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Entropy bounds and the second law for black holes

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A counterexample to the entropy bound proposed by Unruh and Wald and by Page is described. The bound states that the entropy S of any system with energy E and volume V cannot exceed the entropy of an equal volume and energy of unconfined thermal radiation. The bound is found to be violated by thermal field systems whose various dimensions differ by an order of magnitude or more, the violation occurring at intermediate energies. Unruh and Wald used the bound in an argument establishing the validity of the generalized second law when a box containing some entropy is lowered to near a black hole, then opened, and withdrawn open. If the box is thin in one direction, the failure of the bound for its contents makes it appear that a violation of the second law is possible. We show that, in fact, for a thin box the buoyancy effects on which Unruh and Wald's argument relies cancel out. As a result, the second law is fulfilled despite the failure of the bound. It appears from the second law that the bound must nevertheless hold when applied to box plus contents, but a direct proof of this is still lacking. We also consider the alternative entropy bound $S \leq 2\pi ER/\hbar c$ ($2R$ is the largest dimension of the system) proposed earlier. For field systems it is shown to fail at very low energies, but to be valid for complete systems (i.e., box plus confined fields). Further, we show $S \leq 2\pi ER/\hbar c$ is a necessary condition for fulfillment of the second law when a thin system is dropped into a black hole (even in the face of buoyancy) in the sense that if the bound failed, the law would be violated.

I. INTRODUCTION

The ordinary second law of thermodynamics requires a system of given geometry and energy to evolve to a maximum-entropy state. The law does not restrict this maximal entropy. Recently, however, the possibility of setting a general upper bound on the entropy in terms of the system's energy and dimensions alone has been raised in a number of studies[1-5] of the applicability of the generalized second law[6,7] (GSL) in black-hole physics. One suggestion[1] is that the entropy S of an arbitrary system of proper energy E which may be circumscribed by a sphere of radius R should satisfy ($k=c=G=1$)

$$S \leq 2\pi ER/\hbar \ . \quad (1)$$

The need for (1) may be understood as follows. Drop the system into a Schwarzschild hole ($M \gg E$). Provided $R > \hbar E^{-1}$ (Compton length) and $R < 2M$ (hole size), Hawking radiation pressure is found not able to arrest the fall. Thus the black-hole entropy $4\pi M^2/\hbar$ grows by $8\pi ME/\hbar$; this must exceed S (GSL). If R is not tiny compared to $2M$, a bound like (1) is a must, though our simple argument cannot pin down the numerical coefficient.

A second suggestion, made by Unruh and Wald[2] (UW) and, in a closely related form, by Page,[3] is that the entropy S of an arbitrary system of volume V and proper energy E should not exceed the entropy S_r of an equal volume and energy of *unconfined* thermal radiation in the same gravitational field. Unruh and Wald argue that this principle is *sufficient* for the GSL to be satisfied in processes where a box containing a given system is lowered towards a black hole, and then dropped or emptied into it. According to UW, the buoyancy of the box in the "acceleration radiation"[8] felt by it supplies a mechanism whereby the black-hole argument leading to bound (1) may be sidestepped. Their conclusion is that bound (1) is not necessary for the GSL to hold.

The fact remains that bound (1) is supported by direct statistical arguments[1] (aside from a point relating to the choice of zero for the energy which will be discussed forthwith). The status of the principle of UW and Page is less clear. It seems reasonable (though not proved) that thermal radiation should maximize the entropy of a given energy confined within given boundaries. The principle, however, compares a confined system with *unconfined* radiation, and is thus not a corollary. The comparison with unconfined radiation is crucial in UW's specific application of the principle[2]: they assume the entropy of a system to be bounded by the entropy of the ambient radiation it displaces near the hole. Their assumption that the ambient radiation's thermodynamic variables (temperature, pressure, ...) depend only on energy density is reasonable only if this radiation is regarded as unaffected by nearby surfaces, i.e., unconfined.

