{J}(-t, x)) \end{aligned} \quad (\text{C36})$$

and

$$\sigma_i^i(t, x) = \tilde{\sigma}_i(-t, x) = -\sigma_i(-t, x). \quad (\text{C37})$$

Thus the time-reversal operation on the states generates the change in expectation values demanded by the physical meaning of the operation of time reversal.

Suppressing the dependence on all other particles one may write the S operator for the scattering of a Dirac particle as

$$S_{\text{op}} = \int \psi^\dagger(x') G(x'; x) \psi(x) d^4x' d^4x. \quad (\text{C38})$$

The operators $a(p, \lambda, t)$ and $b(p, \lambda, t)$ are normalized by

$$\begin{aligned}
\langle a(p, \lambda, t) a^\dagger(p', \lambda', t) \rangle_0 \\
&= \langle b(p, \lambda, t) b^\dagger(p', \lambda', t') \rangle_0 \\
&= 2\omega(2\pi)^3 \delta^3(\vec{p} - \vec{p}') \delta_{\lambda\lambda'} \delta_{tt'}, \quad (C39)
\end{aligned}$$

where $\omega \equiv (p^2 + m^2)^{1/2}$, and λ and t are the spin- and particle-type labels. Then the S matrix for the scattering of a Dirac particle of type t is

$$\begin{aligned}
S(p', \lambda', t; p, \lambda, t) \\
&= \langle a(p', \lambda', t) S_{\text{op}} a^\dagger(p, \lambda, t) \rangle_0 \\
&= U^\dagger(p', \lambda') G(p', -p) U(p, \lambda), \quad (C40)
\end{aligned}$$

where the type label on G is suppressed. The spinors in (C40) are

$$\begin{aligned}
U(p, \lambda) &= \begin{pmatrix} (v \cdot \sigma)^{1/2} \phi_\lambda \\ (v \cdot \tilde{\sigma})^{1/2} \phi_\lambda \end{pmatrix} \frac{1}{\sqrt{2}}, \\
U^\dagger(p, \lambda) &= \begin{pmatrix} \phi_\lambda (v \cdot \sigma)^{1/2} \\ \phi_\lambda (v \cdot \tilde{\sigma})^{1/2} \end{pmatrix} \frac{1}{\sqrt{2}}, \quad (C41)
\end{aligned}$$

$$V(p, \lambda) = \begin{pmatrix} (v \cdot \sigma)^{1/2} \phi_\lambda^c \\ (-)(v \cdot \tilde{\sigma})^{1/2} \phi_\lambda^c \end{pmatrix} \frac{1}{\sqrt{2}},$$

and

$$V^\dagger(p, \lambda) = \begin{pmatrix} \phi_\lambda^c (v \cdot \sigma)^{1/2} \\ \phi_\lambda^c (v \cdot \sigma)^{1/2} (-1) \end{pmatrix} \frac{1}{\sqrt{2}},$$

where $v = p/m$, $p_0 > 0$, $\phi^c = C\phi$, $\phi_{\lambda\alpha} = \delta_{\lambda\alpha}$, and

$$[(v \cdot \sigma)^{1/2} \phi_\lambda]_\beta = \sum_\alpha (v \cdot \sigma_{\beta\alpha})^{1/2} \phi_{\lambda\alpha},$$

$$[\phi_\lambda (v \cdot \tilde{\sigma})^{1/2}]^\beta = \sum_\alpha \phi_{\lambda\alpha} (v \cdot \tilde{\sigma}^{\alpha\beta})^{1/2},$$

etc.

The M function for the scattering of the Dirac particle is defined by

$$M(p', t; -p, -t) = (v \cdot \sigma)^{1/2} S(p', t; p, t) (v \cdot \sigma)^{1/2}. \quad (C42)$$

The insertion of (C40) and (C41) gives

$$M(p', t; -p, -t) = (v \cdot \sigma) G_{UU}(p', -p) (v \cdot \sigma) + (v \cdot \sigma) G_{UL}(p', -p) + G_{LU}(p', -p) v \cdot \sigma + G_{LL}(p', -p), \quad (C43)$$

where the four-by-four matrix G is written as

$$G = \begin{pmatrix} G_{UU} & G_{UL} \\ G_{LU} & G_{LL} \end{pmatrix} = \begin{pmatrix} 2G'_{UU} & 2G'_{UL} \\ 2G'_{LU} & 2G'_{LL} \end{pmatrix}. \quad (C44)$$

