Massless composite particles and space-time description of gauge transformations

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Space-time symmetries for massless composite particles are studied within the kinematical framework of the relativistic harmonic-oscillator formalism. It is noted first that the Lorentz-invariant harmonic-oscillator equation is separable in the Kalnins-Miller coordinate system in which the helicity and gauge transformation parameters can be separated from other space-time variables. It is shown then that this separation procedure leads to the representations of the Poincaré group which display the same space-time symmetry as those for photons and gravitons. The gauge transformation parameters are shown to be internal space-time parameters in the case of massless composite particles. It is shown in particular that the spin in this case is associated with an orbitlike motion on the plane perpendicular to the momentum of the composite particle.

I. INTRODUCTION

Composite models of "fundamental" particles are of current interest.[1] We are interested in space-time symmetries of these particles. While the massive composite particle is known to have O(3)-like internal space-time symmetries,[2] we are not yet familiar with internal space-time symmetries of massless composite particles.[3,4]

There are at present two different approaches to the space-time structure of composite particles. One way is to look for the dynamical origin of the forces between constituents and then study how they become confined using the Lagrangian field-theoretic method.[5] The other approach is to start with confined constituents and study space-time symmetry properties of the composite system, following closely the line based on the construction of representations of the Poincaré group suggested by Wigner[6] and Dirac.[7]

In the approach of constructing representations of the Poincaré group, the covariant harmonic-oscillator formalism[8] has been shown to be a solution of Dirac's commutator equations for his "instant form" quantum mechanics.[9] In addition, the oscillator model has produced comfortable numerical results in the relativistic quark model.[10–13]

In Wigner's language,[6] the above-mentioned harmonic-oscillator wave functions form the representations of the O(3)-like little group for massive hadrons.[2] For massless particles, the internal space-time symmetry is isomorphic to the two-dimensional Euclidean group or E(2).[6] Therefore, it is an interesting proposition to construct harmonic-oscillator wave functions which form the representations of the E(2)-like little group for massless composite particles.

In this paper, we note first that the harmonic-oscillator equation is separable in many different coordinate systems. In particular, it is separable in one of the orthogonal coordinate systems discussed recently by Kalnins and Miller.[14] In this coordinate system, two of the space-time coordinate variables are conjugate to the generators of the E(2)-like little group for the massless composite particle. We can therefore construct representations of the E(2)-like little group by solving the differential equation using the Kalnins-Miller coordinate system.

We study then properties of the explicit solutions of the harmonic oscillator equation. It is shown that the gauge parameters in this case are space-time parameters, and therefore that the spin of massless composite particles is related to an orbitlike motion in the plane perpendicular to the momentum, as was suspected in one of our previous papers.[15] Although we used the harmonic-oscillator equation for convenience, it turns out that most of the results obtained in this paper, particularly those connected with gauge transformations, are independent of forms of the potential, and are readily applicable to more general cases which share the same space-time symmetry.

We assume throughout this paper that the constituent particles are spinless, and therefore that the resulting massless composite particles can have only integer spins. We realize that composite particles with half-integer helicities would also be interesting.[16] However, in view of our experience in the quark model for hadrons, this spin problem does not appear to be trivial, and definitely requires a separate investigation.

In Sec. II we present the covariant harmonic-oscillator formalism in a form suitable for discussing the internal space-time symmetries for both massive and massless composite particles. Section III contains an explicit construction of the representations having the E(2)-like internal space-time symmetry for massless composite systems. Section IV deals with a physical interpretation of the mathematical formulas derived in Sec. III. It is shown that the gauge transformation is an internal space-time coordinate transformation.

II. FORMULATION OF THE PROBLEM

The standard procedure in studying internal space-time structures is to start from the space-time coordinates of constituent particles. Let us consider for simplicity a composite system consisting of two constituent particles. If their space-time coordinates are x_1 and x_2, we are led to the variables

$$X=(x_1+x_2)/2\ ,\qquad x=(x_1-x_2)/\sqrt{2}\ , \tag{1}$$

where X is the space-time coordinate for the composite particle, and x is the relative space-time separation between the two constituent particles.

In terms of these variables, the generators of the Poincaré group[6] can be written as

$$P_\mu=i\partial/\partial X^\mu\ ,\qquad M_{\mu\nu}=L^{*}_{\mu\nu}+L_{\mu\nu}\ , \tag{2}$$

where

$$L^{*}_{\mu\nu}=i(X_\mu\partial/\partial X^\nu-X_\nu\partial/\partial X^\mu)\ ,\qquad L_{\mu\nu}=i(x_\mu\partial/\partial x^\nu-x_\nu\partial/\partial x^\mu)\ . \tag{3}$$

As was noted in previous papers,[2,17] the translation operator P_μ does not depend on the internal variable x_μ, and the Lorentz transformation generator $M_{\mu\nu}$ is a sum of the generators for X and x coordinates. $L_{\mu\nu}$ is the Lorentz transformation operator acting on the internal coordinate.

