Fermion-monopole system reexamined

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The problem of a Dirac particle in a fixed Abelian monopole field is reexamined. We find that the boundary conditions adopted by Kazama, Yang, and Goldhaber can be generalized. When the generalization is made, we are led to the existence of θ vacua. For massless fermions, chiral symmetry is spontaneously broken. A change in θ is equivalent to a chiral rotation, and the physics is independent of θ. For massive fermions, *CP* invariance is broken except for $\theta=0,\pi$. The vacuum charge for a unit pole obeys the Witten formula $Q=-e\theta/2\pi$ and the monopole becomes a dyon. The results for the two cases are related by an analog of Levinson's theorem. We also note the connection of our results with fractional quantum numbers on solitons, and with the η invariant of Atiyah, Patodi, and Singer.

I. INTRODUCTION

The problem of a Dirac particle[1,2] in a fixed Abelian monopole field has attracted the attention of many authors.[3–10] In particular, Kazama, Yang, and Goldhaber[8] treated the system in detail using the fiber-bundle formalism.[11] In the course of their investigation, they discovered that a boundary condition had to be imposed at the origin (the location of the monopole) for the lowest partial wave and suggested a choice in terms of an additional magnetic moment to the Dirac particle.

We have found, however, that their choice is not the most general that is possible, and the appropriate boundary conditions form a one-parameter family, corresponding to the existence of θ vacua.[12,13] Actually, our work will have considerable overlap with the papers by Goldhaber,[9] Callias,[9] and, in particular, Besson[10]; we believe, however, that we have new results to offer, as well as a simple and explicit treatment. (See added note.)

The organization of the paper is as follows. We introduce the necessary equations in Sec. II. Following Refs. 7–10, we adopt the fiber-bundle description[11] of the monopole to avoid a spurious "string" singularity in the vector potential.

In Sec. III, we treat the case of a massless fermion. We find that the self-adjointness of the Hamiltonian requires chiral symmetry to be broken,[10] thereby allowing charge and angular momentum to be conserved. In this case, the boundary condition serves only to fix the chiral angle, and hence does not affect the physics.

In Sec. IV, we turn to the case of a massive Dirac particle. In this case, the physics is θ dependent. In particular, *CP* invariance is violated apart from two special cases corresponding to $\theta=0$, π. Furthermore, the vacuum acquires an electric charge according to the Witten formula,[14,15] and the monopole actually becomes a dyon.[16,17] We also investigate the Galilean limit. The relation between the two cases is explored in Sec. V. We find that the symmetry breaking in the massless case and the Witten effect in the massive case are related through an analog of Levinson's theorem.[18] We also note the connection of our results with fractional quantum numbers on kinks in one-dimensional systems[19,20] and with the η invariant of Atiyah, Patodi, and Singer.[21]

Finally, in Sec. VI, we briefly discuss various problems associated with the existence of θ. The Appendix deals with some subtleties associated with the Jacobi identity,[17,23] the infinite-volume limit, and the zero-mass limit.

II. THE DIRAC EQUATION IN A MONOPOLE FIELD

The Dirac equation for a charged particle in a monopole field is given by[1–3]

$$i(\partial/\partial t)\psi(\vec{x},t)=H\psi(\vec{x},t)\ , \tag{2.1}$$

$$H=\vec{\alpha}\cdot\vec{\pi}+\beta M\ , \tag{2.2}$$

where we have adopted the minimal prescription

$$\vec{\pi}=-i\,\mathrm{grad}-e\vec{A}\ . \tag{2.3}$$

As emphasized by Wu and Yang,[11] and by Greub and Petry,[11] the vector potential $\vec{A}$ for a monopole should be regarded as a connection on a nontrivial

U(1) bundle; similarly, ψ should be regarded as a section rather than an ordinary function. In our case, it suffices to consider the space outside of the monopole as the union of two overlapping regions R_a and R_b,

$$R_a:\ r>0,\ 0\le\theta<\frac{\pi}{2}+\delta,\ 0\le\phi<2\pi\ , \tag{2.4}$$

$$R_b:\ r>0,\ \frac{\pi}{2}-\delta<\theta\le\pi,\ 0\le\phi<2\pi\ , \tag{2.5}$$

where δ is an angle such that $0<\delta<\pi/2$. The vector potential is then chosen on each region to be

$$A_r\,|_a=A_\theta\,|_a=0,\ A_\phi\,|_a=\frac{g}{r\sin\theta}(1-\cos\theta)\ , \quad (r,\theta,\phi)\in R_a\ , \tag{2.6}$$

$$A_r\,|_b=A_\theta\,|_b=0,\ A_\phi\,|_b=\frac{-g}{r\sin\theta}(1+\cos\theta)\ , \quad (r,\theta,\phi)\in R_b\ , \tag{2.7}$$

where g is the magnetic charge. The two potentials are related in the overlap region $R_a\cap R_b$ by a gauge transformation

$$A_\mu\,|_a=A_\mu\,|_b-\frac{i}{e}S_{ab}\partial_\mu S_{ab}{}^{-1}\ , \tag{2.8}$$

where

$$S_{ab}=e^{2ieg\phi} \tag{2.9}$$

is the transition function. Similarly, we may introduce two wave functions ψ_a and ψ_b, so that the Dirac equation (2.1)–(2.3) is satisfied in each region with the respective vector potentials (2.6) and (2.7). As is well known, it follows from (2.8) that the two functions are related in the overlap region by

$$\psi_a=S_{ab}\psi_b\ . \tag{2.10}$$

The existence of S_{ab} throughout the whole overlap region $R_a\cap R_b$ leads to the celebrated quantization condition[3,16,17]

$$q\equiv eg=\tfrac{1}{2}\times\text{integer}\ . \tag{2.11}$$

The total angular momentum for the system is well known to have the form[24,25]

$$\vec{J}=\vec{x}\times\vec{\pi}-q\frac{\vec{x}}{r}+\tfrac{1}{2}\vec{\sigma},\ \ r\equiv|\vec{x}|\ , \tag{2.12}$$

where the second term is responsible for much of the unusual properties of the system. Since $\vec{J}$ is conserved, we may simultaneously diagonalize H, $\vec{J}^{\,2}$, and J_z, just as in a central potential. We shall denote their eigenvalues as E, $j(j+1)$, and m. It is also easily checked that the effect of a CP inversion

$$A'(\vec{x}\,')\,|_a=A(\vec{x})\,|_b\ , \tag{2.13}$$

$$\psi'(\vec{x}\,')\,|_a=(-1)^{2q+1}i\alpha_2\psi^*(\vec{x})\,|_b\ , \tag{2.14}$$

$$\vec{x}\,'=-\vec{x} \tag{2.15}$$

is to transform a solution $\psi(\vec{x},t)$ of (2.1) with energy E into a solution $\psi'(\vec{x},t)$ with energy $-E$.