In terms of the mathematical momentum-energy vectors k' and k , and the associated vectors $u' = k'/m$ and $u = k/m$, Eq. (C43) can be written

$$M(k', t; k, -t) = (u' \cdot \sigma) G'_{UU}(k', k) (-u \cdot \sigma) + (u' \cdot \sigma) G'_{UL}(k', k) + G'_{LU}(k', k) (-u \cdot \sigma) + G'_{LL}(k', k). \quad (C45)$$

The four-by-four M function is defined by

$$M(k', k) = \begin{pmatrix} M_{UU}(k', k) & M_{UL}(k', k) \\ M_{LU}(k', k) & M_{LL}(k', k) \end{pmatrix} = \begin{pmatrix} 1 & u' \cdot \tilde{\sigma} \\ u' \cdot \sigma & 1 \end{pmatrix} G'(k', k) \begin{pmatrix} 1 & (-u \cdot \sigma) \\ (-u \cdot \tilde{\sigma}) & 1 \end{pmatrix}, \quad (C46)$$

where the type variables t are now suppressed. The original two-by-two M function is $M_{LL}(k', k)$. The other three two-by-two parts are trivially related to $M_{LL}(k', k)$. In particular, one has

$$M(k', k) = \begin{pmatrix} (u' \cdot \tilde{\sigma}) M(k', k) (-u \cdot \tilde{\sigma}) & u' \cdot \tilde{\sigma} M(k', k) \\ M(k', k) (-u \cdot \tilde{\sigma}) & M(k', k) \end{pmatrix}, \quad (C47)$$

where the two-by-two and four-by-four M functions are represented by the same symbol. These relationships, or (C46), entail the Dirac equations

$$(u' \cdot \gamma)(\beta M) = (\beta M) = (\beta M)(-u \cdot \gamma). \quad (\text{C48})$$

The two-by-two M function given in (C42) refers explicitly to the physical process involving an outgoing Dirac particle of type t and an incoming Dirac particle of type t . The function $G(x', x)$ in (C38) describes also the three related processes in which the incoming particle is changed to an outgoing antiparticle or the outgoing particle is changed to an incoming antiparticle, or both. If, for example, the outgoing particle is changed to an incoming antiparticle, then (C40) is replaced by

$$\begin{aligned} S(p', \lambda', -t; p, \lambda, t) &= \langle S_{\text{op}} b^\dagger(p', \lambda', -t) a^\dagger(p, \lambda, t) \rangle_0 \\ &= -[V^\dagger(p', \lambda')] G(-p', -p) U(p, \lambda) \\ &= \phi_\lambda^c [(v' \cdot \sigma)^{1/2} (-1) G'_{UL}(-p', -p) (v \cdot \sigma)^{1/2} \\ &\quad + (v' \cdot \sigma)^{1/2} (-1) G'_{UL}(-p', -p) (v \cdot \tilde{\sigma})^{1/2} \\ &\quad + (v' \cdot \tilde{\sigma})^{1/2} G'_{LU}(-p', -p) (v \cdot \sigma)^{1/2} \\ &\quad + (v' \cdot \tilde{\sigma})^{1/2} G'_{LL}(-p', -p) (v \cdot \tilde{\sigma})^{1/2}] \phi_\lambda. \end{aligned} \quad (\text{C40}')$$

The charge conjugation operator in ϕ^c arises from the convention whereby λ' is to be contracted onto the right-hand index of the spin operator if the particle is an initial particle but onto the left-hand index of the spin operator if the particle is a final particle. The convention for M functions is that the contraction rule is independent of whether the particle involved is initial or final, but that the sign of the mathematical rest-frame spin vector s_r occurring in the spin operator $s_r \cdot \sigma$ is tied to the sign of the associated energy component in the manner specified in (2.53). Then (C3) allows the c on ϕ^c in (C40') to be absorbed into the definition of the spin operators. The S matrix corresponding to the M function is thus the quantity in the brackets in the last line of (C40'). It is converted to the M function of the process, namely, to $M(-p', t; -p, -t)$, by multiplying it by the factors $(v' \cdot \sigma)^{1/2}$ and $(v \cdot \sigma)^{1/2}$, just as in (C.42). This gives

$$M(-p', t; -p, -t) = -v' \cdot \sigma G'_{UU}(-p', -p) v \cdot \sigma - v' \cdot \sigma G'_{UL}(-p', -p) + G'_{LU}(-p', -p) v \cdot \sigma + G'_{LL}(-p', -p). \quad (\text{C43}')$$