In the case of massive hadrons, we can construct representations of the Poincaré group by solving the following harmonic-oscillator equation[2,17,18]:

$$[2(\Box_1+\Box_2)-\tfrac{1}{16}(x_1-x_2)^2+m_0{}^2]\phi(x_1,x_2)=0\ , \tag{4}$$

where the spring constant is assumed to be 1. In terms of the coordinate variables given in Eq. (1), the above Lorentz-invariant differential equation can be written as

$$\{(\partial/\partial X_\mu)^2+m_0{}^2+\tfrac{1}{2}[(\partial/\partial x_\mu)^2-x_\mu{}^2]\}\phi(X,x)=0\ . \tag{5}$$

This equation is separable in the X and x variables, and thus

$$\phi(X,x)=f(X)\psi(x)\ , \tag{6}$$

where $f(X)$ and $\psi(x)$ satisfy differential equations

$$[(\partial/\partial X_\mu)^2+m_0{}^2+\lambda]f(X)=0\ , \tag{7}$$

$$\tfrac{1}{2}[(\partial/\partial x_\mu)^2-x_\mu{}^2]\psi(x)=\lambda\psi(x)\ . \tag{8}$$

The differential equation of Eq. (7) is a Klein-Gordon equation, and its solution is well known. $f(X)$ takes the form

$$f(X)=\exp(\pm ip\cdot X) \tag{9}$$

with

$$p^2=m_0{}^2+\lambda\ ,$$

where p is the four-momentum of the composite particle. p^2 is of course the $(\text{mass})^2$ of the composite particle. The separation constant λ is determined from the solutions of the harmonic-oscillator differential equation given in Eq. (8).

The study of the internal space-time symmetry of composite systems is the study of the little group[6] whose transformations leave the total four-momentum p invariant. In the case of massive hadrons, the little group is isomorphic to O(3), and its representation takes the simplest form in the Lorentz frame in which the composite particle is at rest. The generators of this little group are

$$L_i=\tfrac{1}{2}\epsilon_{ijk}L_{jk}\ . \tag{10}$$

For the case of massive hadrons with positive values of p^2, the harmonic-oscillator equation given in Eq. (8) has solutions which span the representation space of the O(3)-like little group.[2,17]

As was discussed extensively in the literature, the above-mentioned procedure is consistent with the known physical principles of quantum mechanics and special relativity, and is within the framework of Dirac's instant-form quantum mechanics. This procedure is also consistent with the basic observed hadronic phenomena including mass spectra, form-factor behaviors, parton picture, and jet phenomenon. We have therefore a good reason to believe that the solutions of the oscillator equation illustrates the space-time symmetries, if not the detailed dynamics, of relativistic composite particles.

We are then naturally led, still within the framework of this oscillator formalism, to the question of constructing representations of the Poincaré group for massless particles.

If the composite particle is massless, this means that m_0 and λ in Eq. (9) are such that p^2 is zero. In this case, the generators of the little group are different from those in Eq. (10). We shall study whether we can construct representations of this little group by solving the harmonic-oscillator equation given in Eq. (8).

III. CONSTRUCTION OF REPRESENTATIONS FOR MASSLESS COMPOSITE PARTICLES

Let us assume without loss of generality that the momentum of the massless composite particle is in the z direction. Then the little group is generated by L_3, N_1, and N_2, where

$$\begin{aligned} N_1 &= L_{01} - L_2 \ , \\ N_2 &= L_{02} + L_1 \ . \end{aligned} \tag{11}$$

These generators satisfy the commutation relations

$$\begin{aligned} &[N_1, N_2] = 0 \ , \\ &[L_3, N_1] = -iN_2 \ , \\ &[L_3, N_2] = iN_1 \ . \end{aligned} \tag{12}$$

As was noted first by Wigner,[6] the above commutation relations are those for the generators of the E(2) group consisting of a rotation and two translations in a two-dimensional Euclidean plane. N_1 and N_2 represent the two translation operators, while L_3 is the rotation operator. L_3 in this case is the helicity operator. There are two maximal commuting sets of operators in the enveloping Lie algebra. They are

$$\begin{aligned} &\text{(a)}\ \ {}^{2}, N_1, N_2 \ , \\ &\text{(b)}\ N^2, L_3 \ , \end{aligned} \tag{13}$$

where $N^2 = {N_1}^2 + {N_2}^2$. Because we are interested in states with definite helicities, we have to construct wave functions which are diagonal in L_3 and N^2.