In the following, we shall concentrate on the lowest partial wave $j=|q|-\frac{1}{2}$, since this is the case where the subtlety arises. We write the wave section as[8]

$$\psi(\vec{x},t)=\frac{1}{r}\begin{bmatrix}F(r)\ \eta_{jm}\ (\theta,\phi)\\ G(r)\ \eta_{jm}\ (\theta,\phi)\end{bmatrix}e^{-iEt} =\frac{\chi(r)}{r}\eta_{jm}(\theta,\phi)e^{-iEt}\ , \tag{2.16}$$

$$\chi(r)=\begin{bmatrix}F(r)\\ G(r)\end{bmatrix}\ , \tag{2.17}$$

$$\eta_{jm}(\theta,\phi)=\begin{bmatrix}-\left[\dfrac{j-m+1}{2j+2}\right]^{1/2}Y_{q,|q|,m-1/2}\\ \left[\dfrac{j+m+1}{2j+2}\right]^{1/2}Y_{q,|q|,m+1/2}\end{bmatrix}, \tag{2.18}$$

where $Y_{q,l,m}$ are the monopole harmonics defined in Ref. 7. In Eq. (2.16), the nontrivial nature of the U(1) bundle is contained completely in $\eta_{jm}(\theta,\phi)$, so $\chi(r)$ is an ordinary two-component wave function defined on the half-line $r>0$ and with the inner product

$$(\chi_1,\chi_2)=\int_0^\infty dr\,\chi_1^\dagger(r)\chi_2(r)\ . \tag{2.19}$$

Using the formula[8]

$$(\vec{\sigma}\cdot\vec{\pi})\frac{\chi(r)}{r}\eta_{jm}(\theta,\phi) =-\frac{i}{r}\frac{q}{|q|}\frac{d\chi(r)}{dr}\eta_{jm}(\theta,\phi)\ , \tag{2.20}$$

we may reduce Eqs. (2.1)–(2.3) to

$$H_0(r)=E\chi(r)\ , \tag{2.21}$$

$$H_0=-i\frac{q}{|q|}\gamma_5\frac{d}{dr}+\beta M\ . \tag{2.22}$$

We may also check that the effect of a CP inversion is given by

$$\chi(r)\rightarrow\gamma_5\chi^*(r) \tag{2.23}$$

using the formula

$$Y^*_{qlm}\,|_a(\pi-\theta,\phi+\pi)=Y_{qlm}\,|_b(\theta,\phi)\ . \tag{2.24}$$

Furthermore, we may restrict ourselves to $q>0$, since the solutions for $q<0$ may be obtained by a parity transformation

$$\chi(r)\to\beta\chi(r)\ . \tag{2.25}$$

Equations (2.22) and (2.23) are then easily solved to give

$$E=\pm(k^2+M^2)^{1/2}\quad (k>0)\ ,$$
$$\chi_E^{(1)}(r)=\begin{bmatrix}\dfrac{ik}{E-M}\sin kr\\ \cos kr\end{bmatrix},\qquad \chi_E^{(2)}(r)=\begin{bmatrix}\cos kr\\ \dfrac{ik}{E+M}\sin kr\end{bmatrix}, \tag{2.26}$$

in the representation with

$$\gamma_5=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \beta=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{2.27}$$

It is found however, that there also exist normalizable solutions for complex E. The reason is not hard to see; the differential operator (2.22) is not necessarily Hermitian on a half-line.

Fortunately, the problem may be easily resolved. An elementary application of the Weyl-von Neumann theory shows the operator (2.22) is of limit circle type at $r=0$ and of limit point type at $r=\infty$; hence the formal Hamiltonian H_0 has a one-parameter family of self-adjoint extensions.[9]

To use more familiar language, boundary conditions must be imposed so that the Hamiltonian possesses a complete set of eigenfunctions, i.e., acceptable as a physical observable. The appropriate boundary conditions at $r=0$ form a one-parameter family, whereas no boundary conditions are necessary at $r=\infty$.

The desired condition is readily obtained by demanding that

$$\begin{aligned}0&=(\chi_1,H_0\chi_2)-(H_0\chi_1,\chi_2)\\ &=i\chi_1^\dagger(0)\gamma_5\chi_2(0)\\ &=i[F_1^*(0)G_2(0)+F_2(0)G_1^*(0)]\end{aligned} \tag{2.28}$$

or

$$F_1^*(0)/G_1^*(0)=-F_2(0)/G_2(0)\ , \tag{2.29}$$

where we have assumed that χ_1 and χ_2 vanish sufficiently rapidly at infinity (as may be expected for a normalizable wave function). Therefore we find the boundary condition must be of the form[9]

$$F(0)/G(0)=i\tan\left[\frac{\theta}{2}+\frac{\pi}{4}\right] \tag{2.30}$$

or

$$\chi(0)\propto\begin{bmatrix}i\sin\left[\dfrac{\theta}{2}+\dfrac{\pi}{4}\right]\\ \cos\left[\dfrac{\theta}{2}+\dfrac{\pi}{4}\right]\end{bmatrix}\equiv\zeta(\theta)\ , \tag{2.31}$$

where we have chosen to parametrize in terms of angle θ.

As we shall see in the next sections the formal Hamiltonian (2.22) together with the boundary condition (2.31) leads to acceptable (and interesting) physics.

III. THE MASSLESS CASE

For $M=0$, the solutions of (2.22) which satisfy (2.31) are

$$E=k>0:\ u_{k\theta}(r)=e^{i\theta\gamma_5/2}\begin{bmatrix}i\sin\left[kr+\dfrac{\pi}{4}\right]\\ \cos\left[kr+\dfrac{\pi}{4}\right]\end{bmatrix}, \tag{3.1}$$

$$E=-k<0:\ v_{k\theta}(r)=e^{i\theta\gamma_5/2}\begin{bmatrix}-i\sin\left[kr-\dfrac{\pi}{4}\right]\\ \cos\left[kr-\dfrac{\pi}{4}\right]\end{bmatrix}. \tag{3.2}$$

Using the formula

$$\int_0^\infty dr\cos kr=\pi\delta(k)\ , \tag{3.3}$$

it is trivial to show that the solutions (3.1) and (3.2) form a complete orthonormal set.

We also see that the solutions break chiral symmetry in spite of the fact that H_0 formally commutes with γ^5. However, since a shift in θ is equivalent to a chiral rotation, the physics must be independent of θ.

To see this point more clearly, we may consider the scattering amplitude for $j=q-\frac{1}{2}$. Splitting $u_{k\theta}(r)$ into incoming and outgoing waves

$$e^{i\theta/2}\frac{1+i}{2\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}e^{ikr}-e^{-i\theta/2}\frac{1-i}{2\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}e^{-ikr} \tag{3.4}$$

and comparing with[8]

$$e^{-ikz}\begin{bmatrix}0\\1\end{bmatrix}\rightarrow\frac{\sqrt{2q\pi}e^{-ikr}}{ikr}\eta_{j0}(\theta,\phi)+(\text{higher } j) \tag{3.5}$$

for incoming waves and

$$e^{-ikz}\begin{bmatrix}1\\0\end{bmatrix}\rightarrow-\frac{\sqrt{2q\pi}e^{ikr}}{ikr}\eta_{j0}(\theta,\phi)+(\text{higher } j) \tag{3.6}$$

for outgoing waves, we find that there is a change of chirality and a corresponding shift of phase $ie^{i\theta}$. However, chirality is conserved for higher partial waves,[8] so this phase does not give rise to interference effects. (Note that chirality, unlike helicity, is an observable, even if it is not conserved.) Since there are no bound states, we may conclude that θ is not a physical observable.