For initial particles $v = -u$. Thus this equation is a special case of (C45). Thus the two different processes are described by the one function $M(k', t; k, -t)$ defined in (C45), evaluated in different regions in (k', k) space. The function $G(k', k)$ in these different regions is obtained from the Fourier transformation of the single function $G(x', x)$.

The third case is that in which the original incoming particle is changed to an outgoing particle. Then (C40) is replaced by

$$\begin{aligned} S(p', \lambda', t; p, \lambda, -t) &= \langle a(p', \lambda', t) b(p, \lambda, -t) S_{\text{op}} \rangle_0 = -U^\dagger(p', \lambda') G(p', p) V(p, \lambda) \\ &= \phi_\lambda [(v' \cdot \sigma)^{1/2} G'_{UU}(p', p) (v \cdot \sigma)^{1/2} (-1) + (v' \cdot \sigma)^{1/2} G'_{UL}(p', p) (v \cdot \tilde{\sigma})^{1/2} \\ &\quad + (v' \cdot \tilde{\sigma})^{1/2} G'_{LU}(p', p) (v \cdot \sigma)^{1/2} (-1) \\ &\quad + (v' \cdot \tilde{\sigma})^{1/2} G'_{LL}(p', p) (v \cdot \tilde{\sigma})^{1/2}] \phi_\lambda^c. \end{aligned} \quad (\text{C40}'')$$

The S matrix corresponding to this M function is the quantity in the square brackets in (C40''). It is converted to the M function by multiplying it by $(v' \cdot \sigma)^{1/2}$ and $(v \cdot \sigma)^{1/2}$, just as in (C42). This gives

$$M(p', t; p, -t) = v' \cdot \sigma G'_{UU}(p', p) (-v \cdot \sigma) + v' \cdot \sigma G'_{UL}(p', p) + G'_{LU}(p', p) (-v \cdot \sigma) + G'_{LL}. \quad (\text{C43}'')$$

Since both particles are final, one has $u = v$ and $u' = v'$. Thus this is also a special case of the function defined in (C45).

The fourth and final case is similar. The order in which the arguments of the M function are placed is the same as the order of the operators that create or annihilate the corresponding particles from the vacuum. [See (C40), (C40'), and (C40'') and the corresponding equations (C43), (C43'), and (C43'').] Then the fourth case gives

$$\begin{aligned}
-M(p, -t; -p', t) &= M(-p', t; p, -t) \\
&= (-v' \cdot \sigma) G_{UU}^i(-p', p)(-v \cdot \sigma) + (-v' \cdot \sigma) G_{UL}^i(-p', p) \\
&\quad + G_{LU}^i(-p', p)(-v \cdot \sigma) + G_{LL}^i(-p', p), \tag{C43'''}
\end{aligned}$$

where the antisymmetry of M under the interchange of spin- $\frac{1}{2}$ variables is used. The second part of (C43''') is equivalent to (C45), and hence the one function $M(k', t; k, \bar{t})$ describes all four processes.

The formulas given above refer to a single spin- $\frac{1}{2}$ particle. But they immediately generalize to the case of n spin- $\frac{1}{2}$ particles: one treats each such particle in the manner shown above.

