Fluctuations in the homogeneous inflationary universe

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(Received 19 October 1982)

Quantum fluctuations in the homogeneous inflationary universe are shown to give rise to inhomogeneities of the right spectrum and magnitude needed to form galaxies.

I. INTRODUCTION

The idea that the universe may have passed through an exponentially expanding de Sitter phase at very early times—the inflationary universe[1]—has attracted a great deal of attention recently for the following reasons.

(1) It is a necessary consequence of Einstein's equations and the restoration of symmetry that occurs at very high energies and temperatures in grand unified models of the fundamental interactions.[2]

(2) It provides a natural solution to the long-standing homogeneity, horizon, and flatness problems of the standard model,[3] while explaining why the size and entropy of the universe should be so enormous in microscopic units.

(3) It provides a fundamental mechanism (i.e., quantum fluctuations) for generating the primordial inhomogeneous perturbations, with the correct flat spectrum, needed to produce the conspicuous large-scale deviations from homogeneity and isotropy observed in our present universe—namely, galaxies and galactic clusters.

The original inflationary universe suffered from the problem of the "graceful exit" from the de Sitter phase. The "new inflationary universe" provides a possible solution to this problem. However, calculations of the magnitude of density fluctuations in this model indicate a value for $\delta\rho/\rho$ far too large to be consistent with the observed homogeneity of the microwave background.[4]

An alternate inflationary model was proposed in Ref. 5. It makes critical use of the observations of Hawking and Moss[6] on the role of gravity in the phase transition from the de Sitter vacuum. The graceful exit problem is solved automatically by *homogeneous* tunneling over all of physical space rather than the nucleation of small bubbles of new vacuum. The large expansion arises from the small probability for this tunneling to occur and not from a slow rollover from the potential plateau.

The purpose of this paper is to analyze density fluctuations in this homogeneous tunneling inflationary universe and show that their magnitude falls naturally into the range required to generate galaxies at later epochs.

The paper is organized as follows. In Sec. II the growth of the amplitude of density fluctuations in the homogeneous inflationary universe are studied, irrespective of their origin. In Sec. III quantum fluctuations in the de Sitter vacuum are identified as the appropriate origin of primordial fluctuations in the very early universe. The growth of these initially very small fluctuations in the phase transition to the new vacuum is shown to be of the right order of magnitude to produce the inhomogeneities observed in our present universe in Sec. IV.

II. THE GROWTH OF THE FLUCTUATION AMPLITUDE

The background homogeneous solution is specified by the metric

$$ds^2=-dt^2+a^2(t)({}^3g_{ij}dx^idx^j) \tag{2.1}$$

and energy-momentum tensor

$$\begin{aligned} T^0_0&=-E(t),\\ T^i_j&=p(t)\delta^i_j\ ,\\ T^0_i&=T^i_0=0\ , \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} E&=\tfrac{1}{2}\dot{\phi}^2+V(\phi)+\rho_r\ ,\\ p&=\tfrac{1}{2}\dot{\phi}^2-V(\phi)+\rho_r/3\ . \end{aligned} \tag{2.3}$$

Here ϕ is the coherent scalar field variable describing breakdown of the grand unified group from G to $SU(3)\times SU(2)\times U(1)$, and an overdot denotes d/dt.

Einstein's equations are then

$$H^2(t)\equiv(\dot a/a)^2=\frac{C}{a^2}+\frac{1}{3}\kappa E \qquad (2.4)$$

and

$$\dot E=-3H(E+p)\,, \qquad (2.5)$$

with $\kappa\equiv 8\pi/M_{\rm Planck}{}^2$ and C an integration constant.

If $G=\text{SU}(5)$, the Coleman-Weinberg[7] potential appearing in (2.3) may be expressed as

$$V(\phi)=B\phi^4[\ln(\phi^2/\sigma^2)-\tfrac{1}{2}]+\tfrac{1}{2}B\sigma^2-\frac{\xi}{2}R\phi^2\,, \qquad (2.6)$$

where $\sigma=1.2\times10^{15}$ GeV, the symmetry-breaking scale, and

$$B=\tfrac{25}{16}\alpha_{\text{SU}(5)}{}^2=7.7\times10^{-4} \qquad (2.7)$$

are fixed by the strong and electroweak scales and the renormalization group.[8]

The free parameter ξ is naturally set to $\frac{1}{6}$ for classical conformal invariance though any value in the range 0.08 and 0.50 is allowed.

