PHYSICAL REVIEW D VOLUME 27, NUMBER 10 15 MAY 1983

Entropy bounds, acceleration radiation, and the generalized second law

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(Received 30 August 1982)

We calculate the net change in generalized entropy occurring when one attempts to empty the contents of a thin box into a black hole in the manner proposed recently by Bekenstein. The case of a "thick" box also is treated. It is shown that, as in our previous analysis, the effects of acceleration radiation prevent a violation of the generalized second law of thermodynamics. Thus, in this example, the validity of the generalized second law is shown to rest only on the validity of the ordinary second law and the existence of acceleration radiation. No additional assumptions concerning entropy bounds on the contents of the box need to be made.

Although it is understood how the ordinary second law of thermodynamics plausibly arises for a system with a large number of degrees of freedom, there currently exists no proof of the second law based on the known microscopic laws of physics. The general belief in the validity of the second law rests mainly on the repeated demonstrations over the years of the failure of attempts to violate it. In the case of the generalized second law (GSL)—which states that the sum of the entropy of matter outside a black hole plus $\frac{1}{4}$ times the area of the black hole never decreases—considerably less is known since the fundamental microscopic laws of physics—namely, the laws of quantum gravity—remain to be discovered. Nevertheless, semiclassical calculations have provided enough information about the quantum behavior of black holes that *Gedankenexperimente* to test the validity of the GSL can be performed. Such tests are important since the validity of the GSL underlies the relationship between black holes and thermodynamics.

A promising possibility for achieving a violation of the GSL occurs when a box filled with matter is lowered to near the black hole and its contents then are emptied into the black hole. In a classical analysis, the energy of the contents of the box can be "red-shifted away" by lowering the box to the horizon. If this occurs, the energy and area of the black hole will not increase, but the entropy of the matter outside the black hole (i.e., contents of the box) will be lost. Bekenstein proposed a resolution[1] of this apparent violation of the GSL by conjecturing a bound[2] on the entropy in the box in terms of its energy and size. However, this bound does not suffice to rescue the validity of the GSL.[3]

Recently, we analyzed[3] the process of lowering a box toward a black hole, taking fully into account the effects of *acceleration radiation,* i.e., the effective radiation a stationary observer near a black hole would see. We showed that this acceleration radiation produces a buoyancy force which affects the energy-balance calculations and results in more energy being delivered to the black hole than would occur classically. We found that the optimal place for emptying the contents of the box into the black hole was from its "floating point", i.e., the height above the black hole at which

$$E=eV\ , \tag{1}$$

where E and V are the (local measured) energy and volume of the box and e is the energy density of the acceleration radiation. The entropy change in the black hole was calculated to be

$$\Delta S_{\rm bh}=\tfrac{1}{4}\Delta A_{\rm bh}=s(e)V\ , \tag{2}$$

where $s(e)$ is the entropy density of the acceleration radiation. Since the acceleration radiation is thermal and hence maximizes the entropy at fixed energy and volume, we concluded that $\Delta S_{\rm bh}\geq S_{\rm box}$ and hence that the GSL is satisfied. The entropy bound proposed by Bekenstein[2] was not needed in this analysis.

Recently, our analysis and conclusions have been criticized by Bekenstein[4] on the grounds that unconfined thermal radiation need not maximize entropy.

In particular, Bekenstein showed that at sufficiently low temperatures an electromagnetic field in a rectangular box with two large dimensions and one short dimension effectively behaves like a two-dimensional system and has entropy and energy given by

$$S_2=\alpha_2 T^2 A \ , \tag{3}$$

$$E_2=\tfrac{2}{3}\alpha_2 T^3 A \ , \tag{4}$$

where α_2 is a constant and A is the area of the largest face. On the other hand, the usual three-dimensional formulas, applicable to a box with all three of its dimensions large, are

$$S_3=\alpha_3 T^3 V \ , \tag{5}$$

$$E_3=\tfrac{3}{4}\alpha_3 T^4 V \ , \tag{6}$$

where α_3 is a constant and V is the volume of the box. Comparison of Eqs. (3) and (4) and (5) and (6) shows that at fixed volume, we can make $S_2>S_3$ at sufficiently low energy. Bekenstein suggested that in a *Gedankenexperiment* where such a box is lowered near the black hole (so that $S_2>S_3$ at its floating point) a violation of the GSL might be achievable. He argued that the GSL could be saved only by assuming the existence of entropy bounds on confined systems of the type he had previously proposed.