If generally true, the principle should be valid in flat spacetime. Now, in three-dimensional flat spacetime the energy E_r and entropy S_r of a volume V of unconfined thermal radiation at temperature β^{-1} are given by the Boltzmann formulas

$$E_r = N\pi^2\beta^{-4}V\hbar^{-3}/15 , \quad (2)$$

$$S_r = 4\beta E_r/3 , \quad (3)$$

so that

$$S_r(E_r,V) = 1.201(NE_r{}^3V\hbar^{-3})^{1/4} . \quad (4)$$

In the above N is the effective number of (massless) field species (scalar fields enter with weight $\frac{1}{2}$, fermions with weight $\frac{7}{8}$). For flat spacetime the principle thus requires that the entropy S and energy E of any system of volume V comply with the following Page-Unruh-Wald (PUW) bound:

$$S \le S_r(E,V) . \quad (5)$$

Is (5) generally valid? By calculating explicitly S and E of noninteracting thermal fields confined in boxes of various shapes, we shall show (Sec. II) that the PUW bound, when applied to the fields only, can be violated whenever the various dimensions of the system differ by an order of magnitude or more (thin systems). The violations occur at intermediate energies in the energy scale set by the large dimension of the system. The above calculations also disclose violations of bound (1) at extremely low energies in the same scale. This possibility was evident earlier and suggestions on how to cope with it have been made[1,4] and criticized.[2,3,5] How is one to reconcile the above violations with the belief, now widespread, that the GSL is generally valid? Taken at face value, UW's arguments[1] imply a violation of the GSL is possible whenever a system violates the PUW entropy principle. Since the principle can fail, what other factor intervenes to enforce the GSL?

In what follows we present an analysis which reconciles the general validity of the GSL with the mentioned exceptions to bounds (1) and (5). The treatment is confined to thin systems, the only ones for which UW's original treatment is fully applicable.[4] We find it useful to distinguish between a thermal field system, and that system together with the inert box that confines it. The status of bounds (1) and (5) is different for the two cases. We show in Sec. II that the mentioned violation of (1) for thermal fields at very low energies is removed if one applies the bound to the complete system (field plus box). The argument does not depend on details about the box, but only on the assumption that the system cannot be smaller than its own Compton length. The mentioned violations of the PUW bound at intermediate energies may possibly be avoided by applying it only to complete systems. However, no direct argument for this is known.

Turning in Sec. III to the problem of a system which is lowered and then dropped bodily into a hole, we show that bound (1), as applied to the complete system, is a necessary condition for the GSL to be satisfied, despite the role played by buoyancy. Thus, both statistics and the GSL argue for the validity of bound (1). The original UW argument shows that the PUW, as applied to the complete system, also cannot be violated without the GSL being violated. Thus the PUW bound should always be valid for a complete system. However, as mentioned, a direct (i.e., statistical) proof of this is still lacking.

In Sec. IV we consider a thin box containing thermal fields which is lowered towards a black hole, then opened and withdrawn open. This strategy, proposed by UW, relieves one of the need to consider the mass of the box itself in the energy balance for the process. We find that buoyancy has no *net* effect on the problem. As a consequence, UW's result for the overall entropy change must be modified. The GSL is found to be satisfied despite violation of the PUW bound. Thus neither buoyancy nor this bound are generally necessary for the GSL to work. We also indicate why no contradiction exists between the argument demonstrating that bound (1) is a necessary condition for validity of the GSL, and the violation of the bound for fields at extremely low energies.

In Sec. V we show that bound (1) may be applied to complete systems with a very large number of fields. Violations, if any, can occur only for some 10^8 fields or more. It is quite possible that the bound actually applies to an arbitrary large number of fields. Our conclusions are summarized in Sec. VI.

II. EXCEPTIONS TO THE ENTROPY BOUNDS

Consider some noninteracting *massless* fields in flat spacetime confined to a rectangular box of dimensions $a \times b \times c$. The eigenenergies of the fields are

$$\epsilon_{knm} = \pi\hbar(k^2/a^2 + n^2/b^2 + m^2/c^2)^{1/2} , \quad (6)$$

where k, n, and m are non-negative integers, not all of which may vanish.[9] In a thermal state with inverse temperature β, the fields have mean energy and entropy

$$E=\sum_i g_i\epsilon_i(e^{\beta\epsilon_i}\mp 1)^{-1}\ , \tag{7}$$

$$S=\beta E+\sum_i \pm g_i \ln(1\mp e^{-\beta\epsilon_i})\ , \tag{8}$$

where i runs over species as well as (k,n,m), upper (lower) signs apply to bosons (fermions), and g_i denote degeneracy factors. The Boltzmann formulas (2) and (3) are obtained from (7) and (8) by approximating sums by three-dimensional integrals in the well-known fashion. This passage is meaningful if $\beta\hbar\pi/a$, $\beta\hbar\pi/b$, and $\beta\hbar\pi/c$ are all small compared to unity.