In order to treat the fluctuations about this strictly homogeneous background we employ the gauge-invariant formalism of Bardeen.[9] In this formalism the metric fluctuations must first be decomposed into scalar, vector, and tensor harmonics with respect to transformations under ${}^3g_{ij}$. The tensor perturbations couple to anisotropic terms in T^μ_ν and describe gravitational waves. The vector perturbations couple to the divergenceless (vortical) velocity field of the matter-radiation fluid with $\delta E=\delta p=0$. Thus, only the (spatially) scalar perturbations are relevant to the fluctuations in energy density and pressure.

The energy density fluctuation is not general coordinate (gauge) invariant. However, a gauge-invariant amplitude $\epsilon_m(t)Q(\vec{x})$ can be defined which equals $\delta E(\vec{x},t)$ in any gauge in which the matter world lines are orthogonal to $t=$const hypersurfaces. Here $Q(\vec{x})$ is the scalar harmonic with comoving wave number k,

$$\nabla^2Q+k^2Q=0\,. \qquad (2.8)$$

Then the linearized Einstein equations $\delta G_{\mu\nu}=\kappa\delta T_{\mu\nu}$ for the variations from the background solutions of (2.4) and (2.5) may be cast into the gauge-invariant form[10]

$$\ddot z-(\alpha-1)H\dot z-[\alpha+3(1+w)]H^2z=(2+3w)C\frac{z}{a^2}-(k^2-3C)\frac{c_s{}^2z}{a^2}-(k^2-3C)\left[\frac{p\eta a^2}{a^2}\right]. \qquad (2.9)$$

Here the definitions

$$\alpha=-3(1+c_s{}^2)=-3\left[1+\frac{dp}{dE}\right],$$
$$w=p/E\,, \qquad (2.10)$$
$$z=Ea^2\epsilon_m\,,$$

have been used and $\eta(t)$ measures the deviation of the pressure perturbation from that expected from the background equation of state. In addition, if anisotropic stresses are present, i.e., deviation from perfect fluid behavior, then additional terms make their appearance in Eq. (2.9). However, any such terms and all terms on the right side of Eq. (2.9) involve a factor of $1/a^2$ relative to the left side. Thus, once the scale factor $a(t)\propto\exp(\int H\,dt)$ has increased several times these terms may be neglected.

The physics of this statement is apparent if we recall that physical length scales expand with a,

$$\lambda_{\rm phys}(t)=\lambda_0 a(t)\,, \qquad (2.11)$$

while $H^{-1}(t)$, the effective particle horizon, which determines the maximum size of a region with all parts in causal contact with each other, is *constant* during the de Sitter phase:

$$H^{-1}(t)=H_0{}^{-1}=[\tfrac{1}{3}\kappa V(0)]^{-1/2}\,. \qquad (2.12)$$

Thus when $\lambda_{\rm phys}(t)>H^{-1}(t)$ physical perturbations with comoving wavelength λ_0 can no longer be affected by microphysical, i.e., causal processes. The perturbation becomes "frozen in," its subsequent evolution determined solely by the left side of Eq. (2.9). Since the duration of the de Sitter phase is long compared to $H_0{}^{-1}$ (a necessary condition for any inflationary model), the approximate time-translation invariance during this expansion ensures that $z(t_0)$ is independent of t_0. Here t_0 is defined as the time at which

$$\lambda_{\rm phys}(t_0)=H^{-1}(t_0)=H_0{}^{-1}\,. \qquad (2.13)$$

Thus, the fluctuation magnitude is independent of $\lambda_0=2\pi/k$, and the spectrum is scale invariant.

After the phase transition at $t=t_1$ the vacuum energy $V(0)$ is converted into radiation and $H^{-1}(t)$ grows like $2t$ while $a(t)\sim t^{1/2}$. Hence at some later time t_2 the fluctuation will again come within its horizon:

$$\lambda_{\rm phys}(t_2)=H^{-1}(t_2)\ . \tag{2.14}$$

For $t>t_2$ the fluctuations can begin to coalesce into inhomogeneous clumps due to the Jeans instability and form protogalaxies. This is the relevant epoch to ask for the amplitude of the density perturbations.

If we define the growth factor

$$\gamma\equiv\frac{z(t_2)}{z(t_0)}=\frac{Ea^2\epsilon_m\,|_{t_2}}{Ea^2\epsilon_m\,|_{t_0}} \tag{2.15}$$

and use Eqs. (2.4), (2.11), (2.13), and (2.14) we obtain

$$\gamma=\frac{\epsilon_m(t_2)}{\epsilon_m(t_0)}\ . \tag{2.16}$$

Thus the growth factor in z determines the growth of the density perturbation ϵ_m when it reenters the effective horizon.

