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Deterministic model of spin and statistics

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A deterministic model that accounts for the statistical behavior of random samples of identical particles is presented. The model is based on some nonmeasurable distribution of spin values in all directions. The mathematical existence of such distributions is proved by set-theoretical techniques, and the relation between these distributions and observed frequencies is explored within an appropriate extension of probability theory. The relation between quantum mechanics and the model is specified. The model is shown to be consistent with known polarization phenomena and the existence of macroscopic magnetism. Finally the possibility of a thought experiment which indicates a deviation from the predictions of quantum mechanics is described.

I. INTRODUCTION

In 1924 Banach and Tarski proved the following remarkable theorem[1]: *"Using the axiom of choice one can cut a ball into finitely many pieces which can be so rearranged that one obtains two balls of the same size as the original one."*

This so-called "Banach-Tarski paradox" is not a paradox at all. The pieces into which the ball is cut are nonmeasurable sets, that is, one cannot assign them numbers that indicate their volume since this will clearly violate the additivity or invariance of "volume." In spite of this explanation and in spite of independent proofs that nonmeasurable sets exist, the Banach-Tarski result was taken as an unfortunate consequence of the axiom of choice (which is, nevertheless, essential in some fields of "good" mathematics).

Suppose, however, that we reverse this attitude and maintain that the subsets into which the ball is decomposed *exist in physical reality.* These hidden pieces could be detected in two "states." The first is a "one-ball state" and the second a "two-balls state." In each state the pieces do have a "volume" which depends, however, on their mutual configuration. Assume that we have a source that emits five balls in the first state. On the way from the source to a counter two of the balls spontaneously transform to the second state. The counter, which does not distinguish between the states, will detect seven balls. This rather simplistic example serves to indicate that one can "perform miracles" if one is willing to accept the physical reality of some highly abstract set-theoretical objects. In particular, if such assumptions are made, it is possible to account for interference effects in a completely mechanistic way without introducing wavelike nonlocal components to the theory. In a previous letter[2] I have sketched a model for spin-$\frac{1}{2}$ statistics based on some nonmeasurable distribution of spin values in all directions. The purpose of this article is to extend this model, to point out some of its physical consequences, and to supply it with a sound mathematical basis.

Mathematicians, in particular applied mathematicians, were reluctant to take nonmeasurable sets seriously. As a result there exists no mathematical theory that relates nonmeasurable distributions with relative frequencies. Such an extension of probability theory, which I believe is suitable for the spin-statistics case, is developed in Sec. II. In Sec. III I present a model of spin-$\frac{1}{2}$ statistics. The relation between this model and the usual, Hilbert-space, formalism of quantum mechanics is developed and various phenomena such as polarization and macroscopic magnetism are explained. Section IV is an extension of the theory to massive spin-1 particles and finally in Sec. V I point out some cases where the predictions of the model may differ from the predictions of quantum mechanics.

Taken in historical perspective the proposed model belongs to the tradition of "hidden-variable theories." Since the birth of quantum mechanics there have been numerous attempts[3] to explain its statistical correlations by reducing them to deterministic relations among hidden, yet undetected, physical parameters. These challenges to the prevailing Copenhagen interpretation were met, in turn, by some mathematical results to the effect that such an enterprise is impossible. The two most powerful impossibility theorems of this kind are due to Bell[4] and Kochen and Specker.[5]

Bell has argued that spin correlations of identical particles in the singlet state could not be explained by a "local hidden-variable theory." The expectation values (or the corresponding relative frequencies) which are predicted by any local hidden-variable theory should satisfy a certain inequality. This inequality in turn is violated by the expectation values given in quantum mechanics. Various experiments[6] verified quantum mechanics rather than its alternatives. Therefore, any hidden-variable theory that attempts to explain these correlations should include a nonlocal mechanism for transporting influences instantaneously from one particle to its singlet-state companion even when they are far apart. There are many good reasons to reject such a theory. The proofs of the various versions of Bell's inequality all assume the validity of probability theory, in particular the additivity axiom. Nonmeasurable distributions can violate additivity and the resulting *physically observed* relative frequencies can violate Bell's inequality. In fact, in the framework of my model, Bell's argument merely establishes the nonmeasurability of the set of points where the relative frequencies converge to the expected limit. (See theorem 4 below. This set is, however, very "big" so that convergence occurs "almost everywhere" in some nonstandard sense.) Thus if one is willing to pay the price of nonmeasurability, locality could be saved.

The other objection to hidden-variable theories, the Kochen and Specker theorem, applies only in the spin-1 (and higher) cases and is therefore accounted for in Sec. IV.

The proposed model is by no means intended as an alternative to quantum mechanics. It should rather be taken as an attempt to interpret at least a part of quantum mechanics in a deterministic fashion. It should also be noted that the expectation values given in the model are Gallilei invariant. Since spin is a relativistic degree of freedom I feel that a more complete account should be developed in a Lorentz-invariant framework.

II. MATHEMATICAL PRELIMINARIES

In the following, x,y,z,w will denote *unit* vectors in three-dimensional Euclidean space. Let $S^{(2)}$ be the set of all such unit vectors,

$$S^{(2)}=\{x\in E^{(3)} \mid \ |x|=1\} .$$

Let $z\in S^{(2)}$ and $0<\theta<\pi$ denote by $c(z,\theta)$ the set of all unit vectors that form an angle θ with z:

$$c(z,\theta)=\{x\in S^{(2)} \mid x\cdot z=\cos\theta\}$$

(where $x\cdot z$ denotes the scalar product of x and z). $c(z,\theta)$ is a circle on the sphere $S^{(2)}$ with radius $\sin\theta$ and center on the vector z (or $-z$).

The family of Borel-measurable subsets of $S^{(2)}$ is the σ-Boolean algebra generated by the open subsets of $S^{(2)}$.[7] In the following I shall introduce an extension of the concept of Borel measurability.

Let $m_{z\theta}$ be the Lebesgue measure *on the circle* $c(z,\theta)$ so that $m_{z\theta}(c(z,\theta))=2\pi\sin\theta$ denote $p_{z\theta}=(2\pi\sin\theta)^{-1}m_{z\theta}$, then $p_{z\theta}$ is a probability measure on $c(z,\theta)$.

Definition 1. (a) Let f be a real function on $S^{(2)}$. f is *spherically integrable* if for all $z\in S^{(2)}$ and $0<\theta<\pi$ the restriction of f to $c(z,\theta)$ is $p_{z\theta}$ integrable. In this case I shall denote

$$E(f\mid z,\theta)=\int_{c(z,\theta)} f(w)dp_{z\theta}(w) . \tag{2.1}$$

$E(f\mid z,\theta)$ is *the conditional expectation* of f on $c(z,\theta)$.

(b) A spherically integrable function is said to be *totally spherically integrable* if for all $z\in S^{(2)}$ the integral

$$E(f)=\tfrac{1}{2}\int_0^\pi E(f\mid z,\theta)\sin\theta\, d\theta \tag{2.2}$$

is defined and *its value is independent* of z. In this case $E(f)$ is the *total spherical expectation* of f.