The parity transformation P defined in (C12) induces in $G(k', k)$ the change

$$G(k', k) \rightarrow \epsilon(\pi \otimes \beta_i) G(\tilde{k}', \tilde{k}) \left[\prod_i \otimes \beta_i \right], \tag{C49}$$

where ϵ is the product of intrinsic parities at the fields, and i runs over the particle-antiparticle pairs. A little algebra shows that this transformation on G induces the same transformation on the M function defined in (C46):

$$M(k', k) \rightarrow \epsilon \left[\prod_i \otimes \beta_i \right] M(\tilde{k}', \tilde{k}) \left[\prod_i \otimes \beta_i \right]. \tag{C50}$$

Consideration of the no-scattering part entails that the product of the intrinsic parities of $\psi_i(x)$ and $\psi_i^\dagger(x)$ is unity. Thus the intrinsic parities of the spin- $\frac{1}{2}$ fields drop out of ϵ , as in (2.56). Then (2.58) allows (2.55) to be written as

$$\mathcal{P}M(k', t; k, -t) = \epsilon(u' \cdot \sigma) M(\tilde{k}', t; \tilde{k}, -t)(-u \cdot \sigma), \tag{C51}$$

where ϵ is the product of the intrinsic parities of the scalar particles. The parity transformation (C49) on the function G occurring in the expression (C45) for the two-by-two M function induces the transformation $M \rightarrow \mathcal{P}M$, with $\mathcal{P}M$ defined by (C51).

The antiparticle conjugation operation $\psi(x) \leftrightarrow \psi^c(x)$ defined in (C11) replaces S_{op} by

$$\begin{aligned}
S_{\text{op}}^c &= \int [(\psi^c(x'))^\dagger G(x', x) \psi^c(x) d^4x' d^4x = \int [E \psi^\dagger(x')]^\dagger G(x', x) E \psi^\dagger(x) d^4x' d^4x \\
&= \int \psi^\dagger(x) [-E^\dagger G(x', x) E]^\text{Tr} \psi(x) d^4x' d^4x. \tag{C52}
\end{aligned}$$

Comparison to (C38) shows that the antiparticle conjugation operation is equivalent to the operation

$$G(x', x) \leftrightarrow G^c(x', x) = [-E^\dagger G(x, x') E]^\text{Tr} = -E G^\text{Tr}(x, x') E, \tag{C53}$$

where Tr means transpose in spin space.

The momentum-energy space version of (C53) is obtained by replacing x and x' by k and k' , respectively. This transformation on $G(k', k)$ induces on the four-by-four M function defined in (C46) the transformation

$$\begin{aligned}
M(k', t; k', -t) &\leftrightarrow M^c(k', t', k, -t) \\
&= \frac{1}{2} \begin{bmatrix} 1 & u' \cdot \tilde{\sigma} \\ u' \cdot \sigma & 1 \end{bmatrix} \left[-E G^\text{Tr}(k, k') E \right] \begin{bmatrix} 1 & -u \cdot \sigma \\ -u \cdot \tilde{\sigma} & 1 \end{bmatrix} \\
&= -E M^\text{Tr}(k, t; k', -t) E. \tag{C54}
\end{aligned}$$

It induces on the two-by-two M function defined in (C45) the transformation

$$M(k', t; k, -t) \leftrightarrow \mathcal{C}M(k', t', k, -t),$$

where $\mathcal{C}M$ is defined by (2.61).

The parity (or antiparticle conjugation) operation acting on M converts it to the M function that describes processes in a conceivable world in which the amplitude for any process P is equal to the amplitude that the parity inverse (or antiparticle conjugate) of P has in the actual physical world. The analogous time-reversal operation on the four-by-four M function is obtained by making the substitution $S_{\text{op}} \leftrightarrow S_{\text{op}}^t$, defined by (C38) and (C33):

$$\begin{aligned}
S_{\text{op}} \leftrightarrow S'_{\text{op}} &= T^\dagger K U^\dagger S_{\text{op}}^\dagger U K T = T K U^\dagger \int \psi^\dagger(x) G^\dagger(x', x) \psi(x') d^4 x' d^4 x U K T \\
&= \int K [U^\dagger \psi^\dagger(x') U] G^\dagger(x', x) U^\dagger \psi(x'') U K d^4 x' d^4 x = \int K \psi^\dagger(x') \Omega^{\text{Tr}} G^\dagger(x', x) \Omega \psi(x'') K d^4 x' d^4 x \\
&= \int K \psi^\dagger(x') \Omega^{\text{Tr}} G^\dagger(x', x'') \Omega \psi(x) K d^4 x' d^4 x = \int \psi^\dagger(x') \Omega^\dagger G^{\text{Tr}}(x', x'') \Omega \psi(x) d^4 x' d^4 x. \quad (\text{C55})
\end{aligned}$$