To find γ consider Eq. (2.9) with the right side set to zero:

$$\ddot z-(\alpha-1)H\dot z-[\alpha+3(1+w)]H^2z=0\ . \tag{2.17}$$

In pure de Sitter space $w=-1$, $\alpha=0$, and $H=H_0$. Thus,

$$z(t)=z_1+z_2e^{-Ht} \tag{2.18}$$

and $z(t_1)\lesssim z(t_0)$: No significant growth in magnitude can occur prior to the phase transition.

After $V(0)$ has been completely converted to radiation, $w=\frac{1}{3}$, $\alpha=-4$, and $H(t)=\frac{1}{2}(t-t_1)^{-1}$. Thus

$$z(t)=z_1'+z_2'(t-t_1)^{-3/2} \tag{2.19}$$

and again no growth can occur. However, in between the phase transition and the conversion to radiation, ϕ must classically evolve to the new vacuum at $\phi=\sigma$. In that evolution,

$$E+p=\dot\phi^2+\tfrac{4}{3}\rho_r\cong\dot\phi^2$$

and

$$\alpha(t)=-3\frac{d}{dE}(E+p)=\frac{1}{H}\frac{1}{E+p}\frac{d}{dt}(E+p) \tag{2.20}$$

becomes large and positive. Then

$$\alpha\gg 3(1+w)$$

and Eq. (2.17) may be solved explicitly,[10]

$$z(t)=z_1e^{-H_0(t-t_0)}+z_1\int_{t'}^{t}dt'\,H_0e^{-H_0(t-t')}\exp\left[\int_{t_1}^{t'}dt''\,H_0\alpha(t'')\right] \tag{2.21}$$

obeying the initial conditions

$$z(t_1)=z_1,\quad \dot z(t_1)=0\ .$$

Here we have also assumed that $H=H_0=$const since H changes significantly only after $\dot\phi^2$ has become large [cf. Eqs. (2.3)–(2.5)]. On the other hand once this occurs, the conversion to radiation is quite rapid ($t_{\rm conv}<H_0{}^{-1}$) so that then (2.19) becomes applicable.

The significant contribution to (2.21) comes near $t=t_R$, the rollover time ($t_R=t_1+1.55H_0{}^{-1}$). Thus, if we integrate by parts and use $t_2\gg H_0{}^{-1}$ we find

$$\gamma\cong\exp\left[\int_{t_1}^{t_2}dt\,H_0\alpha(t)\right][1+O(e^{-H_0t_2})] \tag{2.22}$$

since $z(t)$ is essentially constant prior to the phase transition and after the conversion to radiation by Eqs. (2.18) and (2.19), Eq. (2.22) can be written as

$$\gamma=\frac{E(t_2)+p(t_2)}{E(t_1)+p(t_1)}=\frac{4f}{3}\frac{V(0)}{\dot\phi^2(t_1)+\frac{4}{3}\rho_r}\ , \tag{2.23}$$

where f is a number of order unity that describes the efficiency of conversion of vacuum energy $V(0)$ into radiation. Since $E(t_0)=V(0)$, Eqs. (2.16) and (2.23) give

$$\epsilon_m(t_2)=\frac{4f}{3}\frac{\delta E(t_0)}{\dot\phi^2(t_1)}\ . \tag{2.24}$$

This determines the scale-invariant magnitude of density perturbations when they reenter their horizon in terms of the magnitude when they left their horizon. We now proceed to calculate $\delta E(t_0/\dot\phi^2(t_1))$.

III. THE INITIAL FLUCTUATION AMPLITUDE

Quantum fluctuations are a natural source for deviations from the classical homogeneous background solution of Eqs. (2.1)–(2.7). The energy-momentum tensor to one-loop order in the de Sitter background is the relevant quantity to consider. Since T_μ^ν is bilinear in the quantum fields and $\langle\phi(x)\phi(x')\rangle$ diverges as $x\to x'$, $\langle T_\mu^\nu\rangle$ will be indeterminate, i.e., infinite at short distances. However, we already know from Eq. (2.9) that the magnitude of the density fluctuations at short distances ($\lambda_{\rm phys}<H^{-1}$) depends on details of the theory and/or initial conditions that are unknown. When $|x=x'|\gg H^{-1}$, $\langle T_0{}^0\rangle$ in the de Sitter background approaches a constant, which is the usual cosmological term [here $\kappa V(0)$] plus an additional one-loop term[11]:

$$-\delta\langle T_0{}^0\rangle=\rho_r=\frac{\pi^2}{30}NT_H{}^4\ , \tag{3.1}$$

where

$$T_H=H/2\pi \tag{3.2}$$

is the Hawking-de Sitter temperature[12] and N is the number of degrees of freedom. Equivalently, if the renormalization procedure is defined by reference to flat spacetime, ρ_r is the one-loop contribution to the effective potential in de Sitter spacetime.