(c) A subset $A\subseteq S^{(2)}$ is *spherically measurable* if its indicator function $\chi_A(w)$ which equals 1 for $w\in A$ and zero for $w\notin A$ is spherically integrable. The set A is *totally spherically measurable* when χ_A is totally spherically integrable. In this case $E(\chi_A)$ is called *the total spherical expectation* of A.

(d) Two spherically integrable functions f,g are *spherically independent* if

$$E(fg\mid z,\theta)=E(f\mid z,\theta)E(g\mid z,\theta) \tag{2.3}$$

for all $z\in S^{(2)}$ and $0<\theta<\pi$.

Every Borel-measurable subset of $S^{(2)}$ is totally spherically measurable. This follows from the fact that the intersection of an open subset of $S^{(2)}$ with the circle $c(z,\theta)$ is open relative to $c(z,\theta)$. Thus every integrable Borel function is totally spherically integrable and its total spherical expectation is identical with its integral with respect to the normalized Lebesgue measure on $S^{(2)}$. To see this, let (r,θ,ϕ) be a set of spherical coordinates and let $z=(1,0,0)$. Then

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} f(\theta,\phi)\sin\theta\, d\phi\, d\theta$$
$$=\tfrac{1}{2}\int_0^\pi\left[\frac{1}{2\pi}\int_0^{2\pi} f(\theta,\phi)d\phi\right]\sin\theta\, d\theta$$
$$=\tfrac{1}{2}\int_0^\pi E(f\mid z,\theta)\sin\theta\, d\theta=E(f) .$$

There are totally spherically measurable sets which are not Borel measurable (not even Lebesgue measurable). We shall see that the family of totally spherically measurable sets is not even a Boolean algebra. Thus there exists two totally spherically measurable sets A,B such that $A\cap B$ is not totally spherically measurable.

Definition 2. A spin-$\frac{1}{2}$ *function* is a function s: $S^{(2)}\to\{-\frac{1}{2},\frac{1}{2}\}$ such that $s(-w)=-s(w)$ for all $w\in S^{(2)}$ and such that s is spherically integrable and

$$E(s\mid z,\theta)=s(z)\cos\theta \tag{2.4}$$

for all $z\in S^{(2)}$ and $0<\theta<\pi$. We have the following.

Theorem 1. There exists a spin-$\frac{1}{2}$ function. The proof is by set-theoretical techniques and is given in the Appendix. The theorem establishes the mathematical consistency of the calculations and model that will follow. We shall see that spin-$\frac{1}{2}$ functions are not measurable in the usual sense but they are, nevertheless, totally spherically integrable and have total spherical expectation zero since

$$\begin{aligned}E(s)&=\tfrac{1}{2}\int_0^\pi E(s\mid z,\theta)\sin\theta\,d\theta\\&=\tfrac{1}{2}s(z)\int_0^\pi\cos\theta\sin\theta\,d\theta\\&=0\,.\end{aligned} \tag{2.5}$$

Let s be a spin-$\frac{1}{2}$ function and denote

$$A_+=\{w\in S^{(2)}\mid s(w)=\tfrac{1}{2}\}$$

and

$$A_-=\{w\in S^{(2)}\mid s(w)=-\tfrac{1}{2}\}\,.$$

Let $s^+=\chi_{A_+}$ and $s^-=\chi_{A_-}$, then $s^+=s+\frac{1}{2}$; and therefore for all $z\in S^{(2)}$ and $0<\theta<\pi$ we have

$$\begin{aligned}E(s^+\mid z,\theta&=E(s+\tfrac{1}{2}\mid z,\theta)\\&=\tfrac{1}{2}+s(z)\cos\theta\\&=\begin{cases}\cos^2(\tfrac{1}{2}\theta) & \text{if } s(z)=\tfrac{1}{2}\,,\\ \sin^2(\tfrac{1}{2}\theta) & \text{if } s(z)=-\tfrac{1}{2}\,.\end{cases}\end{aligned} \tag{2.6}$$

The sets A_+,A_- are totally spherically measurable since for all $z\in S^{(2)}$ we get from formula (2.6)

$$E(s^+)=\tfrac{1}{2}\int_0^\pi E(s^+\mid z,\theta)\sin\theta\,d\theta=\tfrac{1}{2} \tag{2.7}$$

with a similar formula for s^-. Thus the total spherical expectation for "spin up", i.e., $s(w)=\frac{1}{2}$, is $\frac{1}{2}$.

Let s be a spin-$\frac{1}{2}$ function and let O_3 be the group of orthogonal transformations in $E^{(3)}$. For $\alpha\in O_3$ let $s\circ\alpha$ be defined by $s\circ\alpha(w)=s(\alpha(w))$. It is easy to see that for all $z\in S^{(2)}$ and $0<\theta<\pi$

$$E(s\circ\alpha\mid z,\theta)=s(\alpha(z))\cos\theta\,. \tag{2.8}$$

In other words $s\circ\alpha$ is also a spin-$\frac{1}{2}$ function and the conditional expectation in Eq. (2.4) is O_3 invariant. Let s_0 be a fixed spin-$\frac{1}{2}$ function and denote $\mathscr{F}_0=\{s_0\circ\alpha\mid\alpha\in O_3\}$. We have the following.

Theorem 2. $\mathscr{F}_0$ is an infinite set.

Proof: Let O_3^+ be the group of "real" rotations, i.e., orthogonal 3×3 matrices with determinant $+1$. Consider the representation of O_3^+ as a permutation group of the set $\mathscr{F}_0^+=\{s_0\circ\alpha\mid\alpha\in O_3^+\}$ as follows:

$$\beta\to\begin{bmatrix}s_0\circ\alpha\\ s_0\circ(\alpha\beta)\end{bmatrix}\,.$$

The kernel of this representation is a normal (invariant) subgroup of O_3^+ and it cannot be O_3^+ itself [since $s_0(-w)=-s_0(w)$]. But O_3^+ is simple, i.e., it has no nontrivial invariant subgroups. It follows that $\mathscr{F}_0^+$ is infinite since, otherwise, we would have a faithful representation of O_3^+ as a finite permutation group, which is absurd. Q.E.D.

Before I proceed, let me demonstrate, at this stage, some of the "bizarre" features of spin-$\frac{1}{2}$ functions. By similar technique to that in the proof of Theorem 1, one can prove the following (see the Appendix).