It is also instructive to second quantize the Dirac equation. We start as usual from the canonical anticommutation relations

$$\{\psi(\vec{x},t),\psi^\dagger(\vec{y},t)\}=\delta(\vec{x}-\vec{y})\ , \tag{3.7}$$

$$\{\psi(\vec{x},t),\psi(\vec{y},t)\}=\{\psi^\dagger(\vec{x},t),\psi^\dagger(\vec{y},t)\}=0\ , \tag{3.8}$$

together with the Hamiltonian

$$\mathscr{H}(t)=\tfrac{1}{2}\int d\vec{x}\,\psi^\dagger(\vec{x},t)H\psi(\vec{x},t)+\tfrac{1}{2}\int d\vec{x}(H\psi)^\dagger(\vec{x},t)\psi(\vec{x},t)\ . \tag{3.9}$$

The Heisenberg equation of motion is then just the Dirac equation

$$\frac{\partial}{\partial t}\psi(\vec{x},t)=i[\mathscr{H}(t),\psi(\vec{x},t)]=-iH\psi(\vec{x},t)\ . \tag{3.10}$$

Therefore $\psi(\vec{x},t)$ may be decomposed as

$$\psi(\vec{x},t)=\frac{1}{\pi}\int_0^\infty dk\sum_m[b_{km}e^{-ikt}u_k(r)\eta_{jm}(\theta,\phi)+d_{km}^\dagger e^{ikt}v_k(r)\eta_{jm}(\theta,\phi)]+(\text{higher } j)\ , \tag{3.11}$$

where $u_k(r)$ and $v_k(r)$ are arbitrary solutions of (2.21) with $E=\pm k$.

If we also require b and d to satisfy the anticommutation relations

$$\{b_{km},b_{k'm}^\dagger\}=\{d_{km},d_{k'm}^\dagger\}=\pi\delta(k-k')\,,\delta_{mm'} \tag{3.12}$$

$$\text{others}=0\ , \tag{3.13}$$

then $u_k(r)$ and $v_k(r)$ are fixed to be $u_{k\theta}(r)$ and $v_{k\theta}(r)$ (up to phase factors which may be absorbed into b and d).

With this choice both the vector current and the axial-vector current[26] are conserved,

$$\partial^\mu j_\mu=0,\ \ j_\mu\equiv\frac{e}{2}[\bar{\psi},\gamma_\mu\psi]\ , \tag{3.14}$$

$$\partial^\mu j_{\mu5}=0,\ \ j_{\mu5}\equiv\tfrac{1}{2}[\bar{\psi},\gamma_\mu\gamma_5\psi]\ . \tag{3.15}$$

(Note that the anomalous divergence[28–30] vanishes for a pure magnetic field.)

However, the associated charges behave differently. The electric charge is time independent,

$$Q\equiv\int d\vec{x}\,j_0(\vec{x},t)=\frac{e}{\pi}\int_0^\infty dk\sum_m(b_{k\theta m}^\dagger b_{k\theta m}-d_{k\theta m}^\dagger d_{k\theta m})+(\text{higher } j)\ , \tag{3.16}$$

whereas the axial charge is time dependent,

$$\begin{aligned}Q_5(t)\equiv&\int d\vec{x}\,j_{05}(\vec{x},t)\\=&\frac{i}{\pi^2}\int_0^\infty dk\int_0^\infty dk'\frac{\mathscr{P}}{k'-k}e^{i(k-k')t}\sum_m b_{k\theta m}^\dagger b_{k'\theta m}+\frac{i}{\pi^2}\int_0^\infty dk\int_0^\infty dk'\frac{\mathscr{P}}{k'+k}e^{i(k+k')t}\sum_m b_{k\theta m}^\dagger d_{k'\theta m}^\dagger\\&-\frac{i}{\pi^2}\int_0^\infty dk\int_0^\infty dk'\frac{\mathscr{P}}{k'+k}e^{-i(k+k')t}\sum_m d_{k\theta m}b_{k'\theta m}\\&+\frac{i}{\pi^2}\int_0^\infty dk\int_0^\infty dk'\frac{\mathscr{P}}{k'-k}e^{-i(k-k')t}\sum_m d_{k'\theta m}^\dagger d_{k\theta m}+(\text{higher } j)\ ,\end{aligned} \tag{3.17}$$

where we have made use of

$$\int_0^\infty dr\,\sin kr=\frac{\mathscr{P}}{k}\;. \tag{3.18}$$

In fact, strictly speaking, $Q_5(t)$ should be regarded as nonexistent. A chiral rotation induces the Bogoliubov transformation[31]

$$\begin{aligned} b_{k\theta}&=\cos\frac{\theta-\theta'}{2}b_{k\theta'}+\frac{1}{\pi}\sin\frac{\theta-\theta'}{2}\int_0^\infty dk'\left[\frac{\mathscr{P}}{k'-k}b_{k'\theta'}-\frac{1}{k'+k}d^\dagger_{k'\theta'}\right],\\ d_{k\theta}&=\cos\frac{\theta-\theta'}{2}d_{k\theta'}+\frac{1}{\pi}\sin\frac{\theta-\theta'}{2}\int_0^\infty dk'\left[\frac{1}{k'+k}b_{k'\theta'}-\frac{\mathscr{P}}{k'-k}d^\dagger_{k'\theta'}\right]. \end{aligned} \tag{3.19}$$

However, since

$$\langle\theta'\,|\,b^\dagger_{k\theta}b_{k\theta}\,|\,\theta'\rangle=\frac{1}{\pi k}\sin^2\frac{\theta-\theta'}{2}\;, \tag{3.20}$$

the total number of particles is infinite, and the transformation is not unitarily implementable,[32] i.e., the vacua for different θ belong to different Hilbert spaces. These properties are all characteristic of *spontaneously* broken symmetries.[33] The only novel feature is that the relevant long-range field is fermionic, owing to the "abnormal" statistics.[34]

We may also discuss scattering in the language of second quantization. Here we simply remark that we cannot use the solutions of the *free* Dirac equation for a smearout in the asymptotic conditions, etc.; the inner product between sections with different q does not make sense.[11] However, as far as our case is concerned, the usual formalism is still valid if we use the (normalizable) solutions of the *full* one-particle equation (2.1).