Equation (3.1) is just the expression for thermal radiation at temperature T_H, i.e., it is the constant energy per unit volume for volumes large compared to $T_H{}^{-3}$. But on length scales comparable to $T_H{}^{-1}=2\pi H^{-1}$, quantum fluctuations in the radiation field become comparable to the equilibrium value itself. Since Eq. (2.24) requires δE at the time the fluctuation region is H^{-1}, we have the relation

$$\delta E(t_0)\sim\rho_r=\frac{N}{480\pi^2}H_0{}^4\ . \tag{3.3}$$

Because of the neglect of short-distance effects in Eqs. (2.9) and (3.3), this cannot be considered more reliable than an order-of-magnitude estimate.

It is instructive to compare δE with the height of the $R\phi^2$ barrier (Fig. 1):

$$\Delta V=V(\phi_1)-V(0)\ , \tag{3.4}$$

where

$$\phi_1{}^2\ln(\phi_1{}^2/\sigma^2)=-\frac{\xi\kappa\sigma^4}{2}\ . \tag{3.5}$$

For $\xi=\frac{1}{6}$ and $N\sim100$ we find

$$\rho_r/\Delta V\sim10^{-3}\ll1\ . \tag{3.6}$$

The fact that the thermal fluctuations are much smaller than the barrier height is the reason that the action for homogeneous tunneling is large ($A_H\cong410$) and the corresponding probability e^{-A_H} small.[5] In the present context the smallness of (3.6) justifies *a posteriori* the use of ρ_r for pure de Sitter space, since the ϕ quantum fluctuations typified by ρ_r are sensitive to the region of $V(\phi)$ very close to $\phi=0$ only [where $V(\phi)=V(0)=$const]. This is in contrast to the case $\xi=0$ when $\Delta V=0$ and the quantum-thermal fluctuations would be expected to completely destabilize the $\phi=0$ vacuum and dominate the transition to the new vacuum at $\phi=\sigma$. It is precisely this breakdown of the linearized fluctuation equation (2.9) that is signaled by the very large (~50) magnitude of the relative density fluctuations in the new inflationary scenario with $\xi=0$.[4]

When the universe finally does leave the $\phi=0$ vacuum after $\sim A_H/3$ e-foldings of the scale factor,

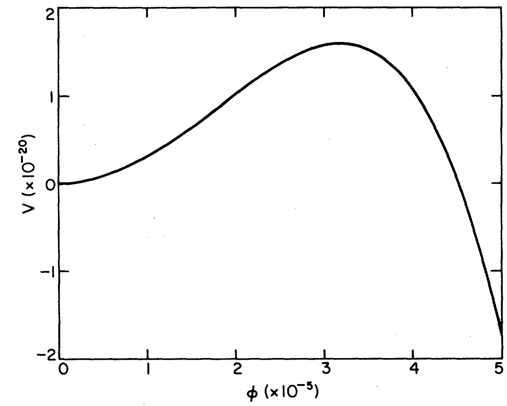

FIG. 1. The form of the potential of Eq. (2.6) near $\phi=0$ in units in which $\sigma=1$.

it emerges on the right of the barrier with

$$E=\tfrac{1}{2}\dot\phi^2+V(\phi)+\rho_r \tag{3.7}$$

equal to its pre-tunneling value. Now, the *classical* bounce equations imply $\dot\phi=0, V(\phi)=V(0)$. However, the action of the Coleman-de Luccia bounce is very close to the Hawking-Moss solution.[5] Hence the ϕ at which the universe emerges from the barrier is uncertain by $\Delta\phi\sim\phi_1$. Equations (3.7) and (3.4) then imply an uncertainty in $\dot\phi^2$ at the time of transition,

$$\Delta\dot\phi^2\Big|_{t_1}\sim2\Delta V$$
$$=\frac{(6\xi)^2H_0{}^4}{2B\ln(\sigma^2/\phi_1{}^2)}\left[1-\frac{1}{2\ln(\sigma^2/\phi_1{}^2)}\right]\ . \tag{3.8}$$