Thus the time-reversal operation on the states or fields is equivalent to the operation on G ,

$$\begin{aligned}
G(x', x) &\leftrightarrow G^\dagger(x', x) \\
&= \Omega^\dagger G^{\text{Tr}}(x', x'') \Omega, \quad (\text{C56})
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
G(k', k) &\leftrightarrow G^\dagger(k', k) \\
&= \Omega^\dagger G^{\text{Tr}}(-\tilde{k}, -\tilde{k}') \Omega. \quad (\text{C57})
\end{aligned}$$

This transformation induces in the four-by-four M function defined by (C46) the analogous transformation

$$\begin{aligned}
M(k', k) \leftrightarrow M^\dagger(k', k) &= \Omega^\dagger M^{\text{Tr}}(-\tilde{k}, -\tilde{k}') \Omega. \\
& \quad (\text{C58})
\end{aligned}$$

The transformation (C57) induces in the two-by-two M function defined by (C45) the transformation

$$\begin{aligned}
M(k', t; k, -t) &\leftrightarrow \mathcal{T} M(k', t; k, -t) \\
&= \tilde{M}(-\tilde{k}, t; -\tilde{k}', -t). \quad (\text{C59})
\end{aligned}$$

In a model where all particles are constructed from spin- $\frac{1}{2}$ particles the intrinsic parity factor ϵ is unity. Then the product of parity inversion, antiparticle conjugation, and time reversal on the four-by-four M function gives, by virtue of (C50), (C54), (C58), and (C13),

$$\begin{aligned}
M(k', t; k, -t) &\rightarrow \mathcal{T} \mathcal{C} \mathcal{P} M(k', t; k, -t) \\
&= -M(-k', t; -k, -t). \quad (\text{C60})
\end{aligned}$$

The same result holds for the two-by-two submatrix $M_{LL} = M$.

The two-by-two M function has one dotted and one undotted index. Thus, by virtue of (2.70), the change of the sign of all of its vector arguments changes its sign. Thus the two-by-two M function is transformed into precisely itself by the product of the \mathcal{T} , \mathcal{C} , and \mathcal{P} transformations.

The formula (C60) refers to a situation involving only one spin- $\frac{1}{2}$ particle. For the case of n such particles the variables k' , k , and t are $4n$ -vectors, or n -vectors, and the minus sign in front of M should be $(-1)^n$. Thus the product of the transformations

$$\begin{aligned}
M(k', t; k, -t) &= \begin{pmatrix} M^{\dot{\alpha}\beta}(k', t; k, -t) & M^{\dot{\alpha}}_{\dot{\beta}}(k', t; k, -t) \\ M_{\alpha\dot{\beta}}(k', t; k, -t) & M_{\alpha\beta}(k', t; k, -t) \end{pmatrix}. \quad (\text{C62})
\end{aligned}$$

\mathcal{T} , \mathcal{C} , and \mathcal{P} again leaves M invariant.

In carrying out the calculations whose results are summarized above it is helpful to recall that the two-by-two function \tilde{M} can be expressed as $C^{-1} M^{\text{Tr}} C$. If M is a product of Pauli matrices, then \tilde{M} is obtained from M by transposing the order of these matrices and replacing each σ_i ($i=1,2,3$) by $-\sigma_i$.

The Lorentz transformation properties of the two-by-two M function is indicated by assigning spinor index types according to the rule

$$\begin{aligned}
M(K', t; k, -t) &= M_{LL}(k', t; k, -t) \\
&\rightarrow M_{\alpha\dot{\beta}}(k', t; k, -t) \\
&= M(k', \alpha, t; k, \dot{\beta}, -t). \quad (\text{C61a})
\end{aligned}$$