We shall have occasion to consider boxes for which a and c are of the same order while $b\ll a$. For these a range of β exists for which

$$\beta\hbar\pi/a \text{ and } \beta\hbar\pi/c \ll 1\ , \tag{9}$$

$$\beta\hbar\pi/b \gg 1\ . \tag{10}$$

Evidently when (9) and (10) are valid, modes with $n\neq 0$ are strongly suppressed in sums (7) and (8) by the smallness of $\exp(-\beta\epsilon_i)$. Then one can approximate (7) and (8) by two-dimensional integrals, taking $n=0$ everywhere. Thus, in effect, the fields become two-dimensional. One then obtains (Appendix A)

$$E=\tilde{N}\zeta_R(3)\beta^{-3}ac\hbar^{-2}/\pi \tag{11}$$

$$S=3\beta E/2\ , \tag{12}$$

$$S=1.089(\tilde{N}E^2ac\hbar^{-2})^{1/3}\ , \tag{13}$$

where $\zeta_R(Z)$ denotes the Riemann zeta function, $\zeta_R(3)=1.202$, and $\tilde{N}$ is a new effective number of species (scalars do not contribute, electromagnetic and neutrino contributions are 1 and $\frac{3}{4}$, respectively). It is instructive to contrast (11)—13) with (2)—(4).

Do (11)—(13) conform with the PUW bound (5)? Constraint (10) with β as given by (11) tells us that

$$b\ll[\pi^2\tilde{N}\zeta_R(3)ac\hbar E^{-1}]^{1/3}\ . \tag{14}$$

But since $V=abc$, the PUW bound (5) with (14) taken into acccount would predict

$$S\ll 1.476(\tilde{N}^{1/4}N^{3/4}E^2ac\hbar^{-2})^{1/3}\ . \tag{15}$$

Now N and $\tilde{N}$ are not very different. For example, for an admixture of one scalar, electromagnetic, and four neutrino fields $(N/\tilde{N})^{1/4}=1.06$; it is thus clear that the true S given by (13) exceeds the PUW bound provided only that (10) is not merely a marginal inequality. Hence the PUW bound, as applied to the box contents, can be violated for a thin box, even by large factors.

To ascertain how small b/a must be for a violation to occur, we have explicitly carried out sums (7) and (8) for a thermal electromagnetic field enclosed in a box of internal dimensions $a\times b\times a$ with various b/a ratios. The sums were performed with a programmable calculator, and included several hundred to several thousand modes in order to achieve good convergence. Allowance was made for the fact that all eigenenergies are doubly degenerate, except for those with one of k, n, or m vanishing; these are nondegenerate. Modes with two or three of k, n, or m vanishing are forbidden by the usual boundary condition (tangential $\vec{E}$ vanishes). Table I displays the results for $b/a=0.1$ and various temperatures. Columns 1 and 2 give the temperature β^{-1} and calculated energy E in units of $\hbar/a$. Column 3 gives the calculated dimensionless entropy. For comparison column 4 gives the PUW bound on S computed from (5) with $N=1$ using the E for each entry. It is seen that the PUW bound is violated for an intermediate range of energy $1\lesssim Ea/\hbar\lesssim 130$. As b/a decreases for given β^{-1}, E and S hardly change because the field is already effectively two-dimensional and "unaware" of the dimension b. By contrast, the PUW bound (5) decreases as $b^{1/4}$ so that the violation becomes larger than in the case displayed in Table I, and extends to a wider range of energy. Thus one passes to the cases described by (9) and (10). We conclude that the PUW bound is violated by pure field systems for intermediate energies whenever various dimensions of the system differ by more than an order of magnitude.