In order to construct solutions of the oscillator wave equation given in Eq. (8), we have to make a judicious choice of the coordinate system in which the differential equation is separable. It is known that the Klein-Gordon equation is separable in many different coordinate systems. In a relatively recent paper,[14] Kalnins and Miller discussed 34 different coordinate systems. Among those, coordinate system number 14 appears to satisfy our requirement. We shall call this the Kalnins-Miller coordinate system.

If the four-vector x is timelike with positive t, the Kalnins-Miller coordinate variables ρ, η, α, and ϕ, are related to x, y, z, and t by

$$\begin{aligned} x &= \rho e^{-\alpha} \eta \cos\phi \ , \\ y &= \rho e^{-\alpha} \eta \sin\phi \ , \\ z &= (\rho/2)[e^{\alpha} + (\eta^2 - 1) e^{-\alpha}] \ , \\ t &= (\rho/2)[e^{\alpha} + (\eta^2 + 1) e^{-\alpha}] \ . \end{aligned} \tag{14}$$

These equations can also be written as

$$\begin{aligned} \rho &= (t^2 - z^2 - r^2)^{1/2} \ , \\ \eta &= r/(t - z) \ , \\ \alpha &= -\ln[(t - z)/(t^2 - z^2 - r)^{1/2}] \ , \\ \phi &= \tan^{-1}(y/x) \ , \end{aligned} \tag{15}$$

where

$$r = (x^2 + y^2)^{1/2} \ .$$

In terms of the Kalnins-Miller variables, L_3 and N^2 take the form

$$\begin{aligned} L_3 &= -i\partial/\partial\phi \ , \\ N_1 &= -i\partial/\partial\eta_1 \ , \\ N_2 &= -i\partial/\partial\eta_2 \ , \\ N^2 &= -(\partial/\partial\eta)^2 - (1/\eta)(\partial/\partial\eta) \\ &\quad - (1/\eta^2)(\partial/\partial\phi)^2 \ , \\ &= -[(\partial/\partial\eta_1)^2 + (\partial/\partial\eta_2)^2] \ , \end{aligned} \tag{16}$$

where

$$\eta_1 = \eta \cos\phi, \quad \eta_2 = \eta \sin\phi \ .$$

The oscillator differential equation of Eq. (8) can then be written as

$$\left[\frac{1}{\rho}\right]^3 \frac{\partial}{\partial\rho}\left[\rho^3 \frac{\partial}{\partial\rho}\psi(x)\right] + \left[\frac{1}{\rho}\right]^2 \left[\left[\frac{\partial}{\partial\alpha}\right]^2 - 2\left[\frac{\partial}{\partial\alpha}\right]\right]\psi(x) - \left[\left[\frac{1}{\rho}\right]^2 e^{-2\alpha} N^2 + \rho^2\right]\psi(x) = 2\lambda\psi(x) \ . \tag{17}$$

In order to separate this differential equation, let us write $\psi(x)$ in the form

$$\psi(x) = G(\rho,\alpha) F(\eta,\phi) \ . \tag{18}$$

If $F(\eta,\phi)$ satisfies the eigenvalue equation

$$N^2 F(\eta,\phi) = b^2 F(\eta,\phi) \ , \tag{19}$$

then, $G(\rho,\alpha)$ should satisfy the differential equation

$$\left[\frac{1}{\rho}\right]^3 \frac{\partial}{\partial\rho}\left[\rho^3\frac{\partial}{\partial\rho}G\right]+\left[\frac{1}{\rho}\right]^2\left[\left[\frac{\partial}{\partial\alpha}\right]^2-2\left[\frac{\partial}{\partial\alpha}\right]-e^{-2\alpha}b^2\right]G-\rho^2G=2\lambda G\ . \quad (20)$$