IV. THE FLUCTUATION AMPLITUDE AT $t=t_2$ AND GALAXY FORMATION

The initial fluctuation in $\dot\phi^2$ at $t=t_1$ determines the growth of the fluctuation magnitude through Eqs. (2.23) and (2.24). These equations together with Eqs. (3.1) and (3.8) yield the magnitude of the fluctuation when it reenters its horizon at $t=t_2$:

$$\epsilon_m(t_2)\cong\frac{4f}{3}\frac{\frac{\pi^2}{30}NT_H{}^2}{(6\xi)^2H_0{}^4}2B\ln(\sigma^2/\phi_1{}^2)$$
$$\times\left[1-\frac{1}{2\ln(\sigma^2/\phi_1{}^2)}\right]^{-1} \tag{4.1}$$

or

$$\epsilon_m(t_2)\cong\frac{fNB}{180\pi^2}\ln\left[\frac{3}{2\pi}\left[\frac{M_p}{\sigma}\right]^2\right] \tag{4.2}$$

for $\xi=\frac{1}{6}$.

Taking $f=\frac{1}{2}$ and $N=100$ then yields

$$\epsilon_m(t_2)\cong 4\times 10^{-4} . \tag{4.3}$$

The uncertainties in this calculation arise from

(1) neglect of short-distance effects in both Eqs. (2.17) and (3.3),

(2) the estimate of $\dot{\phi}^2(t_1)$ by Eq. (3.8), and

(3) the values of f, N, and ξ.

Nevertheless, fluctuations in the homogeneous tunneling model of Ref. 5 are significantly smaller than in the slow-rollover picture. A value of $\epsilon_m(t_2)$ of order of 10^{-4} for galaxy formation has been indicated by purely astrophysical considerations.[13]

It is interesting to calculate t_2 for fluctuations of the appropriate physical scale. Using Eq. (2.14) and assuming a radiation-dominated expansion gives

$$t_2=\frac{\lambda_{\text{phys,now}}{}^2}{2}\left[\frac{8\pi G\rho_{r,\text{now}}}{3c^2}\right]^{1/2},$$
$$=1.3\times 10^7 \text{ sec} , \tag{4.4}$$

for $\lambda_{\text{phys,now}}=10^{24}$ cm. This time is indeed prior to the transition from radiation dominance to matter dominance so that the assumption used in deriving Eq. (4.4) is correct. After t_2 (red-shift of 10^6) the density fluctuations can oscillate as an ordinary pressure sound wave. However, the average magnitude of ϵ_m remains constant: The fluctuations in density cannot grow because of radiation drag.[14] In fact, short-wavelength fluctuations are efficiently damped.[15]

At red-shifts less than 10^4 ($t>10^{11}$ sec) the matter dominates. The nonrelativistic equation of state, $p \ll E$ permits the density fluctuations to grow again: $\epsilon_m \sim t^{2/3} \sim a(t)$ for wavelengths large compared to the Jeans length, which is

$$\lambda_J=\left[\frac{\pi}{G\rho_c}\right]^{1/2} c_s . \tag{4.5}$$

A fluctuation with magnitude given by Eq. (4.3) would grow to $\epsilon_m \sim 1$ at a red-shift of order 4. At this epoch fluctuations on scales between λ_J and the largest λ_{phys} which had entered its horizon (and which had not been previously damped by Silk diffusion) would develop into inhomogeneous clumps of gas. The Silk damping scale for the $\Omega=1$ model that occurs in the inflationary universe corresponds to a mass of 3×10^{13} solar masses.[15] This may be the right order of magnitude for galactic masses, provided there is a large amount of dark, nonbaryonic matter in galactic halos.[14]

An interesting and important test of the entire theory of primordial adiabatic fluctuations is afforded by measurements of the microwave background inhomogeneity. Present observational bounds are of the same order as Eq. (4.3). An improvement of the limit by less than 1 order of magnitude would effectively exclude the present theory. It is remarkable nonetheless that the Zeldovich spectrum and amplitude can arise naturally in the homogeneous tunneling inflationary model, from purely quantum fluctuations, at times when the universe was only 10^{-22} cm large, and, moreover, than these inhomogeneities can eventually give rise to galaxies.

ACKNOWLEDGMENTS

It is a pleasure to thank E. Weinberg, F. Wilczek, and P. Hut for useful conversations and M. S. Turner, J. M. Bardeen, and P. J. Steinhardt for explaining their work to me prior to publication. This research was supported by the National Science Foundation, Grant No. PHY77-27084.

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