The transformation properties of several other two-by-two functions are indicated by the following assignments:

$$\begin{aligned}
\tilde{M}(k, t; k' - t) &\rightarrow \tilde{M}^{\dot{\alpha}\beta}(k, t; k', -t), \\
M(\tilde{k}', t; \tilde{k}, -t) &\rightarrow M^{\dot{\alpha}\beta}(\tilde{k}', t; \tilde{k}, -t), \\
M_{UU}(k', t; k, -t) &\rightarrow M_{UU}^{\dot{\alpha}\beta}(k', t; k, -t), \quad (\text{C61b}) \\
M_{UL}(k', t; -k, -t) &\rightarrow M_{UL\dot{\beta}}^{\dot{\alpha}}(k', t; k, -t), \\
M_{LU}(k', t; -k, t) &\rightarrow M_{LU\alpha}^{\beta}(k', t; k, -t).
\end{aligned}$$

In these equations one may interpret the variables k' and k as $4n$ -vectors, the variables t as n -vectors, and the variables α and β as n -spinors, e.g., $\alpha = (\alpha_1, \dots, \alpha_n)$. Our convention is that the spinor α goes with k' and the spinor β goes with k . The quantities on the right-hand sides of (C61) are invariant under the simultaneous action of the spinor transformation (2.9) and

$$(k', k) \rightarrow [L^{-1}(\Lambda)k', L^{-1}(\Lambda)k].$$

Comparison of (C61) to (C47) shows that one may interpret $(u' \cdot \bar{\sigma})$ and $(-u \cdot \bar{\sigma})$ acting on the left and right, respectively, as operators that simultaneously raise and dot (or undot) the indices α , and β , respectively. Then the subscripts U and L on the two-by-two functions $M(k', t; k, -t)$ can be dropped and the four-by-four M function written as

In formulas (3.5a) and (3.5b) for the ortho-amplitudes and para-amplitudes corresponding to quark closed loops the (quark) M functions are just the products $\prod_i (u_i \cdot \sigma)$ and $\prod_i (-u_i \cdot \sigma)$, respectively. In both cases the individual two-by-two M functions are M_{LL} and their spinor index types are as in $M_{\alpha\beta}$. However, it is possible to use different choices of index type, provided one makes compensating changes in the spin operators that occur in (3.5).

One convenient choice is to use $M^{\dot{\alpha}\beta}$ for the ortho-propagators and $M_{\alpha\dot{\beta}}$ for the para-propagators. Comparison of (C62) to (C47) shows that the ortho-propagator is then $\delta^{\dot{\alpha}\beta}$ and the para-propagator is $\delta_{\alpha\dot{\beta}}$.

The propagators can be considered the analogs of the functions G of field theory. Thus we write

$$G^0 = \begin{pmatrix} 0 & \delta^{\dot{\alpha}\beta} \\ 0 & 0 \end{pmatrix} \quad (\text{C63a})$$

and

$$G^p = \begin{pmatrix} 0 & 0 \\ \delta_{\alpha\dot{\beta}} & 0 \end{pmatrix}. \quad (\text{C63b})$$

The associated spin factors are

$$F_i^0 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ (s_i \cdot \tilde{\sigma} u_i \cdot \sigma)^{\dot{\alpha}\beta} & 0 \end{pmatrix} \quad (\text{C64a})$$

and

$$F_i^p = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & (-u_i \cdot \sigma s_i \cdot \tilde{\sigma})_{\alpha\dot{\beta}} \\ 0 & 0 \end{pmatrix}. \quad (\text{C64b})$$

$$F_i' \beta = \frac{i}{\sqrt{2}} \begin{pmatrix} (-u_i \cdot \sigma)(\tilde{s}_i \cdot \sigma)_{\beta\dot{\alpha}} & 0 \\ 0 & (\tilde{s}_i \cdot \sigma)(u_i \cdot \sigma)_{\dot{\beta}\alpha} \end{pmatrix}. \quad (\text{C69})$$

Then the contributions A^0 and A^p come from the lower right-hand and upper left-hand two-by-two sectors, respectively.