IV. THE MASSIVE CASE

The solutions are now given by

$$E=+(k^2+M^2)^{1/2}:\;\; u_{k\theta}(r)=\frac{k}{[E(E-M\sin\theta)]^{1/2}}\times\left[\cos\left[\frac{\theta}{2}+\frac{\pi}{4}\right]\chi_E^{(1)}(r)+i\sin\left[\frac{\theta}{2}+\frac{\pi}{4}\right]\chi_E^{(2)}(r)\right], \tag{4.1}$$

$$E=-(k^2+M^2)^{1/2}:\;\; v_{k\theta}(r)=\frac{k}{[\,|E|\,(\,|E|+M\sin\theta)]^{1/2}}\times\left[\cos\left[\frac{\theta}{2}+\frac{\pi}{4}\right]\chi_E^{(1)}(r)+i\sin\left[\frac{\theta}{2}+\frac{\pi}{4}\right]\chi_E^{(2)}(r)\right], \tag{4.2}$$

where $\chi_E^{(1)}$ and $\chi_E^{(2)}$ are defined in (2.26). If $\cos\theta<0$, there is also a bound state[9]

$$E=M\sin\theta,\;\;\kappa=M\,|\cos\theta|\;,\qquad B_\theta=\begin{bmatrix} i\sin\left[\frac{\theta}{2}+\frac{\pi}{4}\right]\\ \cos\left[\frac{\theta}{2}+\frac{\pi}{4}\right]\end{bmatrix}\sqrt{2\kappa}\,e^{-\kappa r}\;. \tag{4.3}$$

In general, the solutions of (2.22) satisfy

$$(\chi_E,\chi_{E'})=|\,\chi_E^\dagger(0)(E-\beta M)\chi_{E'}(0)\,|\,\frac{\pi}{k}\delta(E-E')-i\chi_E^\dagger(0)\gamma_5\chi_{E'}(0)\frac{\mathscr{P}}{E-E'}\;. \tag{4.4}$$

In particular, the solutions (4.1)–(4.3) may be checked to be orthonormal,

$$(u_{k\theta},u_{k'\theta})=(v_{k\theta},v_{k'\theta})=\pi\delta(k-k')\;, \tag{4.5}$$

$$(B_\theta,B_\theta)=1\;, \tag{4.6}$$

$$\text{others}=0\;. \tag{4.7}$$

Checking completeness is more involved. We shall not reproduce the details here, since it is similar to later calculations.

It is easy to see from (2.24) and (2.31) that the effect of a CP inversion is to change θ into $-\theta$. In our case of $M\neq 0$, we cannot undo this by a chiral rotation, so CP invariance is broken apart from $\theta=0,\pi$.[35] It may be also checked that the boundary conditions of Ref. 8 correspond precisely to the latter values, as would be expected for a CP-conserving perturbation (an additional magnetic moment). Furthermore, it is trivial to check that if the mass term in the Hamiltonian (2.2) had been $M\beta e^{i\omega\gamma_5}$, the physically significant quantity would be $\bar{\theta}\equiv\theta+\omega$. It is important to note that ω (and hence $\bar{\theta}$) is *not* determined by existing experiments on electromagnetism; ω may always be traded for a term $F_{\mu\nu}\tilde{F}^{\mu\nu}$, which is not expected to have any physical effect in the monopole-free sector of Abelian gauge theories.[36]

Having determined the stationary states, we may apply the rules of hole theory[2] and fill all the negative-energy levels. Since these levels depend on θ, it is natural to ask whether the properties of the vacuum also depend on θ.

To end this, let us calculate the charge density of the Dirac sea. The contribution of the continuum is given by

$$\frac{e}{\pi}\int_0^\infty dk\, v_{k\theta}^\dagger(r)v_{k\theta}(r)$$
$$=\frac{e}{\pi}\int_0^\infty dk-\frac{e}{\pi}\int_0^\infty dk\frac{M}{|E|}\frac{|E|\sin\theta+M}{|E|+M\sin\theta}\cos 2kr+\frac{e}{\pi}\int_0^\infty dk\frac{M}{|E|}\frac{k\cos\theta}{|E|+M\sin\theta}\sin 2kr\ . \tag{4.8}$$

The first term is θ independent and may be discarded; the remaining terms may be grouped together as

$$-\frac{eM}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{|E|}\frac{|E|\sin\theta+ik\cos\theta+M}{|E|+M\sin\theta}e^{2ikr}\ . \tag{4.9}$$

We now deform the contour of integration as in Fig. 1. The cut contributes

$$-\frac{eM\sin\theta}{\pi}\int_M^\infty\frac{d\kappa}{(\kappa^2-M^2)^{1/2}}\frac{\kappa}{\kappa+M\cos\theta}e^{-2\kappa r}\ , \tag{4.10}$$

whereas the pole contributes

$$2eM\cos\theta\, e^{2Mr\cos\theta}\Theta(-\cos\theta)\ , \tag{4.11}$$

which exactly cancels the contribution from the bound state

$$eB_\theta^\dagger(r)B_\theta(r)\Theta(-\cos\theta)\ . \tag{4.12}$$

We note that we could also have started from the charge-symmetric form[26]

$$\left\langle\theta\left|\frac{e}{2}[\psi^\dagger(\vec{x},t),\psi(\vec{x},t)]\right|\theta\right\rangle=-\frac{e}{2\pi}\int_0^\infty dk[u_{k\theta}^\dagger(r)u_{k\theta}(r)-v_{k\theta}^\dagger(r)v_{k\theta}(r)]\frac{2j+1}{4\pi r^2}$$
$$-\frac{e}{2\pi}\mathrm{sgn}(\sin\theta)\Theta(-\cos\theta)B_\theta^\dagger(r)B_\theta(r)\frac{2j+1}{4\pi r^2}\ . \tag{4.13}$$

(Higher partial waves with $j\geq q+\frac{1}{2}$ do not contribute owing to CP invariance.) Since completeness gives

$$\frac{1}{2\pi}\int_0^\infty dk[u_{k\theta}^\dagger(r)u_{k\theta}(r)+v_{k\theta}^\dagger(r)v_{k\theta}(r)]+\tfrac{1}{2}\Theta(-\cos\theta)B_\theta^\dagger(r)B_\theta(r)=\delta(0)\ , \tag{4.14}$$

we are led to the same result for the (three-dimensional) vacuum charge density[37]

$$\rho_\theta(\vec{x})=-\frac{qeM\sin\theta}{2\pi^2r^2}\int_M^\infty\frac{d\kappa}{(\kappa^2-M^2)^{1/2}}\frac{\kappa}{\kappa+M\cos\theta}e^{-2\kappa r}\ . \tag{4.15}$$

The integral reduces to a modified Bessel function when $\theta=\pm\pi/2$ and behaves similarly for other values of θ (except $\pm\pi$). For $r\to 0$,

$$\rho_\theta(\vec{x})\sim\frac{qeM\sin\theta}{2\pi^2r^2}\ln Mr\ , \tag{4.16}$$

whereas for $r\to\infty$

$$\rho_\theta(\vec{x})\sim-\frac{qeM\tan(\theta/2)}{2\pi^2r^2}\left[\frac{\pi}{4Mr}\right]^{1/2}e^{-2Mr}\ . \tag{4.17}$$

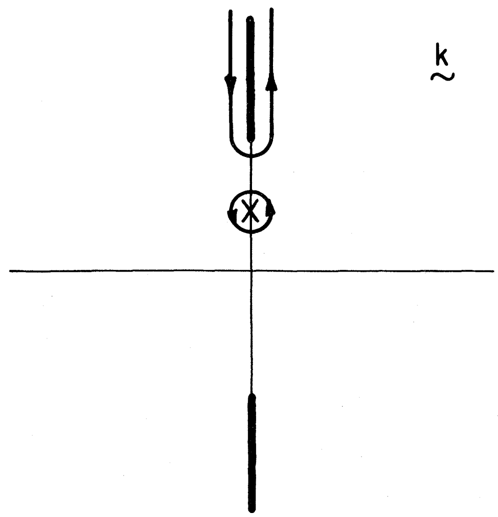

FIG. 1. Contour in the complex k plane.

We may also take the limit $\theta\to\pm\pi$ after an integration by parts; as expected on general grounds,[36,19] the result is one-half the contribution of the zero-energy bound state.

