PHYSICAL REVIEW D VOLUME 27, NUMBER 10 15 MAY 1983

Vector and scalar confining potentials and the Klein paradox

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(Received 15 November 1982)

Confining potentials in equations involving the interaction of fermions lead to no Klein paradoxes if the strength of the vector potential is appropriately limited compared to the scalar potential. For linear potentials the Regge trajectories are asymptotically like those of the harmonic oscillator, namely, $E^2 \sim j$.

I. INTRODUCTION

It is not difficult to introduce potentials which behave as vector exchanges into either the single-particle Dirac equation or the 16-component equation for two interacting fermions. Such potentials also manifest themselves through the particular form of Breit (relativistic) corrections to the Schrödinger equation, as in the quarkonium problem. If a "confining" potential is vectorlike, however, the possibility of a tunneling solution arises, so that such a potential is not in fact confining (Klein paradox). This situation does not occur when the confining potential is a Dirac scalar and indeed, as we shall see, for a scalar potential term of sufficient strength compared to a vector potential term. Moreover, if both potentials grow like r at large r, the leading (large E) part of the spectrum is that of the nonrelativistic harmonic oscillator, with the spring constant a complicated function of the vector and scalar strengths.

In the phenomenology of quarkonium,[1] an equal mixture of vector and scalar confining potential leads to the best fit for the spin-orbit splittings of the spectrum. Such a mixture also has the appealing property that the quark-antiquark potential binds while quark-quark or antiquark-antiquark pairs have no long-range interaction at all. It is thus encouraging that we find such a mixture to be consistent with the absence of Klein paradoxes as well as characteristic of linear Regge trajectories.

In the next section we discuss the single-particle Dirac equation, and in Sec. III the two-fermion equation. We interest ourselves throughout only in the asymptotic behavior in E, since it is only for large E that the solutions with linear potential resemble the nonrelativistic harmonic oscillator.

II. DIRAC EQUATION

We follow the notation[2] of Critchfield in studying the Dirac equation, which has the form

$$(\not{p} - m)\psi = (g_S r + g_V \vec{\alpha}\cdot\vec{r})\psi \ . \tag{2.1}$$

We shall ignore the rest mass m in the following. When the angular dependence is removed, the radial piece of Eq. (2.1) is actually a set of coupled equations for the "large" and "small" components ψ_a and ψ_b,

$$\begin{aligned} &[E-(g_V+g_S)r]\psi_a+\psi_b'+\left[\frac{t+1}{r}\right]\psi_b=0\ , \\ &[E-(g_V-g_S)r]\psi_b-\psi_a'+\left[\frac{t-1}{r}\right]\psi_a=0\ . \end{aligned} \tag{2.2}$$

Here, for l = orbital angular momentum in ψ_a, $t=l+1=j+\frac{1}{2}$ when the total angular momentum $j=l+\frac{1}{2}$, and $t=-l=-(j+\frac{1}{2})$ when $j=l-\frac{1}{2}$. The prime represents d/dr.

Equations (2.2) can be decoupled (except, of course, for boundary condition) in standard fashion to find two Schrödinger-type equations,

$$\psi_a''+\psi_a'\left[\frac{\delta}{E-\delta r}+\frac{2}{r}\right]+\psi_a\left[\frac{-\delta}{E-\delta r}\frac{t-1}{r}-\frac{t(t-1)}{r^2}+E^2-2g_VEr+(g_V{}^2-g_S{}^2)r^2\right]=0\ ,$$

$$\psi_b''+\psi_b'\left[\frac{\sigma}{E-\sigma r}+\frac{2}{r}\right]+\psi_b\left[\frac{\sigma}{E-\sigma r}\frac{t+1}{r}-\frac{t(t+1)}{r^2}+E^2-2g_VEr+(g_V{}^2-g_S{}^2)r^2\right]=0\ . \tag{2.3}$$

We have here defined the quantities

$$\sigma=g_V+g_S,\quad \delta=g_V-g_S\ . \tag{2.4}$$

We remark immediately that (i) only for $g_V{}^2<g_S{}^2$ does the r^2 term have the appropriate sign to be a confining harmonic oscillator, i.e., there is no Klein paradox if $|g_V|<|g_S|$, and (ii) there is a term linear in both r and E which complicates the general solution to this problem. Since we are interested in large E we also drop the terms in Eq. (2.3) of order $1/E$; this approximation will turn out to be internally consistent, although the solutions will be known only to $O(1/E)$.