Theorem 3. Let $\alpha\in O_3$, $\alpha\neq\pm1$. Then there exists a spin-$\frac{1}{2}$ function s such that for all $z\in S^{(2)}$ and all $0<\theta<\pi$

$$\begin{aligned}E(ss\circ\alpha\mid z,\theta)&=E(s\mid z,\theta)E(s\circ\alpha\mid z,\theta)\\&=s(z)s(\alpha(z))\cos^2\theta\,.\end{aligned} \tag{2.9}$$

Each one of the functions $s,s\circ\alpha$ in the above theorem is totally spherically integrable but their product $ss\circ\alpha$ *is not,* since from (2.9)

$$\tfrac{1}{2}\int_0^\pi E(ss\circ\alpha\mid z,\theta)\sin\theta\,d\theta=\tfrac{1}{3}s(z)s(\alpha(z)) \tag{2.10}$$

and therefore the integral on the left-hand side of (2.10) equals $+\frac{1}{12}$ when $s(z)=s(\alpha(z))$ and $-\frac{1}{12}$ when $s(z)\neq s(\alpha(z))$. In the same way, one can show that the intersection

$$\{w\in S^{(2)}\mid s(w)=\tfrac{1}{2}\}\cap\{w\in S^{(2)}\mid s(\alpha(w))=\tfrac{1}{2}\}$$

is not totally spherically measurable even though each one of the intersecting sets is. It follows that

$$\{w\in S^{(2)}\mid s(w)=\tfrac{1}{2}\}$$

is nonmeasurable in the usual sense. The phenomena indicated in theorem 3 is the motivation for the following:

Definition 3. Let $\alpha_1,\alpha_2,\ldots,\alpha_n,\ldots$ be a sequence of orthogonal transformations and let s be a spin-$\frac{1}{2}$ function. Denote $s_i=s\circ\alpha_i$. The sequence $\alpha_1,\alpha_2,\ldots,\alpha_n,\ldots$ is s *random* (and the sequence $s_1,s_2,\ldots,s_n,\ldots$ simply *random*) if, for all $z\in S^{(2)}$ (except maybe finitely many) and all $0<\theta<\pi$,

$$E(s_{i_1}s_{i_2}\cdots s_{i_k}\mid z,\theta)=\prod_{j=1}^{k}E(s_{i_j}\mid z,\theta)=(\cos\theta)^k\prod_{j=1}^{k}s_{i_j}(z) \qquad (2.11)$$

for all $k\geq 2$ and indices $1\leq i_1<i_2<\cdots<i_k<\infty$.[8]

Generalizing the proof of Theorem 3 one can show that random sequences in the above sense exist (see the Appendix).

In standard probability theory a "law of large numbers" is usually invoked in order to establish the relation between finite means of random variables and their mutual expectation values. Such a law usually states that the means of a sequence of random variables converges to their expectation value "almost everywhere" (that is, except on a set of measure zero). In the present case the situation is slightly more complex and a version of the law of large numbers is given below:

Theorem 4. Let $s_1,s_2,\cdots,s_n$ be a random sequence and let $z_0\in S^{(2)}$ be fixed. Let $s'_1,s'_2,\ldots,s'_k$ be the subsequence of the original sequence of all these functions which satisfy $s(z_0)=+\frac{1}{2}$. Denote

$$B=\left\{w\in S^{(2)}\,\middle|\,\frac{1}{k}\sum_{j=1}^{k}s'_j(w)\to\frac{w\cdot z_0}{2}\right\}. \qquad (2.12)$$

Then B is nonmeasurable in terms of the Lebesgue measure on the sphere but nevertheless for all $0<\theta<\pi$

$$E(\chi_B\mid z_0,\theta)=1\ . \qquad (2.13)$$

Proof: I shall prove first that (2.13) obtains. Let θ be fixed. We have $s'_j(z_0)=\frac{1}{2}$ and thus $E(s'_j\mid z_0\theta)=\frac{1}{2}\cos\theta$ for $j=1,2,\ldots$. Hence all the s'_j have the same expectation on $c(z_0,\theta)$. Since spin-$\frac{1}{2}$ functions are bivalued, this also means that the s'_j are equally distributed on $c(z_0,\theta)$. We have assumed that the original sequence is independent on each circle $c(z,\theta)$ [Eq. (2.11)]. Thus the subsequence $\{s'_j\}$ is also independent on each circle and in particular on $c(z_0,\theta)$. It follows from the (strong) law of large numbers[7] that $(1/k)\sum s'_j(w)\to\frac{1}{2}\cos\theta$ for almost all $w\in c(z_0,\theta)$ (where "almost all" is measured in terms of $p_{z_0\theta}$). Thus we have proved (2.13).

In the rest of the proof the term "measurable" refers to the normalized Lebesgue measure on $S^{(2)}$. Assume by negation that B is measurable. Then its measure is 1 since by integrating (2.13) we get

$$E(\chi_B)=\frac{1}{2}\int_0^{\pi}E(\chi_B\mid z_0,\theta)\sin\theta\,d\theta=1\ ,$$

and the measure of a subset of $S^{(2)}$, when defined, is independent of the choice of coordinates. Let $y_0\in B$, $y_0\neq\pm z_0$, be fixed. We can split the sequence $\{s'_j\}$ into two subsequences $\{s'_j\}=\{s^2_j\}\cup\{s^3_j\}$, such that $s^2_j(y_0)=+\frac{1}{2}$ and $s^3_j(y_0)=-\frac{1}{2}$, $j=1,2,\ldots$. Denote

$$C=\left\{w\in S^{(2)}\,\middle|\,\frac{1}{l}\sum_{j=1}^{l}s^2_j(w)\underset{l}{\to}\frac{w\cdot y_0}{2}\right\},$$

$$D=\left\{w\in S^{(2)}\,\middle|\,\frac{1}{m}\sum_{j=1}^{m}s^3_j(w)\underset{m}{\to}-\frac{w\cdot y_0}{2}\right\}.$$

By the law of large numbers we get again

$$E(\chi_C\mid y_0,\theta)=E(\chi_D\mid y_0,\theta)=1$$

and hence also $E(\chi_C\chi_D\mid y_0,\theta)=1$ for all $0<\theta<\pi$. From the definition of s^2_j,s^3_j,

$$\frac{s'_1(w)+\cdots+s'_k(w)}{k}=\left[\frac{l}{k}\right]\left[\frac{s^2_1(w)+\cdots+s^2_l(w)}{l}\right]+\left[\frac{k-l}{k}\right]\left[\frac{s^3_1(w)+\cdots+s^3_{k-l}(w)}{k-l}\right]. \qquad (2.14)$$

We have assumed that $y_0\in B$ thus for large k the ratio (l/k) is approximately the relative frequency of "spin up" in the y_0 direction in the sequence $\{s'_j\}$. In other words

$$\frac{l}{k}\sim\left[\frac{1}{2}+\frac{y_0\cdot z_0}{2}\right].$$

In the same way

$$\frac{k-l}{k}\sim\left[\frac{1}{2}-\frac{y_0\cdot z_0}{2}\right].$$

Assume that $w\in B\cap C\cap D$, then taking the limit $k\to\infty$ in (2.14) we get

$$\frac{w\cdot z_0}{2}=\left[\frac{1}{2}+\frac{y_0\cdot z_0}{2}\right]\left[\frac{w\cdot y_0}{2}\right]+\left[\frac{1}{2}-\frac{y_0\cdot z_0}{2}\right]\left[-\frac{w\cdot y_0}{2}\right].$$

Thus $w\in B\cap C\cap D$ only if $w\cdot z_0=(w\cdot y_0)(y_0\cdot z_0)$. It follows that $B\cap C\cap D$ is measurable and has measure zero. Now put $C\cap D=(B\cap C\cap D)\cup(\bar{B}\cap C\cap D)$ where $\bar{B}=S^{(2)}/B$. We have proved that $B\cap C\cap D$ is measurable and has measure zero. Also, by assumption B is measurable and has measure 1. Therefore, $\bar{B}$ is measurable and has measure zero and thus also $\bar{B}\cap C\cap D$ is measurable and has measure zero. We conclude that $C\cap D$ is measurable and has measure zero, but this is a contradiction since $E(\chi_C\chi_D\,|\,y_0,\theta)=1$ for all $0<\theta<\pi$. Therefore, B is nonmeasurable. Q.E.D.