The purpose of this paper is to give a complete analysis of the *Gedankenexperiment* proposed by Bekenstein. We shall demonstrate again that the effects of acceleration radiation prevent a violation of the GSL. It may be true that a limit on S/E such as that suggested by Bekenstein may hold for ordinary matter but no such additional hypotheses are needed in our analysis.

First, however, we comment briefly on the meaning of the term "thermal radiation" and the assumptions made about it in our previous analysis. We *defined* "thermal radiation" to be the state of matter and radiation which maximizes entropy at fixed energy and volume. We assumed that acceleration radiation is thermal although this has been rigorously proven only for free fields. Hence, the inequality $S\leq Vs(e)$ follows immediately where S denotes the total entropy of the box (including the box walls) at total energy $E=eV$ (including the energy of the box walls). Thus, this inequality is not an additional assumption about the entropy of confined systems when applied to the *total* system.

Nevertheless, Bekenstein's example shows that it is possible to make the entropy per unit volume of the *contents* of a confined system (at fixed energy of the *contents*) greater than that of thermal radiation. Thus, the possibility exists of obtaining a violation of the GSL if we can empty the contents of such a confined system into a black hole in a suitable manner.

There are three possible procedures for doing this which we now shall analyze. After lowering the box containing high-entropy, low-energy material to the desired height we can (i) drop the entire box into the black hole, (ii) open the box completely (or destroy the box) and then return the open box (or its remnants) to our laboratory, or (iii) cut a small hole in the box and return the box (with hole) to our laboratory.

In case (i) the inequality $S\leq Vs(e)$ for the total system is applicable and our previous analysis shows that the GSL cannot be violated.

In case (ii) if the box could be opened or destroyed without energy being added to the contents, then our previous analysis shows that a violation of the GSL could be achieved. However, it is easy to see that a violation of the ordinary second law of thermodynamics also could be achieved by opening such a box inside an ordinary thermal bath in flat spacetime. Thus, if the ordinary second law is valid, energy must be pumped into the contents of the box when the box is opened or destroyed. Indeed, it is straightforward to verify that the validity of the ordinary second law requires that after the box is opened the final entropy S and final energy E of the contents of a box of volume V emptied into a thermal bath of energy density e and volume $v\gg V$ in flat spacetime must satisfy

$$S\leq s(e)V+\frac{1}{T}(E-eV) \ . \tag{7}$$

Thus, applying this result to the black-hole case we find that at the "floating point" $E=eV$, the inequality $S\leq Vs(e)$ again holds for the *contents* of the box *after* the box has been opened (or destroyed), and a violation of the GSL cannot be achieved.

The third procedure, suggested by Bekenstein,[4] for emptying the contents of the box into the black hole is perhaps the most interesting. Although, as argued above, destroying the box must cost energy, it should not cost energy to cut a small hole in the box. Thus, by lowering the box to the desired height, cutting a hole in it, and then returning it to our laboratory we may empty the box without adding in extra energy to its contents. It is instructive to analyze this case in detail. As we shall see, the weight of the contents of the box (with hole) as it is slowly pulled back to infinity—which results from its being "bathed" by acceleration radiation—produces just the right contribution to the energy balance calculation to keep the GSL satisfied.

First, we spell out our assumptions concerning the nature of the box of material used in this *Gedank-*

enexperiment. For a given energy $\mathscr{E}$ of the *contents* of the box, we define $\mathscr{S}(\mathscr{E})$ to be the maximum possible entropy of the contents. Thus, by *definition,* for a box whose contents have energy $\mathscr{E}$ and entropy S, we have

$$S \leq \mathscr{S}(\mathscr{E}) \ . \tag{8}$$

We define $T_{\rm box}{}^{-1} = d\mathscr{S}/d\mathscr{E}$. The only assumption we shall make in our analysis is that after the hole is cut, at each height the box contains energy $\mathscr{E}(T_{\rm box})$ and entropy $\mathscr{S}(T_{\rm box})$ with

$$T_{\rm box} = T = T_{\rm bh}/\chi \ , \tag{9}$$

where $T_{\rm bh}$ is the temperature of the black hole, χ is the red-shift factor, and thus T is the local temperature of the acceleration radiation. We shall place no bounds whatsoever on $\mathscr{S}(\mathscr{E})$ and, in particular, we will *not* assume that $\mathscr{S}(\mathscr{E})$ is bounded by the entropy $Vs(\mathscr{E}/V)$ of a corresponding volume of unconfined thermal radiation at energy $\mathscr{E}$. Note that in the case where no energy is required to open the box completely or destroy it—as was implicitly assumed in our previous analysis[3]—then processes (ii) and (iii) above are equivalent. However, in that case the ordinary second law requires

$$\mathscr{S}(\mathscr{E}) \leq Vs(\mathscr{E}/V)$$

[see Eq. (7)] and our previous results apply. Thus, the calculation given below may be viewed as a generalization of our previous analysis to the case where energy may be required to destroy the box.