Because of the linear term in r the resulting equations remain difficult to solve. It is useful to consider first the case $g_V=0$, when this term is not present. By solving the equations explicitly, one can show in this case that the large eigenvalues for the energy are

$$E^2\approx 2g_Sl\ . \tag{2.5}$$

l is also large here; such a linear Regge trajectory is characteristic of the harmonic oscillator.

We can find an approximate form for large E and l in the more general case, Eq. (2.3), by writing the coefficient of ψ_a (or ψ_b) in the form

$$E^2-2g_VEr+(g_V{}^2-g_S{}^2)r^2-\frac{l^2}{r^2}\equiv E^2-V^2(r)\ , \tag{2.6}$$

where $V(r)$ is an energy-dependent potential. We find the value r_0 of r which minimizes $V^2(r)$, and estimate $E^2\approx V^2(r_0)$. Thus we require

$$-\frac{d}{dr}V^2(r)=2g_VE+2(g_S{}^2-g_V{}^2)r_0-\frac{2l^2}{r_0{}^3}=0\ . \tag{2.7}$$

The self-consistent solution to this equation is found by writing

$$E=\alpha_Er_0\ , \tag{2.8}$$

so that

$$r_0{}^4=\frac{l^2}{g_V\alpha_E+(g_S{}^2-g_V{}^2)}\ . \tag{2.9}$$

α_E is then found from

$$E^2=\alpha_E{}^2r_0{}^2=2g_V\alpha_Er_0{}^2+(g_S{}^2-g_V{}^2)r_0{}^2+\frac{l^2}{r_0{}^2}\ , \tag{2.10}$$

or

$$\alpha_E=\tfrac{3}{2}g_V+\tfrac{1}{2}[9g_V{}^2+8(g_S{}^2-g_V{}^2)]^{1/2} \tag{2.11}$$

and

$$r_0{}^4=l^2\{\tfrac{3}{2}g_V{}^2+\tfrac{1}{2}g_V[9g_V{}^2+8(g_S{}^2-g_V{}^2)]^{1/2}+(g_S{}^2-g_V{}^2)\}^{-1}\ . \tag{2.12}$$

We have thus shown that near the minimum the potential $V^2(r)$ can be written

$$V^2(r)=\frac{l^2}{r^2}+g_{\rm eff}{}^2r^2\ , \tag{2.13}$$

where the effective harmonic-oscillator coupling is

$$g_{\rm eff}{}^2=3g_V{}^2+(g_S{}^2-g_V{}^2)+g_V[9g_V{}^2+8(g_S{}^2-g_V{}^2)]^{1/2}\ . \tag{2.14}$$

The energy eigenvalues are of the form

$$E^2=\tfrac{1}{2}l\{9g_V{}^2+4(g_S{}^2-g_V{}^2)+3g_V[9g_V{}^2+8(g_S{}^2-g_V{}^2)]^{1/2}\}\times\{\tfrac{3}{2}g_V{}^2+(g_S{}^2-g_V{}^2)+\tfrac{1}{2}g_V[9g_V{}^2+8(g_S{}^2-g_V{}^2)]^{1/2}\}^{-1/2} \tag{2.15}$$

$$\underset{g_V=0}{\longrightarrow}2g_Sl \tag{2.16}$$

$$\underset{g_V=g_S}{\sim}\sqrt{3}g_Vl\ . \tag{2.17}$$

E^2 linear in l and in $r_0{}^2$ is again characteristic of the harmonic oscillator. Equation (2.16) agrees with the result of Eq. (2.5). The coefficient of l is a rather complicated function of $\chi \equiv g_V/g_S$ but is numerically quite close to linear in the interesting region $0<\chi<1$.