The set B of directions w for which the sequence $(1/k)\sum_{i=1}^{k} s_i'(w)$ converges to the quantum-mechanical value $z_0\cdot w/2$ is thus "big." This means that if we measure the size of B in a spherical coordinate system with z as axis it appears as if the (normalized) measure of B is 1 or, in other words B is "almost the whole sphere" since $p_{z_0\theta}[B\cap c(z_0,\theta)]=1$ *for all* $0<\theta<\pi$. On the other hand, if we measure the size of B in a different coordinate system B appears "small" as is evident from the proof of Theorem 4. This peculiar "paradoxical" property of B is the analog of the Banach-Tarski paradox in the present context.

What I suggest is to utilize this paradoxical property of B and maintain that the averages do converge on their quantum-mechanical expected value almost everywhere (since B is, in some measure-theoretic source, "almost the whole sphere"). This move does not involve the logical contradiction derived from Bell's inequality precisely because B is "small" in another sense. The same observation applies to random sequences $s_1,s_2,\ldots,s_n,\ldots$ which are not necessarily polarized in a particular direction as follows from the following:

Theorem 5. Let $s_1,s_2,\ldots,s_n,\ldots$ *be a random sequence. Denote*

$$A=\left\{w\in S^{(2)}\,\middle|\,\frac{1}{n}\sum_{i=1}^{n}s_i(w)\rightarrow 0\right\}.$$

Then for all $z\in S^{(2)}$: $E(\chi_A\,|\,z,\pi/2)=1$. *Moreover if* $z\in A$ *then for all* $0<\theta<\pi$: $E(\chi_A\,|\,z,\theta)=1$.

Proof: For a given $z\in S^{(2)}$ divide the sequence $\{s_i\}$ into two subsequences $\{s_i\}=\{s_i'\}\cup\{s_i''\}$ where $s_i'(z)=+\frac{1}{2}s_i''(z)=-\frac{1}{2}$. From Theorem 4 it follows that

$$k^{-1}\sum_{i=1}^{k}s_i'(w)\rightarrow\frac{w\cdot z}{2}$$

almost everywhere on $c(z,\theta)$ for all $0<\theta<\pi$. By the same argument

$$l^{-1}\sum_{i=1}^{l}s_i''(w)\rightarrow-\frac{w\cdot z}{2}$$

almost everywhere on $c(z,\theta)$. Now.

$$\frac{1}{n}\sum_{i=1}^{n}s_i(w)=\left[\frac{k}{n}\right]\left[\frac{1}{k}\sum_{i=1}^{k}s_i'(w)-\frac{w\cdot z}{2}\right]+\left[\frac{n-k}{n}\right]\left[\frac{1}{n-k}\sum_{i=1}^{n-k}s_i''(w)+\frac{w\cdot z}{2}\right]+\left[\frac{2k-n}{n}\right]\frac{w\cdot z}{2}\,. \tag{2.15}$$

For $\theta=\pi/2$ we have $w\cdot z=0$ and thus the right-hand side of (2.15) converges to 0 almost everywhere on $c(z,\pi/2)$ as n increases. If $z\in A$ we have

$$\frac{1}{n}\sum_{i=1}^{n}s_i(z)\rightarrow 0\,,$$

thus for large n, $k/n\sim\frac{1}{2}$ and $(n-k)/n\sim\frac{1}{2}$ and again the right-hand side converges to zero almost everywhere on $c(z,\theta)$ for all $0<\theta<\pi$. Q.E.D.

Theorems 4 and 5 combined give us the answer to the following question: "What is the relative frequency of spin up in the x and y direction in a random sequence of spin function?" The answer is "with probability 1" it is the relative frequency of spin up in the x direction (which is $\frac{1}{2}$, see Theorem 5) times the relative frequency of spin up in the y direction among those functions in the sample which have spin up in the x direction (which is $\frac{1}{2}+x\cdot y/2$, Theorem 4). What is peculiar here is the term "with probability 1" (or "almost everywhere") which utilizes the rather paradoxical properties of the sets A and B. As it turns out two events may have probability 1 in my sense but their conjunction only probability zero (see proof of Theorem 4) which is indeed bizarre. The model that follows is a solution to the quantum-statistical paradoxes only as far as one is willing to accept the premise that "strange" physical phenomena [such as the Einstein-Podolsky-Rosen (EPR) paradox] calls for some nonstandard explanation. One can, as it were, reduce a physical paradox to a mathematical "pathology."

III. FERMI-DIRAC STATISTICS

I shall use the electron as an example of a spin-$\frac{1}{2}$ particle, but the following may apply to other spin-$\frac{1}{2}$ fermions as well. The model is based on the following assumptions:

(a) Each electron at any given moment has a definite spin in all directions and its spin values are given by a spin-$\frac{1}{2}$ function.

(b) All electron spin-$\frac{1}{2}$ functions belong to a family of the form $\mathscr{F}_0=\{s_0\circ\alpha\,|\,\alpha\in O_3\}$ for some fixed (yet unknown) spin-$\frac{1}{2}$ function s_0.

(c) If s_1,s_2 are spin-$\frac{1}{2}$ functions of two uncorrelated electrons then s_1,s_2 are taken as spherically independent, i.e., they satisfy (2.9).

Condition (b) can, in fact, be dropped to obtain a somewhat weaker model. In the following I shall assume, however, that all three conditions are satisfied. As a result of a (low-energy) interaction the electron spin-$\frac{1}{2}$ function s is transformed to another spin-$\frac{1}{2}$ function s', but the transformation is always of the form $s'=s\circ\alpha$ for $\alpha\in O_3$. I assume that the transformation α depends on the dynamic variables other than the spin and on the type of interaction. This assumption turns the model into a deterministic one.