Below, we shall make the same "thin-box" approximation as previously made.[3] The generalization to the case of a "thick box"—where the distribution of matter within the box must be taken into account—is given in the Appendix.

We calculate the change in black-hole entropy by the same type of energy-balance analysis used previously.[3] The energy delivered to the black hole is the difference between the energy in the box initially and the work done in raising and lowering the box. This work is composed of three parts: (1) the work done due to the weight of the box walls, (2) that due to the buoyancy force of the acceleration radiation, and (3) that due to the weight of the contents of the box. Instead of assuming that contributions (2) and (3) cancel during the process of pulling the box out, we will treat the three contributions separately.[4] The work due to the weight of the walls of the box will cancel on descent and ascent, and will give a net contribution of zero. Similarly, since by assumption the box has not changed in size or shape, the contributions of the buoyancy force will cancel. This leaves only the weight of the contents. On the way down, the work delivered to infinity on account of the weight of the contents is

$$W_1 = -\int_\infty^{l_0} E_i \frac{d\chi}{dl} dl = E_i(1-\chi_0) \ , \tag{10}$$

where E_i is the energy of the contents of the closed box, l_0 is the point at which the hole is cut in the box and χ is again the red-shift factor. On the way back up, the contents of the box are in thermal contact with the exterior through the small hole. Hence, as mentioned above, the contents will have energy $\mathscr{E}(T)$, with T given by Eq. (9). The work done to raise the contents is then given by

$$W_2 = -\int_{l_0}^{\infty} \mathscr{E}(T) \frac{d\chi}{dl} dl \ . \tag{11}$$

We can integrate by parts to get

$$W_2 = \mathscr{E}(T_0)\chi_0 - \mathscr{E}(T_\infty)\chi_\infty + \int_{l_0}^{\infty} \frac{d\mathscr{E}}{dl}\chi \, dl \ . \tag{12}$$

We will neglect the energy at infinity assuming that T_∞ is negligible. The last term may be rewritten as

$$\begin{aligned} \int_{l_0}^{\infty} \frac{d\mathscr{E}}{dl}\chi \, dl &= \int_{l_0}^{\infty} \frac{d\mathscr{E}}{dT}\frac{dT}{dl}\chi \, dl \\ &= \int_{l_0}^{\infty} T\frac{d\mathscr{S}}{dT}\frac{dT}{dl}\chi \, dl \ , \end{aligned} \tag{13}$$

where $\mathscr{S}(T)$ is the entropy of the enclosed thermal radiation. Using Eq. (9), we obtain

$$W_2 = \mathscr{E}(T_0)\chi_0 - T_{\rm bh}\mathscr{S}(T_0) \ . \tag{14}$$

Thus, the net work delivered to infinity during the lowering and the raising is

$$W_1 + W_2 = E_i + \chi_0[\mathscr{E}(T_0) - E_i] - T_{\rm bh}\mathscr{S}(T_0) \tag{15}$$

and the net increase in entropy of the black hole is

$$\begin{aligned} \Delta S_{\rm bh} &= \frac{1}{T_{\rm bh}}[E_i - (W_1 + W_2)] \\ &= \mathscr{S}(T_0) + \frac{1}{T_0}[E_i - \mathscr{E}(T_0)] \ . \end{aligned} \tag{16}$$

This is minimized if the hole is cut in the box when

$$\begin{aligned} 0 = \frac{d(\Delta S_{\rm bh})}{dl_0} &= \frac{d}{dl_0}\left\{ \mathscr{S}(T_0) + \frac{1}{T_0}[E_i - \mathscr{E}(T_0)] \right\} \\ &= \left[\frac{d\mathscr{S}}{dT} - \frac{1}{T}\frac{d\mathscr{E}}{dT} \right]_{T=T_0} \frac{dT_0}{dl_0} \\ &\quad - \frac{1}{T_0{}^2}[E_i - \mathscr{E}(T_0)]\frac{dT_0}{dl_0} \ . \end{aligned} \tag{17}$$

The first term is zero by the definition of T. The minimization condition therefore is simply

$$E_i = \mathscr{E}(T_0) \ . \tag{18}$$

We can rephrase this condition by saying that the optimal height is that at which the temperature which the material in the box would have if fully thermalized is equal to the temperature of the acceleration radiation outside the box.