The total charge is given by

$$Q=\frac{-2qeM\sin\theta}{\pi}\int_M^\infty\frac{d\kappa}{2(\kappa^2-M^2)^{1/2}}\frac{1}{\kappa+M\cos\theta}\,. \tag{4.18}$$

Changing the integration variable $\kappa=M\cosh 2x$ and using the formula[38]

$$\int_0^\infty\frac{dx}{\cosh^2x-\sin^2(\theta/2)}=\frac{\theta}{\sin\theta}\quad(\,|\,\theta\,|<\pi) \tag{4.19}$$

we find[37]

$$Q=-\frac{e\theta}{2\pi}2q\,. \tag{4.20}$$

For unit pole strength ($q=\frac{1}{2}$), this is precisely the Witten formula,[14,15] and we conclude that the monopole actually becomes a dyon for $M\neq 0$. (The zero-mass limit is discussed in the Appendix.)

Similarly, we may calculate the vacuum expectation values of other fermion bilinears; of particular interest is the axial charge density

$$\langle\theta\,|\,j_{05}(\vec{x},t)\,|\,\theta\rangle=0\,. \tag{4.21}$$

Another quantity of interest is the vacuum energy density; it is given by

$$\frac{qM\sin\theta}{4\pi^2r^2}\delta(r)-\frac{q\cos\theta}{2\pi^2r^2}\int_M^\infty d\kappa\frac{(\kappa^2-M^2)^{1/2}}{\kappa+M\cos\theta}e^{-2\kappa r}\,. \tag{4.22}$$

Unlike the charge however, even the energy difference between different θ vacua remains divergent after renormalization.[10] This is gratifying; otherwise, one may have worried about charged collective excitations. We may also note that according to (4.16) and (4.17), the electrostatic energy of the charge distribution is finite.

Since the Witten effect is due to the Dirac sea, it is also interesting to consider the Galilean limit.[9] As in Sec. I, the Pauli equation

$$i\frac{\partial}{\partial t}\psi(\vec{x},t)=\frac{1}{2M}(\vec{\sigma}\cdot\vec{\pi})^2\psi(\vec{x},t) \tag{4.23}$$

may be reduced to

$$\tilde{H}_0\chi(r)=E\chi(r)\,, \tag{4.24}$$

$$\tilde{H}_0=-\frac{1}{2M}\frac{d^2}{dr^2}\,, \tag{4.25}$$

where $\chi(r)$ now has only one component.

Again, an application of the Weyl-von Neumann theory indicates the existence of a one-parameter family of self-adjoint extensions for H_0; the required boundary condition is

$$\chi'(0)/\chi(0)=-\kappa\,, \tag{4.26}$$

where κ is an arbitrary real quantity with the dimension of momentum. In particular, if κ is positive, there exists a bound state

$$E=-\frac{\kappa^2}{2M}:\ \tilde{B}_\kappa(r)=\sqrt{2\kappa}e^{-\kappa r}\,. \tag{4.27}$$

[As shown in the Appendix, $\delta(\vec{x})$ may be set equal to zero for a wave section.] To determine the value of κ, we consider (4.20) as the limiting case of (2.1). The standard procedure[39,27] is to perform a unitary transformation on (2.1) so that only the upper or lower components of the Dirac wave function are nonvanishing.

This prescription appears feasible in our case also; a formal calculation gives

$$e^{iS}He^{-iS}=c\beta[(\vec{\sigma}\cdot\vec{\pi})^2+M^2c^2]^{1/2}\,, \tag{4.28}$$

$$S=\frac{-i\beta}{2}\arctan\frac{\vec{\alpha}\cdot\vec{\pi}}{Mc}\,. \tag{4.29}$$

(We have reinserted the velocity of light.) However, in the $j=q-\frac{1}{2}$ sector, S reduces to

$$\tfrac{1}{2}\arctan\left[\frac{-i\rho_2}{Mc}\frac{d}{dr}\right], \rho_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \tag{4.30}$$

and we again need a boundary condition to ensure that e^{iS} is unitary.[40] As in Sec. III, there exists a one-parameter family e^{iS_ω} with the corresponding eigenfunctions

$$e^{i\omega\rho_2/2}\begin{bmatrix}\cos kr\\ -\sin kr\end{bmatrix}, \tag{4.31}$$

where k now ranges over both positive and negative values.

After some calculation, we find that for $E \geq Mc^2$,

$$e^{iS_\omega}\chi_E^{(1)}(r)=\frac{i}{\sin\delta_0}\begin{bmatrix}\sin kr\\ 0\end{bmatrix}-\frac{i(1+e^{-i\omega})}{2\pi}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}\int_{Mc}^{\infty}d\kappa\frac{\kappa+(E/c)-Mc}{\kappa^2+k^2}\left[\frac{\kappa-Mc}{\kappa+Mc}\right]^{1/4}e^{-\kappa r}$$
$$-\frac{i(e^{i\omega}+1)}{2\pi}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}\int_{Mc}^{\infty}d\kappa\frac{\kappa-(E/c)+Mc}{\kappa^2+k^2}\left[\frac{\kappa+Mc}{\kappa-Mc}\right]^{1/4}e^{-\kappa r}, \tag{4.32}$$

where the first term is the naive result with

$$\tan\delta_0=\frac{kc}{E+Mc^2}, \quad |\delta_0|<\frac{\pi}{2}. \tag{4.33}$$

Similarly,

$$e^{iS_\omega}\chi_E^{(2)}(r)=\frac{1}{\cos\delta_0}\begin{bmatrix}\cos kr\\ 0\end{bmatrix}+\frac{1-e^{-i\omega}}{2\pi}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}\int_{Mc}^{\infty}d\kappa\frac{\kappa-E/c+Mc}{\kappa^2+k^2}\left[\frac{\kappa-Mc}{\kappa+Mc}\right]^{1/4}e^{-\kappa r}$$
$$+\frac{e^{i\omega}-1}{2\pi}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}\int_{Mc}^{\infty}d\kappa\frac{\kappa+E/c-Mc}{\kappa^2+k^2}\left[\frac{\kappa+Mc}{\kappa-Mc}\right]^{1/4}e^{-\kappa r}. \tag{4.34}$$

Comparing with (4.1) and (4.2), we find that the extra terms can be made to vanish only when $\theta=\pi/2$, $\omega=0$ or $\theta=-\pi/2$, $\omega=\pm\pi$. In other words, the positive-energy solutions cannot be decoupled from the negative-energy solutions. This is not surprising in view of the Witten effect (4.20); what is unexpected is that the exception occurs at the CP-violating values $\theta=\pm\pi/2$ rather than at the CP-conserving values $\theta=0, \pi$.

In view of such circumstances, we conclude that a reduction to the Pauli equation is not very useful. If one insists, however, the best thing to do is presumably to use the naive result

$$e^{iS}u_{k\theta}(r)\simeq 2i\left[\frac{E}{E-M\sin\theta}\right]^{1/2}\begin{bmatrix}1\\ 0\end{bmatrix}\left[\cos\left[\frac{\theta}{2}+\frac{\pi}{4}\right]\cos\delta_0\sin kr+\sin\left[\frac{\theta}{2}+\frac{\pi}{4}\right]\sin\delta_0\cos kr\right] \tag{4.35}$$

but with the boundary condition (4.26) imposed at some value r_0 greater than $1/Mc$. It follows from (4.33) and (4.35) that the appropriate choice of κ for $kr_0 \ll 1$ is

$$\kappa=-2Mc\cot\left[\frac{\theta}{2}+\frac{\pi}{4}\right], \tag{4.36}$$

which reduces to the conventional choice[41] $\chi(0)=0$ ($\kappa=\pm\infty$) in the limit $Mc\to\infty$.