The relation between the proposed model and the usual Hilbert-space formulation of quantum mechanics is as follows: Let z be a fixed unit vector in physical space and (r,θ,ϕ) a system of spherical coordinates such that $z=(1,0,0)$. Let $|+\rangle=\binom{1}{0}$ and $|-\rangle=\binom{0}{1}$ be the vectors in the two-dimensional complex Hilbert space H_2 which correspond to the states "spin up" and "spin down" in the z direction, respectively. Let $x=(1,\theta,\phi)$ be an arbitrary unit vector in physical space. The orthogonal matrix

$$\alpha=\begin{bmatrix}\cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi\\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi\\ -\sin\theta & 0 & \cos\theta\end{bmatrix} \tag{3.1}$$

transforms z to x. This transformation is represented on H_2 by the unitary matrix

$$D_{1/2}(\alpha)=\begin{bmatrix}\cos(\frac{1}{2}\theta)e^{-i\phi/2} & -\sin(\frac{1}{2}\theta)e^{-i\phi/2}\\ \sin(\frac{1}{2}\theta)e^{i\phi/2} & \cos(\frac{1}{2}\theta)e^{i\phi/2}\end{bmatrix}. \tag{3.2}$$

Thus the first and second columns of $D_{1/2}(\alpha)$ correspond with the states spin up and spin down in the x direction, respectively. Assume that s is the electron spin-$\frac{1}{2}$ function. For all $x\in S^{(2)}$ define the vector $|s(x)\rangle\in H_2$ as follows:

$$|s(x)\rangle=\begin{bmatrix}\cos(\frac{1}{2}\theta)e^{-i\phi/2}\\ \sin(\frac{1}{2}\theta)e^{i\phi/2}\end{bmatrix}\quad \text{if } s(x)=\tfrac{1}{2}$$

and (3.3)

$$|s(x)\rangle=\begin{bmatrix}-\sin(\frac{1}{2}\theta)e^{-i\phi/2}\\ \cos(\frac{1}{2}\theta)e^{i\phi/2}\end{bmatrix}\quad \text{if } s(x)=-\tfrac{1}{2}\,.$$

The complete description of the electron spin state is given by the collection of vectors $\{|s(x)\rangle \mid x\in S^{(2)}\}$. Note that every unit vector of H_2 has a representation as the first or second column of $D_{1/2}(\alpha)$. This is not true in the three-dimensional spin-1 case, a fact which causes some complications (see Sec. IV below). From (3.3) we get that $|\langle s(x)|s(z)\rangle|^2=\cos^2(\frac{1}{2}\theta)$ when $s(x)=s(z)$ and $|\langle s(x)|s(z)\rangle|^2=\sin^2(\frac{1}{2}\theta)$ for $s(x)\neq s(z)$. Hence

$$|\langle s(x)|s(z)\rangle|^2=p_{z\theta}[\{w\in c(z,\theta)\mid s(w)=s(x)\}]=\tfrac{1}{2}+2s(x)s(z)\cos\theta \tag{3.4}$$

for $\theta=\arccos(x\cdot z)$. In other words, $|\langle s(x)|s(z)\rangle|^2$ *is the conditional expectation of "spin equals* $s(x)$*" on the circle* $c(z,\theta)$. Formula (3.4) establishes the interpretation of the quantum-mechanical expectation values in terms of the conditional expectations given by the model.

With the use of principle (c) of the model one can account for the quantum-mechanical addition rules as well. Let s_1,s_2 be the spin-$\frac{1}{2}$ functions of two noninteracting electrons. The quantum-mechanical description of the two electron systems is given by the tensor product $H_2\otimes H_2$. Let $|s_1(x)\rangle$, $|s_2(x)\rangle$ be defined as above. Then the four-dimensional vector

$$|s_1(x)s_2(x)\rangle=|s_1(x)\rangle\,|s_2(x)\rangle$$

refers to the state "spin $s_1(x)$ for electron 1 *and* spin $s_2(x)$ for electron 2 in the x direction." Since s_1s_2 are spherically independent we get

$$\begin{aligned}|\langle s_1(x)s_2(x)|s_1(z)s_2(z)\rangle|^2&=p_{z\theta}[\{w\in c(z,\theta)\mid s_1(w)=s_1(x)\}]p_{z\theta}[\{w\in c(z,\theta)\mid s_2(w)=s_2(x)\}]\\ &=p_{z\theta}[\{w\in c(z,\theta)\mid s_1(w)=s_1(x)\}\cap\{w\in c(z,\theta)\mid s_2(w)=s_2(x)\}]\,,\end{aligned} \tag{3.5}$$

where $\theta=\arccos(x\cdot z)$. For example, if $s_1(z)=s_2(z)=\frac{1}{2}$ and $s_1(x)=-s_2(x)=\frac{1}{2}$ we get from (2.6) and (3.5)

$$|\langle s_1(x)s_2(x)|s_1(z)s_2(z)\rangle|^2=\cos^2(\tfrac{1}{2}\theta)\sin^2(\tfrac{1}{2}\theta)$$

in agreement with quantum mechanics. Thus *taking the tensor product* $H_2\otimes H_2$ *is equivalent to the assumption that* s_1,s_2 *are spherically independent.*

When the electrons are correlated we cannot any longer assume spherical independence. In this case, the electron spin-$\frac{1}{2}$ functions are *spherically correlated* in a way given by the triplet and singlet states. This case will be described in a separate article. (A hint of the triplet-state correlations may be found in the discussion of spin-1 particles below.)

Theorems 4 and 5 above are the keys to the explanation of the statistical behavior of large samples of identical spin-$\frac{1}{2}$ fermions. Suppose that we are given a random sample of electrons. Let $s_1, s_2, \ldots, s_n$ be the sequence of its spin-$\frac{1}{2}$ functions. The randomness of the sample is accounted in the present model in terms of the randomness of the sequence $s_1, s_2, \ldots, s_n$ (Definition 3). Let z be a fixed direction and $s'_1, \ldots, s'_k$ the subsequence of all those functions in the sample which have spin up in the z direction. For large n we have with probability 1: $k \sim n/2$ (Theorem 5). Let x be an arbitrary direction. Then for large n we have with probability 1:

$$k^{-1}\sum_{i=1}^{k} s'_i(x) \sim \frac{x\cdot z}{2} ,$$

where the probability is measured in terms of $p_{z\theta}$ for $\theta = \arccos(x\cdot z)$ (Theorem 4). Thus, the answer to the question: "In a random sample of electrons what is the relative frequency of spin up in the z direction and spin up in the x direction" is "with probability 1 it is $\frac{1}{2}\cos^2(\frac{1}{2}\theta)$", where probability is measured as before. This explains the frequencies observed in various polarization experiments as well as EPR types of arrangements.[2]