How does bound (1) fare in the comparison? Column 5 of Table I gives $S(ER/\hbar)^{-1}$ where S and E are the computed values, and $R\equiv\frac{1}{2}(2a^2+b^2)^{1/2}$ is the circumscribing radius. According to bound (1), entries in this column should not exceed 2π. The tabulated results bear this out for the wide range $10^{-9}\lesssim Ea/\hbar\lesssim 250$. The bound also works for larger E; this is easy to see. For β^{-1} much larger than in Table I, conditions (9) as well as $\beta\hbar\pi/b\ll 1$ will be satisfied (for $b/a=0.1$). Then (4) may be used to show that

$$S(Ea/\hbar)^{-1}\approx 0.68(\hbar/Ea)^{1/4}\ll 1 \tag{16}$$

so that bound (1) is easily satisfied. However, for extremely low energies $(Ea/\hbar<10^{-9})$ bound (1) is violated. This was expected, for S/E has no upper bound if the ground-state energy of the system in question is taken as zero.[1]

In Ref. 1 we proposed to extend the validity of bound (1) to all energies by including in the energy the vacuum state's contribution on the assumption that it would always be positive. Because negative vacuum energies appear in some calculations,[10] this

TABLE I. Energy and entropy as functions of temperature for a thermal electromagnetic field in a box with dimensions $a \times a \times 0.1a$.

$\beta^{-1}a/\hbar$	$Ea/\hbar$	S	$S_{\mathrm{PUW}}(E,V)$	$S(ER/\hbar)^{-1}$
0.20	1.00×10^{-9}	5.23×10^{-9}	1.20×10^{-7}	7.37
0.25	8.50×10^{-8}	3.59×10^{-7}	3.36×10^{-6}	5.96
0.50	6.26×10^{-4}	1.39×10^{-3}	2.67×10^{-3}	3.14
1	6.80×10^{-2}	8.18×10^{-2}	8.99×10^{-2}	1.70
2	1.426	0.9298	0.8810	0.92
3	6.359	2.865	2.704	0.64
4	17.69	6.068	5.825	0.48
5	39.96	10.98	10.73	0.39
6	80.04	18.23	18.07	0.32
7	146.6	28.42	28.45	0.27
8	249.9	42.16	42.44	0.24

suggestion has been attacked.[3,11] The basic idea is, however, still viable. One realizes that the vacuum energy is as much a property of the field as of the box which confines it (the box's geometry determines the vacuum energy). If in endeavoring to delineate a complete system, we include the vacuum energy, we must also include the minimum mass the box must have in order to be able to confine the field. It is plausible that the complete system (field + box) always has positive energy.[4] It is equally plausible that this energy is sufficient for the system's dimensions to be larger than the corresponding Compton length (otherwise the box could not be localized). Thus, for the complete system the energy is $E+E_0$ where E, as always, is the thermal energy computed from (7), while E_0 is a minimal "ground" energy never smaller than about $\hbar/R$. It is seen from Table I that, for the complete system, bound (1) is easily satisfied for *all* β. This is actually true in general. There exists a detailed argument[1] showing that if the lowest energy of a field system of arbitrary shape is not very small compared to $\hbar/R$, that system obeys bound (1) for all energies. Thus although bound (1), as applied to the fields only, breaks down for extremely low energies, it is generally valid when applied to a complete system.

The violation inherent in the PUW bound is not so easy to correct because it is not confined to very low energies. If we try to apply this bound to a complete system, we find that requiring E_0 to be no smaller than $\hbar/R$, or even than $\hbar/b$, does not suffice to make the bound work. For as we raise the thermal energy, but insist that the system be thin, i.e., that condition (10) remain valid, we must make b smaller, but the decrease need not be steeper than that of $E^{-1/3}$ [see (10) and (11)]. Thus an E_0 of order $\hbar/b$ can soon become a negligible part of the total system energy, and cannot much raise the value of the bound. The bound is still violated. By comparing (15) with $E\to E_0+E$ with (13) we see that violations of the PUW bound will be avoided if E_0 must always be at least of order E. It is not unreasonable that this should be required: the box might *have* to be that massive to avoid bursting under the pressure of the (relativistic) fields it confines. However, we have not found an airtight way of seeing this.

III. DROPPING THE BOX INTO A BLACK HOLE

Bound (1) was suggested by a *Gedankenexperiment*[1] in which an entropy-bearing system is dropped into a black hole in such a way as to minimize the energy added to the hole. The bound was found to be a necessary condition for the validity of the GSL (if it is violated, the law is violated). Unruh and Wald called attention[2] to the importance of buoyancy due to acceleration radiation in a version of the *Gedankenexperiment* in which a rectangular box bearing entropy S is lowered on a string from infinity to near a Schwarzschild hole of mass M, and then dropped in. According to UW the buoyancy of the box so affects the energy balance that the need to invoke bound (1) in order to enforce the GSL no longer arises.