III. TWO INTERACTING FERMIONS

Here we show that for an analogous equation for two interacting fermions, the structure of the solution is much the same as in the single-particle equation. The equation we use was first introduced by Kemmer,[3] had several treatments for the angular momentum $j=0$ case,[4] and was finally separated into angular and radial pieces for general j by Koide.[5] We use the notation of the last reference, and refer the reader there for a more detailed treatment. We note that the connection of this particular equation with relativistic field theory is not direct, and in this sense a more recent equation[6] with a more direct connection may be more relevant for comparison with data. For our modest purposes the Kemmer equation is sufficient. This equation takes the form

$$\left[i(\vec{\alpha}^{(1)}-\vec{\alpha}^{(2)})\cdot\vec{\nabla}+\beta^{(1)}m_1+\beta^{(2)}m_2 + \sum_a O_a V_a(r)\right]\psi(r)=E\psi(r)\ , \quad (3.1)$$

where $\vec{\nabla}$ operates on the relative coordinate $\vec{r}=\vec{r}_1-\vec{r}_2$, ψ is a 16-component spinor and the superscripts (1) and (2) refer to the Dirac spaces of particles 1 and 2. The V_a are potentials of tensorial nature determined by the O_a. We are interested only in the case $V_S(r)=g_S r$, $V_V(r)=g_V r$, for which

$$O_S=\beta^{(1)}\beta^{(2)},\quad O_V=1-\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}\ . \quad (3.2)$$

We stress that the strengths g_S and g_V are not necessarily to be identified with the strengths used in Sec. II. This identification could be made by taking the large-mass limit for one of the particles in the Kemmer equation and then comparing this limit with the case of the one-particle equation. This exercise does not interest us here because we can arrive at our result without the connection. Moreover, we treat only the limit $E \gg$ rest masses. The form of the trajectories in this limit need not be the same in detail as those which might follow from taking first one mass infinite and then the large-E limit.

Through a series of transformations and decompositions in vector spherical harmonics, Eq. (3.1) becomes a set of eight (radial) equations for eight functions. Four of these equations are merely algebraic; the remaining four are fully coupled first-order equations for four unknown functions of r. Moreover, this set of equations is different for parity states $P=(-1)^j$ as opposed to $P=(-1)^{j+1}$. We do not write the full set here because they are rather complicated. Instead we write the four coupled radial equations for the case $P=(-1)^j$, and for fermions of equal mass m:

$$[E+(g_S-4g_V)r]a_3+2\left[f_2'+\frac{2}{r}f_2\right]-\frac{4\sqrt{j(j+1)}}{r(E-g_S r)}\left[mg_1+\frac{\sqrt{j(j+1)}}{r}a_3\right]=0\ , \quad (3.3a)$$

$$[E+(g_S-2g_V)r]b_0+2\left[g_1'+\frac{1}{r}g_1\right]-\frac{4\sqrt{j(j+1)}}{r[E-(g_S+2g_V)r]}\left[mf_2+\frac{\sqrt{j(j+1)}}{r}b_0\right]=0\ , \quad (3.3b)$$

$$(E-g_S r)f_2-2a_3'-\frac{4m}{E-(g_S+2g_V)r}\left[mf_2+\frac{\sqrt{j(j+1)}}{r}b_0\right]=0\ , \quad (3.3c)$$

$$[E-(g_S+2g_V)r]g_1-2\left[b_0'+\frac{1}{r}b_0\right]-\frac{4m}{E-g_S r}\left[mg_1+\frac{\sqrt{j(j+1)}}{r}a_3\right]=0\ . \quad (3.3d)$$

Here $a_3(r)$, $f_2(r)$, $g_1(r)$, and $b_0(r)$ are four functions occurring in the decomposition of ψ.

We now consider the large-E limit. We are anticipating that, as in the one-particle equation, E will be proportional to $\sqrt{j}$ as well as to a potential minimum r_0, so that in Eqs. (3.3) we can drop all $1/E$ terms except those containing the numerator factor $j(j+1)$. Equations (3.3) then decouple into two independent pairs of equations: Eqs. (3.3a) and (3.3c) become

$$[E+(g_S-4g_V)r]a_3+2\left[f_2'+\frac{2}{r}f_2\right]-\frac{4j(j+1)}{r^2(E-g_S r)}a_3=0\ , \quad (3.4a)$$

$$(E-g_S r)f_2-2a_3'=0 \quad (3.4b)$$

and Eqs. (3.3b) and (3.3d) become

$$[E+(g_S-2g_V)r]b_0+2\left[g_1'+\frac{1}{r}g_1\right]-\frac{4j(j+1)}{r^2[E-(g_S+2g_V)r]}b_0=0\;, \tag{3.5a}$$

$$[E-(g_S+2g_V)r]g_1-2\left[b_0'+\frac{1}{r}b_0\right]=0\;. \tag{3.5b}$$

While we have explicitly written the equal-mass case, the unequal-mass case also gives the two sets of Eqs. (3.4) and (3.5) for $E \gg$ fermion masses.