If the meson corresponding to vertex i is a spin-0 meson, then $s_i = u_i$, and (C67) becomes

$$F_i' \beta = \frac{i}{\sqrt{2}} \begin{pmatrix} (-u_i \cdot \sigma)(u_i \cdot \tilde{\sigma}) & \\ & (u_i \cdot \tilde{\sigma})(u_i \cdot \sigma) \end{pmatrix} = \frac{i\gamma_5}{\sqrt{2}}. \quad (\text{C70a})$$

On the other hand, if the meson associated with vertex i is a spin-1 meson then $u \cdot s = 0$ and

$$F_i' \beta = \frac{i}{\sqrt{2}} \begin{pmatrix} (s_i \cdot \sigma)(u_i \cdot \tilde{\sigma}) & 0 \\ 0 & (s_i \cdot \tilde{\sigma})(u_i \cdot \sigma) \end{pmatrix} = s^\mu \sigma_{\mu\rho} u^\rho / \sqrt{2}, \quad (\text{C70b})$$

where

$$\sigma_{\mu\rho} = \frac{i}{2} (\gamma_\mu \gamma_\rho - \gamma_\nu \gamma_\rho). \quad (\text{C71})$$

Thus the coupling of the spin-0 particle to the

Then one may write, in place of (3.5),

$$A(G^0, k, s) = - \left[\text{Tr} \prod_i (F_i^0 G^0) \right] f(k) \quad (\text{C65a})$$

and

$$A(G^p, k, s) = - \left[\text{Tr} \prod_i (F_i^p G^p) \right] f(k). \quad (\text{C65b})$$

The sum of the two amplitudes is then

$$A^0 + A^p = - \left[\text{Tr} \prod_i (F_i' G) \right] f(k), \quad (\text{C66})$$

where

$$G = G^0 + G^p \text{ and } F_i' = F_i^0 + F_i^p.$$

Equation (C40) shows that the four-by-four matrix G occurs in the form $U^\dagger G U$. The matrix βG occurs in $\bar{U} \beta G U$, where $\bar{U} = U^\dagger \beta$. Correspondingly, (C66) may be written in the alternative form

$$A^0 + A^p = - \left[\text{Tr} \prod_i (F_i' \beta) (\beta G) f(k) \right], \quad (\text{C67})$$

where

$$\beta G = \begin{pmatrix} \delta_{\alpha\dot{\beta}} & 0 \\ 0 & \delta^{\dot{\alpha}\beta} \end{pmatrix} \quad (\text{C68})$$

and

ortho-quarks and para-quarks are via the matrices

$$i\gamma_5(1 + \gamma_5)/\sqrt{2}$$

and

$$i\gamma_5(1 - \gamma_5)/\sqrt{2},$$

respectively. The couplings of the spin-1 particle to the ortho-quarks and para-quarks are via the matrices

$$s^\mu \sigma_{\mu\nu} u^\nu (1 + \gamma_5) / \sqrt{2}$$

and

$$s^\mu \sigma_{\mu\nu} u^\nu (1 - \gamma_5) / \sqrt{2},$$

respectively.

This separation of the ortho-quark and para-quark contributions into the two orthogonal parts of spin space means that at the zero-entropy level the ortho-quarks and para-quarks are, in effect, distinct entities: they are represented by orthogonal states. The rules shown in Fig. 8 mean that there are also, in effect, two different kinds of mesons, one com-

posed of a quark and antiquark of ortho-quark type, the other composed of a quark and antiquark of para-quark type.

The close analogy at the zero-entropy level between the ortho-quark—para-quark types and the flavor types suggests that one should allow, at the zero-entropy level, also the mesons built from an ortho-quark and a para-antiquark, or from a para-quark and an ortho-antiquark. The coupling of these two new types of mesons will be obtained by filling the two empty spaces in the coupling matrix $F_i \beta$ of (C69). Indeed, if one goes back to the two-component formalism, and follows the normal and natural practice of imposing parity invariance on each quark propagator individually, rather than on the process as whole, then the function $F_i \beta$ becomes

$$\Gamma_i = F_i \beta = \frac{i}{\sqrt{2}} \begin{pmatrix} [(-u_i \cdot \sigma)(s_i \cdot \tilde{\sigma})]_{\beta\alpha} & [(-u_i \cdot \sigma) s_i \cdot \tilde{\sigma} (u_i \cdot \sigma)]_{\beta\alpha} \\ s_i \cdot \tilde{\sigma}^{\beta\alpha} & [(s_i \cdot \tilde{\sigma})(u_i \cdot \sigma)]_{\beta\alpha} \end{pmatrix}. \quad (C72)$$