The differential equation of Eq. (19) is a two-dimensional Helmholtz equation if b^2 does not vanish. It is a Laplace's equation if $b^2=0$. As was discussed in our previous papers,[15,16] b^2 has to vanish in order that the representation be finite dimensional.[19] The solution then becomes

$$F(\eta,\phi)=\eta^m\exp(\pm im\phi)\ , \quad (21)$$

where m is an integer and is the magnitude of the angular momentum. Since $b^2=0$, the differential equation of Eq. (20) becomes

$$\left[\frac{1}{\rho}\right]^3 \frac{\partial}{\partial\rho}\left[\rho^3\frac{\partial}{\partial\rho}G\right]+\left[\frac{1}{\rho}\right]^2\left[\left[\frac{\partial}{\partial\alpha}\right]^2-2\left[\frac{\partial}{\partial\alpha}\right]^2\right]G-\rho^2G=2\lambda G\ . \quad (22)$$

The solution of the above differential equation will then take the form[20]

$$G_{\mu n}(\rho,\alpha)=[\rho^n\exp(-\rho^2/2)]L_\mu^{(n+1)}(\rho^2)A_n^{(\pm)}(\alpha)\ , \quad (23)$$

where

$$A_n^{(+)}(\alpha)=\exp[(n+2)\alpha]\ ,$$
$$A_n^{(-)}(\alpha)=\exp(-n\alpha)\ .$$

$L_\mu^{(n+1)}(\rho^2)$ is the generalized Laguerre function.[21] The eigenvalue on Eq. (22) takes

$$\lambda=-(n+2\mu+1)\ , \quad (24)$$

where n and μ take integer values. In order that the composite particle be massless, the above eigenvalue and $m_0{}^2$ should satisfy the condition of Eq. (9) for massless particles:

$$p^2=m_0{}^2+\lambda=0\ . \quad (25)$$

We have so far been working for the case where x is timelike with positive values of t. If t is negative, we can reverse the sign of the Cartesian coordinate variables given in Eqs. (14) and (15). If x is a spacelike vector, z and t of Eq. (14) have to be modified to[14]

$$z=(\rho/2)[e^{\alpha}-(\eta^2-1)e^{-\alpha}]\ ,$$
$$t=(\rho/2)[e^{\alpha}-(\eta^2+1)e^{-\alpha}]\ . \quad (26)$$

Consequently, two of the equations in Eq. (15) are modified to

$$\rho=(r^2+z^2-t^2)^{1/2}\ ,$$
$$\eta=r/(z-t)\ , \quad (27)$$
$$\alpha=-\ln[\rho/(z-t)]\ .$$

The process of separating and solving the differential equation is the same as in the case of timelike region. We can use the form of Eq. (21) for $F(\eta,\phi)$, and Eq. (23) for $G(\rho,\alpha)$. However, the eigenvalue λ in this case takes the values

$$\lambda=n+2\mu+1\ . \quad (28)$$

It is important that the masslessness condition of Eq. (25) be satisfied for the above values of λ. Since λ's for the timelike and spacelike regions have opposite signs, $m_0{}^2$ will also have different signs. This does not cause any conceptual difficulty, because the timelike region never mixes with the spacelike region under Poincaré transformations.

IV. INTERPRETATION OF REPRESENTATIONS FOR MASSLESS COMPOSITE PARTICLES

In studying space-time symmetries of the solution of the wave equation obtained in Sec. III, we note that the internal wave function $\psi(x)$ is a product of $G(\rho,\alpha)$ and $F(\eta,\phi)$, as is given in Eq. (18). This allows us to deal with F and G separately.

Let us first discuss the F function. It is not difficult to see that ϕ in Eq. (21) is the angle variable specifying the rotation around the z axis with momentum $\pm m$. This is known as the helicity for the massless particle. Since physically observable states are expected to be helicity eigenstates, they are invariant under the rotation around the z axis.

In terms of the η_1 and η_2 variables defined in Eq. (16), F can be written as

$$F(\eta,\phi)=F(\eta_1,\eta_2)=(\eta_1\pm i\eta_2)^m\ , \quad (29)$$

with

$$\eta_1=x/(t-z),\ \ \eta_2=y/(t-z)\ ,$$

for the timelike region. The operators N_1 and N_2 given in Eq. (16) now generate translations in the $\eta_1\eta_2$ plane. Since both of these "translation" operators commute with N^2, the differential equation of Eq. (17) is invariant under this transformation. We can replace η_1 and η_2 in $F(\eta_1,\eta_2)$ of Eq. (29) by η_1' and η_2', respectively, where

$$\eta_1'=\eta_1+\nu_1\ ,$$
$$\eta_2'=\eta_2+\nu_2\ , \quad (30)$$

without changing the differential equation.