It is also apparent, however, that the limit should *not* be taken, since it is nonuniform in θ. In other words, real electrons may have a large de Broglie wavelength, but not a zero Compton wavelength.[42]

V. THE CONNECTION BETWEEN THE MASSLESS CASE AND THE MASSIVE CASE

In this section, we investigate the properties of Eq. (2.21) in the complex E plane. It is convenient at first to enclose the system in a sphere of radius R. (Some subtleties associated with this point are discussed in the Appendix.) It is then necessary to impose boundary conditions at $r=R$ as well as at $r=0$. A suitable choice is

$$\chi(R)\propto\zeta(\theta')=\begin{bmatrix} i\sin\left[\dfrac{\theta'}{2}+\dfrac{\pi}{4}\right] \\ \cos\left[\dfrac{\theta'}{2}+\dfrac{\pi}{4}\right]\end{bmatrix}, \tag{5.1}$$

where θ' is arbitrary, so that H_0 has a discrete set of real eigenvalues $\{E_n\}$.

The vacuum charge density may be now written as

$$\rho_\theta(\vec{x})=-\frac{qe}{4\pi r^2}\left[\sum_{E_n>0}u_n^\dagger(r)u_n(r)-\sum_{E_n<0}v_n^\dagger(r)v_n(r)\right]. \tag{5.2}$$

A formal integration over space then gives

$$Q=-qe\left[\sum_{E_n>0}1-\sum_{E_n<0}1\right]. \tag{5.3}$$

Evidently, the quantity in the large parentheses of (5.3) measures the *spectral asymmetry* of the operator $H_0(H)$; this is essentially the definition of the η invariant of Atiyah, Patodi, and Singer.[21] To be more precise, one first introduces a ζ-function regularization[43]

$$\eta(s)\equiv\sum_n \operatorname{sgn}(E_n)\,|E_n|^{-s}. \tag{5.4}$$

The right-hand side defines an analytic function for Re s sufficiently large; $\eta(s)$ is then defined for other values by analytic continuation. It turns out in particular that $\eta(s)$ is regular at $s=0$; its value there defines the η invariant $\eta\equiv\eta(0)$. Since η is formally equal to the quantity in the large parentheses of (5.3), we may expect

$$\eta=\theta/\pi. \tag{5.5}$$

To show that (5.5) is actually true we follow standard techniques[44,45] and consider the solutions of (2.21) for general (complex) values of E.

We have already introduced two solutions $\chi_E^{(1)}$ and $\chi_E^{(2)}$ in (2.27). For our purposes, it is more convenient to take the linear combinations $\chi_E^{(1\pm i2)}\equiv\chi_E^{(1)}\pm i\chi_E^{(2)}$ corresponding to the boundary conditions for $\theta=0,\ \pi$,

$$\chi_E^{(1\pm i2)}(0)=\begin{bmatrix}\pm i\\ 1\end{bmatrix}. \tag{5.6}$$

We note that both solutions are analytic in E as expected for E-independent boundary conditions. We shall also introduce two other solutions. One satisfies the boundary condition at $r=0$,

$$w_1(r;E,\theta)=\chi_E^{(1+i2)}(r)-\tan\frac{\theta}{2}\chi_E^{(1-i2)}(r). \tag{5.7}$$

The other satisfies the boundary condition at $r=R$,

$$w_2(r;E;\theta')=\chi_E^{(1+i2)}(r)+l_R(E,\theta')\chi_E^{(1-i2)}(r). \tag{5.8}$$

Generically denoting these solutions as χ_E and ψ_E, we may record the conjugation formula

$$\beta\chi_E^*(r)=-\chi_{E^*}(r), \tag{5.9}$$

the Wronskian formula

$$\frac{d}{dr}\psi_{E^*}^\dagger(r)\gamma_5\chi_E(r)=i\psi_{E^*}^\dagger(r)H_0\chi_E(r)-i(H_0\chi_{E^*})^\dagger(r)\chi_E(r)=0, \tag{5.10}$$

and the Green's formula

$$2i\,\mathrm{Im}E\int_0^R dr\,\psi_E^\dagger(r)\chi_E(r)=-i\int_0^R dr\frac{d}{dr}\psi_E^\dagger(r)\gamma_5\chi_E(r)=-i\psi_E^\dagger(R)\gamma_5\chi_E(R)+i\psi_E^\dagger(0)\gamma_5\chi_E(0). \tag{5.11}$$

It follows from the definition that

$$0=\zeta^\dagger(\theta')\gamma_5 w_2(R;E,\theta')=\zeta^\dagger(\theta')\gamma_5\chi_E^{(1+i2)}(R)+l_R(E,\theta')\zeta^\dagger(\theta')\gamma_5\chi_E^{(1-i2)}(R) \tag{5.12}$$

or

$$-l_R(E,\theta')=\zeta^\dagger(\theta')\gamma_5\chi_E^{(1+i2)}(R)/\zeta^\dagger(\theta')\gamma_5\chi_E^{(1-i2)}(R). \tag{5.13}$$

Hence the function

$$f_R(E,\theta,\theta')=l_R(E,\theta')+\tan\frac{\theta}{2} \tag{5.14}$$

is meromorphic in E. Also by virtue of (5.9) we have

$$f_R^*(E,\theta,\theta')=f_R(E^*,\theta,\theta'). \tag{5.15}$$

Obviously, for fixed R,θ,θ', $f_R(E,\theta,\theta')$ can vanish if and only if E is equal to an eigenvalue E_n under the boundary conditions (2.31) and (5.1). On the other hand, $f_R(E,\theta,\theta')$ will have a pole if and only if

$$\zeta^\dagger(\theta')\gamma_5\chi_E^{(1-i2)}(R)=0 \; , \tag{5.16}$$

i.e., when E is also an eigenvalue E_n', but for the boundary-value problem with $\theta=\pi$.