Each pair of equations can now be decoupled to give one second-order Schrödinger-type equation; namely, for Eqs. (3.4),

$$f_2''+\frac{2}{r}f_2'+f_2\left[\tfrac{1}{4}(E-g_Sr)[E+(g_S-4g_V)r]-\frac{2+j(j+1)}{r^2}\right]=0\;, \tag{3.6}$$

and, for Eqs. (3.5),

$$g_1''+\frac{1}{r}g_1'+g_1\left[\tfrac{1}{4}[E+(g_S-2g_V)r][E-(g_S+2g_V)r]-\frac{1+j(j+1)}{r^2}\right]=0\;. \tag{3.7}$$

Exactly the same procedure for the case $P=(-1)^{j+1}$ yields second-order equations for two more functions,

$$f_3''+\frac{2}{r}f_3'+f_3\left[\tfrac{1}{4}(E+g_Sr)[E-(g_S+4g_V)r]-\frac{2+j(j+1)}{r^2}\right]=0\;, \tag{3.8}$$

$$g_0''+\frac{1}{r}g_0'+g_0\left[\tfrac{1}{4}[E-(g_S+2g_V)r][E+(g_S-2g_V)r]-\frac{1+j(j+1)}{r^2}\right]=0\;. \tag{3.9}$$

We can now treat these equations as in the one-particle case by writing the coefficient of the function in the form $\frac{1}{4}[E^2-V^2(r,E,j)]$ for large E and j both. For $g_V=0$, V^2 is a function of r and j alone; indeed it is the same function for all four equations, namely, $g_S{}^2r^2+4j^2/r^2$. This is the normal harmonic oscillator as described in Sec. II. For $g_V\neq0$, V^2 contains a linear term in E. We have for Eqs. (3.6)–(3.9), respectively,

$$V^2=(g_S{}^2-4g_Vg_S)r^2+4g_VEr+\frac{4j^2}{r^2}\;, \tag{3.10a}$$

$$V^2=(g_S{}^2-4g_V{}^2)r^2+4g_VEr+\frac{4j^2}{r^2}\;, \tag{3.10b}$$

$$V^2=(g_S{}^2+4g_Vg_S)r^2+4g_VEr+\frac{4j^2}{r^2}\;, \tag{3.10c}$$

$$V^2=(g_S{}^2-4g_V{}^2)r^2+4g_VEr+\frac{4j^2}{r^2}\;. \tag{3.10d}$$

Since we want the coefficient of r^2 to be positive in order to avoid Klein paradoxes, we require

$$|g_S|>4|g_V|\;. \tag{3.11}$$

The energy eigenvalue determination also follows from the smallest coefficient of the unknown function in the two Schrödinger-type equations for each parity, since the large-r asymptotic behavior of the solutions corresponds to this coefficient. Thus Eq. (3.6) rather than (3.7) determines the $P=(-1)^j$ eigenvalues, and Eq. (3.8) rather than (3.9) determines the $P=(-1)^{j+1}$ eigenvalues. To find these eigenvalues we follow the procedure of Sec. II, namely, we find [from Eq. (3.6)] the minimum r_0 for $V^2(r)$ as given in Eq. (3.10a) with the estimate $E^2\approx V^2(r_0)$. Setting $E=\alpha_Er_0$, we find

$$r_0{}^4=\frac{4j^2}{2g_V\alpha_E+(g_S{}^2-4g_Vg_S)}\;, \tag{3.12}$$

$$\alpha_E=3g_V+[9g_V{}^2+2(g_S{}^2-4g_Vg_S)]^{1/2}\;. \tag{3.13}$$

This corresponds to an effective coupling, as in Eq. (2.13), of

$$V^2=\frac{4j^2}{r^2}+g_{\rm eff}{}^2r^2\;,$$

with

$$g_{\rm eff}{}^2=g_S{}^2-4g_Vg_S+12g_V{}^2+4g_V[9g_V{}^2+2(g_S{}^2-4g_Vg_S)]^{1/2}\;. \tag{3.14}$$