In the same way I can also explain the properties of macroscopic magnetic fields. I have assumed that an electron has a definite spin in all directions, therefore it also has a definite magnetic moment in each direction x, given by $\mu_0 s(x)$ where μ_0 is the Bohr magneton and s the electron spin-$\frac{1}{2}$ function. Thus microscopic magnetic moments are not vectors. Macroscopic magnetic moments, on the other hand, behave like vectors. This could be explained as follows: Assume that we have a sample of electrons all polarized "up" in the z direction and all confined to a relatively small portion of space. Let $s_1, s_2, \ldots, s_n$ be the electrons' spin-$\frac{1}{2}$ function then $s_j(z_0) = +\frac{1}{2}$. The magnetic moment in the z direction due to the electrons in the sample is approximately

$$\mu_z \sim \sum_{j=1}^{n} \mu_0 s_j(z) = \frac{\mu_0 n}{2} .$$

Let $x \neq z$, then

$$\mu_x \sim \sum \mu_0 s_j(x) = \frac{\mu_0 n}{2}(x\cdot z) + \mu_0 r(n) ,$$

where

$$r(n) = \sum s_j(x) - \frac{n}{2} x\cdot z .$$

As n increases, with probability 1, the component $\mu_0 r(n)$ represents a decreasing percentage of the effect [since $(1/n)\sum s_i(w) \to (w\cdot z)/2$ for almost all $w \in c(z,\theta)$, where $\theta = \arccos(x\cdot z)$]. Hence for large n

$$\mu_x \sim \frac{\mu_0 n}{2}(x\cdot z) = \mu_z(x\cdot z) .$$

Therefore, *the vectorlike behavior of macroscopic magnetic moments is a statistical phenomenon* in my model. One may speculate at this stage what would have happened if the distribution of spins on the sphere had not been given by a spin-$\frac{1}{2}$ function but rather by another type of function. In that case, macroscopic magnetic fields would not have behaved like vectors. This suggests that the macroscopic space-time structure that we observe results, in fact, from a statistical distribution of hidden variables of various kinds.

IV. BOSE-EINSTEIN STATISTICS

I shall construct a model for Bose-Einstein statistics along the same lines as in the Fermi-Dirac case.

Definition 4. A spin-1 function is a function j: $S^{(2)} \to \{-1,0,1\}$ such that $j(-w) = -j(w)$ for $w \in S^{(2)}$ and such that

$$E(j \mid z,\theta) = j(z)\cos\theta , \tag{4.1}$$

$$E(j^2 \mid z,\theta) = \sin^2\theta + j^2(z)\tfrac{1}{2}(3\cos^2\theta - 1) . \tag{4.2}$$

Again using the same set-theoretical techniques, one can prove that spin-1 functions exist and thus the model is mathematically consistent (see the Appendix). From (4.1) and (4.2) it is obvious that both j and j^2 are totally spherically integrable and their total spherical expectation is

$$E(j) = 0 , \quad E(j^2) = \tfrac{2}{3} . \tag{4.3}$$

Note that the conditional expectation of spin $\neq 0$ on $c(z,\theta)$ is $\sin^2\theta$, when $j(z) = 0$, but for $j(z) \neq 0$ one has to add to this value a "dipole moment effect," $\frac{1}{2}(3\cos^2\theta - 1)$. Note also that the conditions (4.1) and (4.2) are O_3 invariant, i.e., if j is a spin-1 function, so is $j \circ \alpha$ for $\alpha \in O_3$.

Let $A_+ = \{w \in S^{(2)} \mid j(w) = 1\}$ and similarly define A_0 and A_-. Let j^+, j^0, j^- be the indicator functions of $A_+ A_0 A_-$, respectively, then $j^2 = j^+ + j^-$, $j = j^+ - j^-$, and $j^0 = 1 - j^2$. Thus we can calculate the expectation values for spin up, spin zero, and spin down on $c(z,\theta)$ from (4.1) and (4.2). These values are given in the following formulation:

	$E(j^+\mid z,\theta)$	$E(j^0\mid z,\theta)$	$E(j^-\mid z,\theta)$
$j(z)=1$	$\cos^4(\frac{1}{2}\theta)$	$\frac{1}{2}\sin^2\theta$	$\sin^4(\frac{1}{2}\theta)$
$j(z)=0$	$\frac{1}{2}\sin^2\theta$	$\cos^2\theta$	$\frac{1}{2}\sin^2\theta$
$j(z)=-1$	$\sin^4(\frac{1}{2}\theta)$	$\frac{1}{2}\sin^2\theta$	$\cos^4(\frac{1}{2}\theta)$.

(4.4)

We are now in the position to introduce the physical assumptions. As before I shall assume that each (massive) spin-1 particle has definite spin value in all directions and its spin values are given by a spin-1 function. I shall also assume that the spin-1 functions of a particular species of spin-1 particles all belong to a family of the form $\{j_0 \circ \alpha \mid \alpha \in 0_3\}$ for some fixed j_0. Finally I assume that the spin-1 functions of two uncorrelated identical particles are spherically independent.

Let H_3 be the three-dimensional complex Hilbert space and let

$$\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \begin{bmatrix}0\\1\\0\end{bmatrix},\quad \begin{bmatrix}0\\0\\1\end{bmatrix}$$

be the H_3 vectors representing the states spin up, zero, and down in the z direction, respectively. The orthogonal transformation α in (3.1) is represented on H_3 by the unitary matrix

$$D_1(\alpha)=\begin{bmatrix}\cos^2(\frac{1}{2}\theta)e^{-i\phi} & -\frac{1}{\sqrt{2}}\sin\theta\, e^{-i\phi} & \sin^2(\frac{1}{2}\theta)e^{-i\phi}\\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta\\ \sin^2(\frac{1}{2}\theta)e^{i\phi} & \frac{1}{\sqrt{2}}\sin\theta\, e^{i\phi} & \cos^2(\frac{1}{2}\theta)e^{i\phi}\end{bmatrix} . \tag{4.5}$$

Let $x=(1,\theta,\phi)$ be an arbitrary unit vector in the physical space and let j be a spin-1 function. Define $|j(x)\rangle \in H_3$ to be the first, second, or third column of $D_1(\alpha)$ in case $j(x)=1,0,-1$, respectively. Then we get again

$$|\langle j(x)\,|\,j(z)\rangle|^2=p_{z\theta}[\{w\in c(z,\theta)\mid j(w)=j(x)\}] \tag{4.6}$$

for $\theta=\arccos(x\cdot z)$. This could easily be verified from (4.4). The rest of the spin-1 case follow the same lines as the spin-$\frac{1}{2}$ case. There is, however, one complication in this case that does not arise in the spin-$\frac{1}{2}$ case. Not all the vectors in H_3 can be represented as one of the columns of the matrix (4.5). From the principle of superposition we know, on the other hand, that every unit vector in H_3 corresponds to a possible physical state. Thus quantum mechanics is much richer in states than my model. This observation is related to a result due to Kochen and Specker,[9] who proved that the no bivalued homomorphism exists on the lattice of subspaces of the three-dimensional Hilbert space. Thus the set of *all* H_3 states could not be recovered by a hidden-variable theory, not even one such as I have suggested. The way to circumvent this difficulty is to observe that the principle of superposition makes sense only in the *absence* of a detailed hidden-variable theory. I suggest, in other words, that we introduce the following:

Superselection Rule: The states realizable by a single (massive) spin-1 particle are all of the form $|j(x)\rangle$ for $x\in S^{(2)}$ and some spin-1 function j.[10]

What about the rest of the unit vectors of H_3? They represent some *statistical information* about a particle that *belongs to a collection* which has some distinguished features. For example, the vector

$$\frac{1}{\sqrt{2}}e^{i\pi/4}|+\rangle+\tfrac{1}{2}|0\rangle+\tfrac{1}{2}e^{i\pi/2}|-\rangle$$

represents the following statistical information about a spin-1 particle:

(1) It belongs to a collection in which the relative frequencies of spin up, zero, and down in the z direction are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, respectively.