Therefore, with the choice of contours $C_\pm$ as in Fig. 2, $\eta(s)$ may be represented as

$$\eta(s)-\eta(s)\,|_{\theta=\pi}=\left[\frac{1}{2\pi i}\int_{C_+}\frac{dE}{E^s}+\frac{1}{2\pi i}\int_{C_-}\frac{dE}{(-E)^s}\right]\frac{1}{f_R(E,\theta,\theta')}\frac{df_R(E,\theta,\theta')}{dE} \; . \tag{5.17}$$

Since E is complex, the limit $R\to\infty$ may be safely taken to give

$$f(E,\theta)\equiv\lim_{R\to\infty}f_R(E,\theta,\theta')=l(E)+\tan\frac{\theta}{2} \; ,$$
$$l(E)=\begin{cases}-\dfrac{M-ik}{E} & (\mathrm{Im}k>0)\,,\\[2mm] -\dfrac{M+ik}{E} & (\mathrm{Im}k<0)\,.\end{cases} \tag{5.18}$$

We note that θ' dropped out. Hence $\eta(s)\,|_{\theta=\pi}$ may be taken to be zero by CP invariance, and we obtain

$$\eta(s)=\left[\frac{-1}{2\pi i}\int_{C_+}\frac{dE}{E^s}+\frac{1}{2\pi i}\int_{C_-}\frac{dE}{(-E)^s}\right]\frac{1}{f(E,\theta)}\frac{df(E,\theta)}{dE} \; . \tag{5.19}$$

Having disposed of R and θ', we now take the limit $s\to 0$. Noting that

$$l(+\infty\pm i0)=l(-\infty\pm i0)=\pm i \; , \tag{5.20}$$

we find

$$\eta=\frac{-1}{2\pi i}\ln\frac{f(\infty+i0,\theta)}{f(\infty-i0,\theta)}+\frac{1}{2\pi i}\ln\frac{f(-\infty-i0,\theta)}{f(-\infty+i0,\theta)}=\frac{\theta}{\pi}+\text{integer} \; . \tag{5.21}$$

Continuity and symmetry for $-\pi<\theta<\pi$ fixes the integer to be zero, and we recover (5.5) as expected.

To cast (5.21) into a more familiar form, we first eliminate the reference to negative-energy states by rewriting in terms of particles (f) and antiparticles ($\bar{f}$),

$$\eta=\frac{1}{\pi}[\arg f(\infty-i0,\theta)-\arg\bar{f}(\infty-i0,\theta)] \; . \tag{5.22}$$

To expose the physical meaning of f we note that by virtue of (5.11),

$$0<(\chi_E^{(1+i2)}+l(E)\chi_E^{(1-i2)},\ \chi_E^{(1+i2)}+l(E)\chi_E^{(1-i2)})=\frac{2\,\mathrm{Im}l(E)}{\mathrm{Im}E}<\infty \tag{5.23}$$

for complex E. This implies that the solutions for real E,

$$\chi_E^{(\pm)}(r)\equiv\chi_E^{(1+i2)}(r)+l(E\pm i0)\chi_E^{(1-i2)}(r) \; , \tag{5.24}$$

must be of the form $e^{\pm ikr}$ for $E\geq M$. Explicitly,

$$\chi_E^{(\pm)}(\underline{R})=2i\begin{bmatrix}E+M\\ \pm k\end{bmatrix}e^{\pm i(kr-\delta_0)} \; , \tag{5.25}$$

where δ_0 is defined as in (4.33). Since

$$u_{k\theta}(r)\propto[f(E-i0,\theta)\chi^{(+)}(r)$$
$$-f(E+i0,\theta)\chi^{(-)}(r)] \; , \tag{5.26}$$

the phase shift is given by

$$e^{2i\delta(E)}=-\frac{f(E-i0,\theta)}{f(E+i0,\theta)} \; , \tag{5.27}$$

and we find that $f(E,\theta)$ is essentially the Jost function.[45]

The fruit of our excursion into the complex plane is therefore the following simple formula:

$$\eta=\frac{1}{\pi}[\delta(\infty)-\bar{\delta}(\infty)] \; , \tag{5.28}$$

where

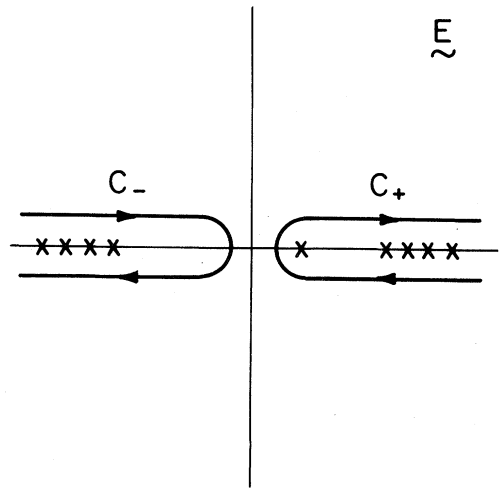

FIG. 2. Contour in the complex E plane.

$$\tan\delta(E)=\frac{k}{E+M}\tan\left[\frac{\theta}{2}+\frac{\pi}{4}\right],$$
$$\tan\bar{\delta}(E)=\frac{k}{E+M}\tan\left[-\frac{\theta}{2}+\frac{\pi}{4}\right]. \tag{5.29}$$

The similarity with Levinson's theorem[18] is obvious; condensed matter physicists may prefer a form closer to the Friedel sum rule[18]

$$Q=-\frac{e}{2}\frac{2j+1}{\pi}[\delta(\infty)-\bar{\delta}(\infty)]. \tag{5.30}$$

In any case, the intriguing aspect of the equality is that it establishes a connection between the Witten effect for $M\neq 0$ and the chiral symmetry breaking for $M=0$. Masses are negligible at high energies, and the phase shift approaches its value in the massless theory, which in turn is governed by the chiral symmetry breaking as we have seen in Sec. III. (The argument may also be regarded as a "physics proof" for the invariance of η.)

It is also worth noting that our results are closely connected with the occurrence of fractional charges on a kink in one dimension.[19,20] The reason is not hard to see. Let us ignore the fact that Eqs. (2.22) and (2.23) were derived from the three-dimensional theory (2.1) and (2.2). Then we may consider two systems back to back which have different values of θ for $r>0$ and $r<0$. (The latter is parity reflected.) By putting θ into the mass term rather than the boundary condition, the wave function may be joined continuously, whereas the Hamiltonian becomes

$$H_0=-i\gamma_5\frac{d}{dr}+M\beta e^{i\theta(r)\gamma_5}, \tag{5.31}$$

$$\theta(r)=\begin{cases}\theta_+, & r>0\\ \theta_-, & r<0.\end{cases} \tag{5.32}$$

If we also smooth out the discontinuity of $\theta(r)$ at $r=0$, we obtain a Hamiltonian which can be considered as a field-theoretic analog of polyacetylene.

Going through the same calculations as before, it is not hard to show that

$$\eta(-\infty,\infty)=-\eta(\infty,0)-\eta(0,\infty), \tag{5.33}$$

where $\eta(-\infty,\infty)$ is the η invariant for the Hamiltonian (5.31) defined on the whole line, whereas $\eta(-\infty,0)$ and $\eta(0,\infty)$ are the η invariants for the same Hamiltonian but defined on the two half-lines with a common boundary condition of the form (2.31). We may remark that (5.33) does not depend on the angle used in the boundary condition; owing to (5.10), that also means that we could have imposed the boundary condition at some point other than $r=0$.

It is also easy to show that the charge on a kink is minus one-half the η invariant. Hence

$$Q=\tfrac{1}{2}\eta(-\infty,0)+\tfrac{1}{2}\eta(0,\infty). \tag{5.34}$$

Here we notice a slight difference between monopoles and kinks. For the latter, the charge is not related to the true phase shifts (which go to zero as $E\rightarrow\infty$), but to the apparent phase shifts at $r=\pm\infty$ for a standing wave with infinite energy and zero chirality.

Finally, we note that instantons[13] in four-dimensional Euclidean gauge theories may also be reduced to a one-dimensional kink problem.[46] It is evidently of interest to find what precisely all these systems have in common and to see whether it may be formulated as a general principle.