Energy eigenvalues take the form

$$E^2=2j\{2(g_S{}^2-4g_Vg_S+9g_V{}^2)+6g_V[9g_V{}^2+2(g_S{}^2-4g_Vg_S)]^{1/2}\}$$
$$\times\{6g_V{}^2+g_S{}^2-4g_Vg_S+2g_V[9g_V{}^2+2(g_S{}^2-4g_Vg_S)]^{1/2}\}^{-1/2} \quad (3.15)$$

$$\underset{g_V\to 0}{\longrightarrow} 4g_Sj \quad (3.16)$$

$$\underset{g_V=\frac{1}{4}g_S}{\sim} 12\sqrt{3}g_Vj\ . \quad (3.17)$$

One more point remains to be resolved in our approximate solutions of these Dirac-type equations. Namely, with any ordinary set of boundary conditions at infinity, numerical integration for the solution runs into difficulty because of the singularities at $E=g_Sr$ in Eq. (3.4a) and at

$$E=(g_S+2g_V)r$$

in Eq. (3.5a). Terms of this type do not appear in the single-particle equations (2.2). Strict solution of the equations would require a special way to handle these singularities. However, our technique for handling the solution allows us to avoid this question, because as we discuss below our bound state in each case lies inside these singularities. This is possible because as E increases, so does r_0. This result is easliy established by calculating the quantities

$$D_1=E-g_Sr$$

and

$$D_2=E-(g_S+2g_V)r\ ,$$

substituting $r=r_0$, $E=\alpha_Er_0$, and checking that D_1 and D_2 are positive for all possible values of g_S and g_V. We forego the explicit exercise here but remark only that the minimum value of either D_1 or D_2 occurs for both g_S and g_V positive, when $D_2=0$ for $g_S=4g_V$, and otherwise D_1 and D_2 are positive definite. By the required inequality (3.11) $g_S\neq 4g_V$, and our result is established.

IV. CONCLUSION

Quarkonium phenomenology is done using a Schrödinger equation, with the tensorial nature of the interaction revealed only through the Breit corrections due to relativity. Equal mixtures of scalar and vector confining potentials are preferred.

In this brief paper we have shown first that it is possible to treat mixtures of scalar and vector potentials in equations involving fermions, and that there is no Klein paradox if the strength of the vector potential is limited compared to the strength of the scalar potential. Second, a linear potential in these equations gives at large E and angular momentum linear Regge trajectories, characteristic of harmonic forces in the Schrödinger equation, with slope a calculable function of the vector and scalar strengths.

It is interesting to ask whether Klein paradoxes occur in other treatments of QCD which follow from more fundamental points of view or at least from a relativistic starting point. One such treatment is that of Mandelstam,[7] who by truncation of the Dyson-Schwinger equations is able to set up a Bethe-Salpeter equation for $q\bar{q}$ bound states. This equation has a vectorlike kernel whose Fourier transform is a linear confining potential; it is easy to see how the equation can be generalized to include scalarlike kernels. Careful treatment of the Mandelstam equation does indeed reveal a Klein paradox which is resolved in appropriate combinations of vector and scalar kernels; we shall report on this in more detail elsewhere.

Still another relativistic equation, an equal-time equation, proposed by Suura,[6] is known to have Klein paradoxes much like those we have discussed in this paper. The general conclusion we draw is clear; such paradoxes are difficult to avoid, and the solution to them seems to point towards limits on the tensorial mixture of allowed interactions. This fact will have bearing on phenomenology through the fine structure of the spectrum, and at least in the equations which have been studied these phenomenological implications do not seem unreasonable.

ACKNOWLEDGMENTS

We thank the Aspen Center for Physics for their hospitality. P.M.F. was supported in part by the U.S. National Science Foundation under Grant No. PHY-8100257, S.G.G. by the U.S. Department of Energy under Contract No. DE-AC02-82ER 40051, and P.K. by the U.S. Department of Energy under Contract No. DE-AM03-76SF00034.

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