In the optimal case (18) the entropy change (16) of the black hole is just $\mathscr{S}(T_0)$. Thus, using Eq. (8) we have proven that

$$\Delta S_{\rm bh} \geq \mathscr{S}(T_0) \geq S \ , \tag{19}$$

where S is the original entropy of the contents of the box. This shows that the generalized second law of thermodynamics cannot be violated in this *Gedankenexperiment.*

It is worth emphasizing three points concerning the above analysis. First, in the analysis we needed to consider only the properties of the material within the box. We did not need to compare the entropy within the box with the entropy of the displaced radiation as was done in our previous analysis. Second, the optimal height at which to open the box is no longer the floating point of the contents of the box. It is rather that point where the temperature of the acceleration radiation equals the temperature the material inside the box would have if it were fully thermalized. Third, if done at the optimal height and if the material which we lower into the black hole is fully thermalized, then the process is reversible. Thus, again we can, in principle, mine energy from a black hole. However, there is a further payoff here. Since the energy density inside the type of box considered here can be much higher than that of unconfined radiation at the same temperature, one gets more material per scoop than one would have expected on the basis of our previous analysis.[3,5]

The above analysis, of course, rules out only a particular class of proposals for violating the GSL. However, because of acceleration radiation, in any quasistatic process the region exterior to the black hole behaves as though it were filled with thermal radiation in hydrostatic equilibrium. Hence, for any quasistatic process involving a black hole, there will be an analogous process for a self-gravitating star composed of (real) thermal radiation. Thus, if it is possible to violate the GSL by quasi-static processes, it should also be possible to violate the ordinary second law for self-gravitating stars.

In summary, our analysis does not prove the general validity of the GSL, nor does it explain how it may arise from the microscopic laws of physics. However, the failure of the above *Gedankenexperimente* to produce violations of the GSL provides evidence for the validity of the GSL of the same nature as has led to the firm belief in the validity of the ordinary second law of thermodynamics. We emphasize that our arguments for the validity of the GSL rest only on the validity of the ordinary second law and on the existence of acceleration radiation.

This research was supported in part by the National Science and Engineering Council of Canada, by the Alfred P. Sloan Foundation, and by National Science Foundation Grant No. PHY 80-26043 to the University of Chicago.

APPENDIX: THICK BOXES

In the analysis above as well as in our previous analysis, we assumed for simplicity that the box is sufficiently thin that changes in χ and $d\chi/dl$ across the box are small compared with their average values. This assumption has been criticized by Bekenstein.[6] The purpose of this appendix is to analyze the case of a "thick box," where the above simplifying assumption is not made. As expected, we find that the conclusions of our analysis remain unchanged.

Consider a box of arbitrary size containing energy density ρ. As the box is lowered into the black hole, the energy density will depend both on the height l of the center of the box above the horizon, and the position within the box, y, as measured from the center.

The energy of the box as seen at infinity is given by

$$E_\infty(l) = \int \rho(l,y)\chi(l+y)dy \tag{A1}$$

whereas the weight of the box at infinity is

$$w = \int \rho(l,y)\frac{\partial\chi(l+y)}{\partial l}dy \ . \tag{A2}$$

The condition that no extra energy is fed into or extracted from the box as it is lowered is

$$\begin{aligned} 0 &= \frac{dE_\infty(l)}{dl} - w \\ &= \int \frac{\partial\rho(l,y)}{\partial l}\chi(l+y)dy \ . \end{aligned} \tag{A3}$$

Using Eq. (A3) we find that the work done during the downward trip is

$$W_1 = -\int_\infty^{l_0} w\, dl = E_i - \int \rho(l_0,y)\chi(l_0+y)dy \ , \tag{A4}$$

where E_i is the initial energy in the box.

After the hole has been cut, the box will be filled with a thermal distribution of matter. The entropy density σ of thermal matter *within* the box may, in general, be a function of the energy density ϵ at the given point, the position l of the box, and the position y within the box, $\sigma=\sigma(\epsilon,l,y)$. An important relation satisfied by σ can be derived from the ordinary second law of thermodynamics as follows. Consider an *insulated* box filled with thermal matter which is adiabatically raised and/or lowered on a string (with no energy flow between the string and box) in a static gravitational field. Since this process of raising and lowering the box is adiabatic, no change in the total entropy $\mathcal{S}$ in the box can occur. Therefore, we have

$$\begin{aligned} 0=\frac{d\mathcal{S}}{dl}&=\frac{d}{dl}\int \sigma(\epsilon,l,y)dy \\ &=\int \left[\frac{\partial\sigma}{\partial\epsilon}\frac{\partial\epsilon(l,y)}{\partial l}+\frac{\partial\sigma}{\partial l}\right]dy \\ &=\int \left[T^{-1}\frac{\partial\epsilon}{\partial l}+\frac{\partial\sigma}{\partial l}\right]dy \ . \end{aligned} \tag{A5}$$