VI. DISCUSSION

In the previous sections, we have seen that the monopole-fermion system leads to θ vacua, and the monopole becomes a dyon for $M\neq 0$. Evidently, many questions remain to be answered, some of which are the following.

(1) What are the conditions for the appearance of θ? For example, what would happen if there were two or more monopoles, or if we considered a monopole-boson system instead of a monopole-fermion system?

(2) What would happen if we included the electric interaction between the dyon-type monopole and the fermion? What would happen if the monopole were also quantized?[47] Is there any systematic way of

treating such effects?

(3) What is the relation between Abelian monopoles and non-Abelian monopoles[48] (dyons[49])? In particular, what is the relevance of our results for monopole-induced proton decay?[15,50,51]

We hope to answer some of these questions in the near future. In the meanwhile, we may also ask the following question: Do monopoles actually exist?

Note added in proof. While the revised version of the paper was being typed, Dr. B. Grossman informed us that he has independently obtained many of our results.[52]

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APPENDIX

In this appendix, we would like to deal with three subtleties which have led to some confusion in the literature (as well as the author).

The first one is associated with the quantity[17,23]

$$[[\pi_x,\pi_y],\pi_z]+\text{cyclic permutations}\ . \tag{A1}$$

The Jacobi identity requires it to be zero; however, explicit computation using

$$[\pi_x,\pi_y]=+ieB_z,\ \text{etc.}\ ,$$
$$\sum_i [\pi_i,B_i]=-i\,\mathrm{div}\vec{B} \tag{A2}$$

gives $-4\pi q\delta(\vec{x})$. Some authors have therefore concluded that the wave section must vanish at the origin, a condition not satisfied by any of the solutions for the lowest partial wave.

Unfortunately, the conclusion is incorrent. If the π_i's are regarded as differential operators acting on distributions, it is sufficient that the *test* sections should vanish at the origin. Since wave sections in $R^3-\{0\}$ with compact support do vanish as $r\to 0$ (the support cannot "touch" the origin), it is consistent to set $\delta(\vec{x})=0$.

On the other hand, we may also regard π_i as self-adjoint operators (observables) acting on the Hilbert space of wave sections.

The situation then becomes subtle owing to domain problems[17]; this is particularly so for commutators.[53] However, it is important to recognize that such subtleties in the Jacobi identity may be relevant to interferences in the measurement of $\vec{\pi}$,[54] but cannot affect the problem of the correct choice of the Hamiltonian. The latter is determined by the requirement of self-adjointness, which is sufficient to give a consistent quantum mechanics.

The conclusion is also expected on general grounds. Since whatever physical effects the monopole may have besides its magnetic charge are completely specified by the boundary condition, we could have imposed it at a small distance away from the origin; then there would obviously be no difficulties.

The second subtlety is concerned with the infinite volume limit. In Sec. V, we had to take the limit $R\to\infty$ before the limit $s\to 0$; the other order gave the divergent expression (5.3).[37] As a consequence, the derivation of the equality

$$Q=-qe\eta \tag{A3}$$

was heuristic; we had to compute both sides explicitly and compare.

However, in view of the significance of (A3), we shall give a more direct derivation. To this end, we introduce the Green's function (resolvent)

$$(H_0-E)G(r,r';E)=\delta(r-r') \tag{A4}$$

under the boundary conditions (2.31) and (5.1). As is well known, the Green's function has the representations

$$G(r,r';E)=\begin{cases}\dfrac{w_2(r;E,\theta')w_1^\dagger(r';E^*,\theta)}{f_R(E,\theta,\theta')} & (r>r')\ ,\\[2ex] \dfrac{w_1(r;E,\theta)w_2^\dagger(r';E^*,\theta')}{f_R(E,\theta,\theta')} & (r<r')\ ,\end{cases} \tag{A5}$$

$$G(r,r';E)=\sum_{E_n>0}\frac{u_n(r)u_n^\dagger(r')}{E_n-E}+\sum_{E_n<0}\frac{v_n(r)v_n^\dagger(r')}{E_n-E}\ . \tag{A6}$$

Actually, (A6) is divergent; however, we may still use it to represent $\eta(s)$ as

$$\int_0^R dr\left[\frac{-1}{2\pi i}\int_{C_+}\frac{dE}{E^s}+\frac{1}{2\pi i}\int_{C_-}\frac{dE}{(-E)^s}\right]$$
$$\times\mathrm{tr}G(r,r+0;E) \tag{A7}$$

since

$$\sum_{E_n>0}\frac{u_n(r)u_n^\dagger(r')}{E_n^s}-\sum_{E_n<0}\frac{v_n(r)v_n^\dagger(r')}{|E_n|^s} \tag{A8}$$

is convergent for sufficiently large Re s.

The other form (A5) for $G(r,r';E)$ then gives

$$\int_0^R dr \left[\frac{-1}{2\pi i} \int_{C_+} \frac{dE}{E^s} + \frac{1}{2\pi i} \int_{C_-} \frac{dE}{(-E)^s} \right] \times \frac{w_2^\dagger(r;E^*,\theta')w_1(r;E,\theta)}{f_R(E,\theta,\theta')} . \quad \text{(A9)}$$

Using the definitions (5.7) and (5.8) and the analyticity of χ_E, we may bring (A10) into the form

$$\int_0^R dr \left[\frac{-1}{2\pi i} \int_{C_+} \frac{dE}{E^s} + \frac{1}{2\pi i} \int_{C_-} \frac{dE}{(-E)^s} \right] \times \frac{w_1^\dagger(r;E^*,\theta)w_1(r;E,\theta)}{f_R(E,\theta,\theta')} . \quad \text{(A10)}$$

We may now take the limit $R \to \infty$ and pinch the contour to find

$$\eta(s) = \int_0^\infty dr \left[\int_{E>0} d\sigma(E) - \int_{E<0} d\sigma(E) \right] \times w_1^\dagger(r;E,\theta)w_1(r;E,\theta)\,|E|^{-s} , \quad \text{(A11)}$$

where

$$\frac{d\sigma(E)}{dE} \equiv -\mathrm{Im}\frac{1}{f(E+i0,\theta)} \geq 0 \quad \text{(A12)}$$

by (5.23). Comparing (4.1) and (4.2) with (5.7) and (5.18), we find

$$\left[\int_{E>0} - \int_{E<0} \right] d\sigma(E) w_1^\dagger(r;E,\theta)w_1(r;E,\theta) = \frac{1}{\pi} \int_0^\infty dk [u_{k\theta}^\dagger(r)u_{k\theta}(r) - v_{k\theta}^\dagger(r)v_{k\theta}(r)] + (\text{bound state}) . \quad \text{(A13)}$$

Hence, as $s \to 0$, (A11) reduces to (A3) as desired.[55]

It is not hard to see now why we need to take the limit $R \to \infty$ before $s \to 0$. For finite R, there is a charge distribution near $r = R$ (as well as $r = 0$) which must be sent off to infinity first.

The remark also allows us to resolve a paradox associated with the zero-mass limit. The Witten formula (4.20) is independent of M; on the other hand, there should be no θ dependence for $M = 0$. The answer[51] is that the total charge remains fixed as $M \to 0$, but it becomes spread over a volume of $O(M^{-3})$ so that the charge density goes to zero.

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