As was emphasized in our previous papers,[15,16] the above-mentioned N_1 and N_2 transformations are equivalent to gauge transformations.[22] Then what is the gauge transformation in terms of the conventional space-time variables? The translation of Eq. (30) causes the following changes in the η and ϕ variables:

$$\eta \to \eta' = [(\eta_1+\nu_1)^2+(\eta_2+\nu_2)^2]^{1/2} ,$$
$$\phi \to \phi' = \tan^{-1}[(\eta_2+\nu_2)/(\eta_1+\nu_1)] . \tag{31}$$

Another way to interpret the above transformations is to regard Eq. (29) as a rotation around the origin in the $\eta_1\eta_2$ plane. Then the transformation of Eq. (30) is to shift the center of rotation from the origin to the coordinate point $(-\nu_1,-\nu_2)$.[16]

In order to see the effect of the N_1 and N_2 transformations in terms of the Cartesian space-time variables, let us write Eq. (14) for the timelike region as

$$\begin{aligned} &r = \rho\eta e^{-\alpha} , \\ &y/x = \tan\phi , \\ &t+z = \rho(e^{\alpha}+\eta^2 e^{-\alpha}) , \\ &z-t = -\rho e^{-\alpha} . \end{aligned} \tag{32}$$

It is apparent that the $(t-z)$ variable remains invariant under the gauge transformation. The effects of Eqs. (30) or (31) on other variables are

$$\begin{aligned} &y'/x' = \tan\phi' , \\ &r'/r = \eta'/\eta , \\ &(z'+t')/(z+t) = (e^{2\alpha}+\eta'^2)/(e^{2\alpha}+\eta^2) . \end{aligned} \tag{33}$$

The variable $\rho=(t^2-z^2-r^2)^{1/2}$ is a gauge-invariant quantity.

Since $(t-z)$ is invariant under the N_1 and N_2 transformations, it is clear from Eq. (29) that the x and y coordinate variables are directly proportional to η_1 and η_2, respectively. Indeed, for $b^2=0$, the differential equation given in Eq. (19) can be written as

$$[(\partial/\partial x)^2+(\partial/\partial y)^2]F(x,y)=0 , \tag{34}$$

with the solution

$$F(x,y)=(x\pm iy)^m .$$

The mathematics of this form is quite familiar to us, and does not require any further explanation. The point is that the gauge transformation parameters are now directly related to the x and y coordinate variables, and the spin of the massless composite particle is indeed due to the above orbitlike form. The gauge transformation in this case is a translation of the rotation axis from the origin to another point in the xy plane.[23]

As for the normalization of $F(\eta,\phi)$, the ϕ dependence is just like the case of hydrogen atom. The Hilbert space and the normalization of wave function associated with this variable are well known. The η dependence is not normalizable,[4] and there is no Hilbert space associated with this variable. As was noted before,[15] this is due to the fact that gauge transformation is not measurable.

Let us next discuss properties of the $G(\rho,\alpha)$ function given in Eq. (23). This function is a product of two separate functions. The ρ dependence is normalizable, and the wave function is concentrated within a hyperbolic region near the light cones. On the other hand, the α dependence, which measures the $(t-z)$ variable for fixed ρ, is not normalizable.[4] However, this does not introduce any additional difficulty to the overall wave function which is not normalizable due to the nonobservability of gauge transformations.

The above discussion has so far been restricted to x in the forward light cone. By changing the sign of the Cartesian variables given in Eq. (14), we can give the same reasoning for the backward light cone. By replacing the z and t variables by those given in Eq. (26), we can give a similar treatment for the spacelike region.

V. CONCLUDING REMARKS

Starting from the harmonic-oscillator equation designed to describe a massless composite system consisting of two particles bound together by a harmonic-oscillator force,[2,7–13,18] we have constructed the representations of the Poincaré group satisfying the prescribed requirements.[6] It has been shown that the wave function can be factorized into the part whose form is expected from those of photons and gravitons,[15,16] and into the part which depends on the form of potential.

We have studied in detail the part containing the space-time symmetries expected from those of photons and gravitons. This part is independent of the form of potential. It has been shown that the gauge transformation is a space-time transformation in the internal coordinate system which specifies the relative space-time separation between the constituent particles.

Unlike the case of massive composite particles, wave functions are not normalizable.[4] However, this should not alarm us. The transverse coordinates in this case are proportional to gauge parameters. In the case of massless composite particles, the fact that gauge transformation is not observable is translated into the lack of Hilbert space associated with the transverse coordinate variables.

As for the fundamental question of whether massless composite particles exist in nature, or whether the existing massless particles such as photons and gravitons are ultimately composite, we are not able to provide the answer at this time. Yet, it is of interest to study their internal space-time symmetries, particularly from the standpoint of constructing representations of the Poincaré group.

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