However, for thermally distributed matter, we have[7] $T=T_\infty/\chi$. Thus, we obtain

$$0=\frac{1}{T_\infty}\int \chi\frac{\partial\epsilon}{\partial l}dy+\int\frac{\partial\sigma}{\partial l}dy \ . \tag{A6}$$

But the first term vanishes by Eq. (A3). Thus, we obtain the general restriction on the functional dependence of $\sigma(\epsilon,l,y)$ on l:

$$0=\int\frac{\partial\sigma}{\partial l}dy \ . \tag{A7}$$

Using Eq. (A7), we find that if the matter in a box is always thermally distributed but we no longer assume that the box is insulated—in particular, if matter may flow in and out of the box—then we have

$$\frac{d\mathcal{S}}{dl}=\frac{1}{T_\infty}\int \chi\frac{\partial\epsilon}{\partial l}dy \ . \tag{A8}$$

In our case, on the way up, the box will be filled with thermal matter at temperature

$$T=T_{\rm bh}/\chi \ . \tag{A9}$$

The work done against the weight of this energy is

$$\begin{aligned} W_2&=-\int_{l_0}^{\infty}\int \epsilon\frac{\partial\chi}{\partial l}dy\,dl \\ &=\int \epsilon(l_0,y)\chi(l_0+y)dy \\ &\quad+\int_{l_0}^{\infty}\int\frac{\partial\epsilon}{\partial l}\chi\,dy\,dl \ , \end{aligned} \tag{A10}$$

where the boundary contribution from infinity in the integration by parts was neglected. Hence, using Eq. (A8), we find

$$\begin{aligned} W_2&=\int \epsilon(l_0,y)\chi(l_0+y)dy+T_{\rm bh}\int_{l_0}^{\infty}\frac{d\mathcal{S}}{dl}dl \\ &=\int \epsilon(l_0,y)\chi(l_0+y)dy-T_{\rm bh}\mathcal{S}(l_0) \ . \end{aligned} \tag{A11}$$

The change in entropy of the black hole is thus

$$\begin{aligned} \Delta S_{\rm bh}&=\frac{1}{T_{\rm bh}}(E_i-W_1-W_2) \\ &=\frac{1}{T_{\rm bh}}\int[\rho(l_0,y)-\epsilon(l_0,y)]\chi(l_0+y)dy \\ &\quad+\mathcal{S}(l_0) \ . \end{aligned} \tag{A12}$$

Minimizing this with respect to l_0 and using Eqs. (A3) and (A8), we obtain

$$0=\frac{1}{T_{\rm bh}}\int[\rho(l_0,y)-\epsilon(l_0,y)]\frac{\partial\chi}{\partial l}(l_0+y)dy \ . \tag{A13}$$

This equation states that the energy delivered to the black hole is minimized when the weight of the energy in the box equals the weight the thermal radiation has at the temperature appropriate to that level.

The equality of the weights does not guarantee the equality of the energies, however. If $\rho(l_0,y)$ is distributed differently from $\epsilon(l_0,y)$, the energies need not be equal. However, if they are distributed differently, we can do even better toward maximizing the entropy loss of the box and minimizing the entropy gain of the black hole. Suppose that at the optimal level of equal weights, the difference between the energies is not zero,

$$\int(\rho-\epsilon)\chi\,dy\neq 0 \ .$$

Then, clearly, the matter in the box is not distributed thermally. Now allow the material within the box to thermalize. This will preserve the energy, increase the entropy, and, in general, change the weight. We can then reoptimize the work by moving the box to the new optimal height corresponding to the new energy distribution. In this process, we increase the entropy contained in the box (and thus increase the entropy loss when its contents go into the black hole) and decrease the entropy gain of the black hole. It follows that the optimal procedure for attempting to violate the GSL with a "thick box" is to have the matter thermally distributed in the box at the optimal height l_0. In that case, we have $\rho(l_0,y)=\epsilon(T,l_0,y)$, and, hence,

$$\Delta S_{\rm bh}=\mathcal{S}(l_0)=S_{\rm box} \ . \tag{A14}$$

Thus, the GSL is satisfied.

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