For the case of a spin-0 meson, where $s_i = u_i$, the two new terms reduce to

$$F_i'' \beta = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -u_i \cdot \sigma \\ u_i \cdot \tilde{\sigma} & 0 \end{pmatrix} \\ = i \gamma_5 (\gamma_\mu \cdot u_i^\mu) / \sqrt{2}. \quad (C73a)$$

For the case of a spin-1 meson, where $u_i s_i = 0$, the two new terms reduce to

$$F_i'' \beta = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & s_i \cdot \sigma \\ s_i \cdot \tilde{\sigma} & 0 \end{pmatrix} = i \gamma_\mu \cdot s_i^\mu / \sqrt{2}. \quad (C73b)$$

The explicitly appearing spin vector s^μ in (C70b) and (C73b) can be eliminated, since the index μ against which it is contracted can play an equivalent role. Recall that the summation over the three physical spin-1 states is represented, as in (3.19), by

$$\sum_{e=1}^3 s_e^{\text{phys}\mu} s_e^{\text{phys}\nu} = (-g^{\mu\nu} + v^\mu v^\nu) \\ = \sum_{e=1}^3 (i s_e^{\text{math}\mu})(i s_e^{\text{math}\nu}). \quad (C74)$$

Thus the vectors $i s_e^{\text{math}\mu}$ in (C70b) and (C73b) can be omitted and the index μ of $\sigma_{\mu\rho}$ or γ_μ contracted directly onto the metric tensor $(-g^{\mu\nu} + v^\mu v^\nu)$.

The result can now be compared to the results of Bardakci and Halpern,²⁶ who use the standard four-component formalism. The complete coupling of the spin-0 meson is via the factor

$$\Gamma_j = i \gamma_5 (1 + \gamma \cdot \mu_j) / \sqrt{2}, \quad (C75a)$$

whereas the complete coupling of the spin-1 meson is via the factor

$$\Gamma_j = (\gamma_\mu - i \sigma_{\mu\rho} u_j^\rho) / \sqrt{2}, \quad (C75b)$$

where

$$u_j = k_j / m = v_j \text{ sign } u_j^0,$$

and v_j is the four-velocity of the meson j .

The sum of all of the zero-entropy amplitudes corresponding to a fixed cyclic order of the meson variables is the common scalar factor $f(k)$ times the trace of the cyclically ordered product of factors Γ_i :

$$A(k_1, \dots, k_n) \\ = -\text{Tr}(\Gamma_1 \Gamma_2, \dots, \Gamma_n) f_n(k_1, \dots, k_n), \quad (C76)$$

These couplings are the couplings associated with positive-metric pseudoscalar and vector mesons. These mesons are the mesons that are the basic particles of the ordered Hilbert space, and thus of the physical Hilbert space.

The factorization property does not hold for the sum of zero-entropy amplitudes discussed above. It holds rather for the individual zero-entropy amplitudes. An individual zero-entropy amplitude is obtained by assigning to each quark line segment of the closed loop of, say, Fig. 1 an ortho-quark or para-quark label, and inserting an associated factor of $(1 + \gamma_5)/2$ or $(1 - \gamma_5)/2$, respectively, between the corresponding factors Γ_i and Γ_{i+1} of the trace in (C76). Notice that when a meson is coupled to a zero-entropy function only one or the other of the two terms of (C75a) or (C75b) will contribute, and

this term will be the same at the vertices lying on the two ends of the meson connection line, by virtue of the identity represented in Fig. 8, and the two similar identities associated with the two other ortho-para type mesons. Thus the only coupling matrices that enter are those associated with positive metric pseudoscalar and vector mesons.

The present theory thus resolves simultaneously four serious difficulties that have long plagued this kind of approach. These problems are^{5,26} first, the apparent necessity for a doubling of the pseudosca-

lar and vector particles, second, the apparent necessity for a parity-doublet partner of each of the above-mentioned particles, third, the apparent demand that each of these parity-doublet partners have the wrong metric (i.e., be a ghost), and fourth, the lack of any rationale for the empirically observed $SU(6)_W$ symmetry of vertices. This latter symmetry emerges automatically in the present theory for the amplitudes formed as the sum of zero-entropy amplitudes, provided all momentum vectors \vec{p}_j are parallel to the third coordinate axis.

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