(2) The collection is polarized in such a way that it has no particles with spin-0 in the $x=(1,\theta,\phi)$ direction for $\phi=0$ and $\theta=\arctan(\sqrt{2})$.

It can easily be proved that every vector in H_3 could be interpreted in a similar way. Note that the particle has a *definite* spin in all directions and the information given by the unit vector merely represents our ignorance of these values.

V. PRINCIPLES FOR TESTING THE MODEL

A random sample of electrons is described by a random sequence of spin-$\frac{1}{2}$ functions. The model, on the other hand, is assumed to be deterministic, that is, one can calculate, in principle, the exact transformation $s \to s \circ \alpha$ which the electron spin-$\frac{1}{2}$ function undergoes during an interaction. Using such a calculation one can design a sample of electrons in which randomness has been destroyed. As a result the sample will manifest statistical behavior which deviates from the predictions of quantum mechanics.

Suppose that we pass an electron through a Stern-Gerlach apparatus oriented in the z direction. Let s be the electron spin-$\frac{1}{2}$ function before it enters the apparatus. After it leaves the apparatus the electron Fermi function is $s \circ \alpha$ for $\alpha \in O_3$. We know that in *every* such experiment the electron spin in the z direction does not change. Thus we can assume that in this case α leaves z invariant $\alpha(z)=z$. The orthogonal transformation α depends on the dynamic variables of the electron other than the spin itself and on the properties of the magnetic field of the apparatus. I do not have a theory of the nature of this dependence but in the following I shall assume that such a theory is available.

Consider the following thought experiment: Take an electron and pass it through a Stern-Gerlach apparatus in the z direction. Suppose it goes "up". Now further polarize it in the x direction for x *orthogonal* to z. Suppose it goes "up" again. Finally polarize it a third time in the z direction. Let s be the original spin-$\frac{1}{2}$ function of the electron. It undergoes three orthogonal transformations:

$$s \to s \circ \alpha_1 \to s \circ (\alpha_2 \alpha_1) \to s \circ (\alpha_3 \alpha_2 \alpha_1) ,$$

where $\alpha_1(z)=\alpha_3(z)=z$ and $\alpha_2(x)=x$. If we can control the dynamic variables and magnetic fields so that α_2 is a 180° rotation about the x axis then *the electron will definitely go "down" through the third apparatus.* This occurs since

$$\alpha_3\alpha_2\alpha_1(z)=\alpha_3\alpha_2(z)=\alpha_3(-z)=-z$$

and hence

$$s(\alpha_3\alpha_2\alpha_1 z)=s(-z)=-s(z) .$$

Quantum mechanics as well as my model predicts that in a *random* sample of electrons about $\frac{1}{8}$ of the electrons will go up through all three apparatus. Thus we have a thought experiment which exhibits about 12% deviation from the predictions of quantum mechanics. The thought experiment is not intended, of course, as an instruction for the experimentalist. It indicates, I believe, that any theory of the nature of the transformations $s \to s \circ \alpha$ could be tested in principle. I believe that such a theory could be developed in a relativistic framework. I have already mentioned[2] the possibility that the orthogonal transformation α may have something to do with the "Wigner rotation", which is essentially a rotation of the spin-$\frac{1}{2}$ function due to a Lorentz transformation from one reference frame to another.

Note that even if the present model fails such a test, one can still weaken it by dropping condition (b) of the model, that is, by rejecting the idea that two electron spin-$\frac{1}{2}$ functions are related by an orthogonal transformation. This weakened version of the model is not deterministic. It is, at best, a "hidden-variable" interpretation of the expectation values of quantum mechanics.

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APPENDIX

Proof of Theorem 1: I shall use two "heavy" axioms from set theory, the axiom of choice[1] and the continuum hypothesis.[11] For $z \in S^{(2)}$ and $0 < \theta \le \pi/2$ denote

$$\mathscr{R}(z,\theta)=c(z,\theta) \cup c(-z,\theta) \cup \{-z,z\} .$$

$\mathscr{R}(z,\theta)$ is the set composed from two poles $-z,z$ and two circles $c(z,\theta)$ and $c(-z,\theta)$ which collapse onto one circle in the case $\theta=\pi/2$.

Let K be the family of all sets of the form $\mathscr{R}(z,\theta)$, i.e.,

$$K=\{\mathscr{R}(z,\theta) \mid z \in S^{(2)}, 0<\theta \le \pi/2\} .$$

By the axiom of choice (or rather its equivalent the principle of well ordering) there is a well ordering of the family K. Moreover since K has the power of the continuum it follows from the continuum hypothesis that there is such well ordering in which every element $\mathscr{R}(z,\theta)$ has only countably many predecessors. I shall define a spin-$\frac{1}{2}$ function s on $S^{(2)}$ by induction on such an order.

(1) Let $\mathscr{R}(z_1,\theta_1)$ be the first element in the order. Put $s(z_1)=-s(-z_1)=\frac{1}{2}$. Divide the circle $c(z_1,\theta_1)$ into two arbitrary disjoint subsets whose measure, on the circle $c(z_1,\theta_1)$ is $\cos^2(\theta_1/2)$ and $\sin^2(\theta_1/2)$ and define s to be $+\frac{1}{2}$ on the larger set [the one with measure $\cos^2(\theta_1/2)$] and $-\frac{1}{2}$ on the smaller set.

For $w \in c(-z_1,\theta_1)$ put $s(w)=-s(-w)$. [In the case $\theta=\pi/2$ one should divide the circle $c(z_1,\pi/2)$ into two disjoint subsets of measure $\frac{1}{2}$ such that the first set is the reflection of the second.]

(2) Suppose we have defined s on all the elements of K up to but not including a certain element $\mathscr{R}(z_\lambda,\theta_\lambda)$ in the order. If $\mathscr{R}(\theta_\lambda,z_\lambda)$ is identical with

some previous element in the order then s has already been defined, otherwise there are two possibilities: (a) z_λ (and thus $-z_\lambda$) do not belong to any preceding element in the order. In that case define $s(z_\lambda)=+\frac{1}{2}=-s(-z_\lambda)$. Since $\mathscr{R}(z_\lambda,\theta_\lambda)$ is preceded by at most countably many elements and since the intersection of two nonidentical circles contains at most two points it follows that s has already been defined on at most countably many points of $c(z_\lambda,\theta_\lambda)$ hence only on a subset of $c(z_\lambda,\theta_\lambda)$ of measure zero. Thus we can divide the rest of $c(z_\lambda,\theta_\lambda)$ into two subsets whose measures are $\cos^2(\theta_\lambda/2)$ and $\sin^2(\theta_\lambda/2)$, respectively, and define s to be $+\frac{1}{2}$ on the larger set, $-\frac{1}{2}$ on the smaller set, and for $w\in c(-z_\lambda,\theta_\lambda)$ put $s(w)=-s(-w)$. (b) z_λ (and thus $-z_\lambda$) belong to some preceding element in the order. In that case $s(z_\lambda)$ and $s(-z_\lambda)$ has already been defined. If $s(z_\lambda)=+\frac{1}{2}$ we follow the procedure of case (a). If $s(z_\lambda)=-\frac{1}{2}$ we follow the same procedure reversing the roles of z_λ and $-z_\lambda$.

The function s which results from this construction satisfies

$$E(s\mid z,\theta)=s(z)\left[\cos^2\left[\frac{\theta}{2}\right]-\sin^2\left[\frac{\theta}{2}\right]\right]$$

$$=s(z)\cos\theta$$

for all $z\in S^{(2)}$ and $0<\theta<\pi$. Q.E.D.

A proof of Theorem 3 and the existence of random sequences. I shall sketch a proof for the existence of random sequences, the proof of Theorem 3 is much simpler and follows the same principles. Let $x_0\in S^{(2)}$ be fixed. Let α be a rotation of space around the x_0 axis by an angle ϕ such that $\phi\pi^{-1}$ is an irrational number. We have $\alpha^n\neq 1$ for $n=\pm1,\pm2,\ldots$ and $\alpha(x_0)=x_0$. I shall prove that there is a spin function s such that the sequence $s_n=s\circ\alpha^n$ is random. I shall use the notations of the proof of Theorem 1 above. For $z\in S^{(2)}$ and $0<\theta\leq\pi/2$ put

$$L(z,\theta)=\bigcup_{n=-\infty}^{\infty}\mathscr{R}(\alpha^n(z),\theta)\ .$$

Let $z\neq\pm x_0$ and consider the set $L(z,\theta)$, ignoring for a while the rest of the sphere. I shall show that it is possible to define a spin-$\frac{1}{2}$ function s *on* $L(z,\theta)$ such that the sequence $s_n=s\circ\alpha^n$ satisfies

$$E(s_{i_1}s_{i_2}\cdots s_{i_k}\mid z,\theta)=\prod_{j=1}^{k}E(s_{i_j}\mid z,\theta)$$

for all indices $0\leq i_1<i_2<\cdots<i_k<\infty$. To establish that well order the countable family

$$\{\mathscr{R}(\alpha^n(z),\theta)\mid n=0,\pm1,\pm2,\ldots\}$$

and define s by usual induction on the order. The procedure follows the same principle as the induction in Theorem 1 with one extra constraint. If s has already been defined on the first $k-1$ sets in the order

$$\mathscr{R}(\alpha^{n_1}(z),\theta)\ ,\mathscr{R}(\alpha^{n_2}(z),\theta),\ldots,\mathscr{R}(\alpha^{n_{k-1}}(z),\theta)$$

then on the kth element one should make sure that the equation

$$E(s_{n_{i_1}}s_{n_{i_2}}\cdots s_{n_{i_j}}s_{n_k}\mid z,\theta)=E(s_{n_{i_1}}\mid z,\theta)E(s_{n_{i_2}}\mid z,\theta)\cdots E(s_{n_{i_j}}\mid z,\theta)E(s_{n_k}\mid z,\theta)$$

will be satisfied for all choices of indices $1\leq i_1<i_2<\cdots<i_j<k$. This definition is possible since $z\neq x_0$ and thus up to the kth stage the function s has already been defined only for finitely many points of $c(\alpha^{n_k}(z),\theta)$. It follows that the division of $c(\alpha^{n_k}(z),\theta)$ into $+\frac{1}{2}$ and $-\frac{1}{2}$ subsets is arbitrary from a measure-theoretic standpoint.

To construct a function s on a sphere as a whole note that if z,z' are two directions, we have $L(z,\theta)=L(z',\theta)$ only if $z'=\pm\alpha^n z$ for some $n=0,\pm1,\pm2,\ldots$. In all other cases the intersection $L(z,\theta)\cap L(z',\theta)$ contains only countably many points. Thus we can well order the family

$$\{L(z,\theta)\mid z\in S^{(2)}\ ,0<\theta\leq\pi/2\}$$

and define s by induction on the order. At each step we use the above construction taking into account the fact that s has already been defined only on countably many points of $L(z,\theta)$. The only exceptions to this procedure are the cases $z=\pm x_0$, $0<\theta\leq\pi/2$ where we have $L(x_0,\theta)=\mathscr{R}(x_0,\theta)$. In these particular cases we can define s in a way similar to a typical step in the proof of Theorem 1.

It is important to note that even for $z=x_0$ Theorem 4 still holds; that is, the relative frequencies converge to the expected limit. In this case the result follows from the ergodic theorem, not the law of large numbers.

A Note on the existence of spin-1 functions. Again by the same induction as in Theorem 1 one divides each circle into *three* disjoint subsets whose measures are given by (4.4).

[1]This nontechnical formulation is due to T. J. Jech, in *Handbook of Mathematical Logic,* edited by J. Barwise (North-Holland, Amsterdam, 1977), where the reader may find proofs and further references.

[2]I. Pitowsky, Phys. Rev. Lett. 48, 1299 (1982); 48, 1768 (E) (1982). See also my answer to comments [by N. D. Mermin, Phys. Rev. Lett. 49, 1214 (1982); A. L. Macdonald, *ibid.* 49, 1215 (1982)] Phys. Rev. Lett. 49, 1216 (1982).

[3]For details, see F. J. Belinfante, *A Survey of Hidden Variable Theories* (Pergamon, New York, 1973); J. Bub, *The Interpretation of Quantum Mechanics* (Reidel, Dordrecht, 1974).

[4]J. S. Bell, Physics (N.Y.) 1, 195 (1964).

[5]S. Kochen and E. P. Specker, J. Math. Mech. 17, 59 (1967).

[6]For a survey of experimental results, see J. F. Clauser and A. Shimony, Rep. Prog. Phys. 49, 349 (1978).

[7]The reader may find details on measure and probability theories in Y. S. Chow and H. Teicher, *Probability Theory, Independence Interchangability, Martingales* (Springer, New York, 1978).

[8]Condition (2.1) can be replaced by the somewhat weaker assumption of asymptotic independence, that is, the difference between the right- and left-hand sides of (2.11) tends very rapidly to zero as $k \to \infty$.

[9]Reference 5. An interpretation of the Kochen and Specker result, different from the one presented here, is in I. Pitowsky, Philos. Sci. 49, 380 (1982).

[10]A somewhat similar suggestion was made previously by B. O. Hultrgen and A. Shimony, J. Math. Phys. 18, 381 (1977).

[11]The continuum hypothesis is, in fact, dispensable. One can prove the theorem from a much weaker axiom.