Their reasoning is as follows. Buoyancy effectively lightens the box so the energy that can be gotten by lowering it in the hole's field is reduced from the naive result.[1,6] Eventually the box "floats" in the acceleration radiation and no more energy can be extracted. The minimal unextracted energy is found to be[2]

$$\epsilon_{\min}=T_{\mathrm{bh}}S_r(E_r=E_0+E,V)\ , \tag{17}$$

where $T_{\mathrm{bh}}=\hbar/8\pi M$ is the black-hole temperature, and S_r is the entropy of the radiation displaced by the box at the floating point. This occurs when[2,4]

$$E_0+E=E_r \,, \quad (18)$$

where E_0+E is the total proper energy of the box, and E_r the proper energy of the displaced radiation. Now, if the box is dropped from the floating point, it will increase the black-hole entropy by

$$\Delta S_{\rm bh}=\epsilon_{\rm min}/T_{\rm bh}=S_r(E_0+E,V) \,. \quad (19)$$

Thus UW's entropy principle [$S \le S_r$ when (18) is valid] is found to be sufficient for the GSL to be respected ($\Delta S_{\rm bh}-S \ge 0$). If the drop is made from another point, $\Delta S_{\rm bh}$ can only exceed (19) and we again have $\Delta S_{\rm bh}-S>0$.

There seems to be no need here to invoke bound (1). However, as we show below, bound (1) is still a necessary condition for the GSL to hold in the logical sense: if bound (1) failed, consideration of buoyancy could not "save" the GSL. Thus, although we need not invoke bound (1) explicitly, its truth is implied by the GSL. This is consistent with our conclusion (Sec. II) that bound (1) holds for each *complete* system (box + fields). Note, however, that because bound (1) is a *necessary* condition for the GSL, it cannot be used to provide a proof of the GSL. Providing such a proof was one of UW's objectives. However, since we have given counterexamples to the PUW bound for fields in flat spacetime, doubt has been cast on the crux of their argument. [Is

$$S \le S_r(E_0+E,V)$$

at the floating point even for a thin box?]

We now examine all these points. We focus on a box with arbitrary contents of dimensions $a\times b\times c$ with $a\sim c$ but $b \ll a$, because it is for such geometry that the PUW bound is in doubt. An additional reason is that UW's analysis culminating in formula (17) is valid only if at every stage of the descent of the box, the gravitational potential $\chi[\chi=(1-2M/r)^{1/2}$ for Schwarzschild geometry] is essentially constant throughout the volume it displaces.[2,4] If l denotes radial proper distance, this implies that

$$b(d\chi/dl) \ll \chi \quad (20)$$

holds.[2,4] (We shall consider the box to be lowered with its short side in the radial direction.) It turns out that (18) can be satisfied only if $b \ll a$.[4] Further, since the box is to fall into the hole we demand $R<M$.

The radiation displaced by the box is all at essentially uniform local temperature which we may write as $T_{\rm bh}\chi^{-1}$. This allows us to rewrite UW's result (19) in an alternative manner. Using relation (3) for unconfined radiation we have $S_r=4E_r\chi T_{\rm bh}{}^{-1}/3$ at the floating point. In view of (18) and (19) we get

$$\Delta S_{\rm bh}=4\chi_0(E_0+E)T_{\rm bh}{}^{-1}/3 \,, \quad (21)$$

where χ_0 denotes the gravitational potential at the floating point. To determine χ_0 we write down the floating condition (18) explicitly with help of (2):

$$E_0+E=N\pi^2T_{\rm bh}{}^4\chi_0{}^{-4}V\hbar^{-3}/15 \,. \quad (22)$$

Solving for $\chi_0T_{\rm bh}{}^{-1}$ and substituting in (21) we get

$$\Delta S_{\rm bh}=1.201(E_0+E)[Nabc\hbar(E_0+E)^{-1}]^{1/4}\hbar^{-1} \,. \quad (23)$$

Evidently, the box must be larger than its Compton length, i.e., $\hbar(E_0+E)<R$. Because

$$2R=(a^2+b^2+c^2)^{1/2}$$

one finds that

$$abc<(\tfrac{4}{3})^{3/2}R^3 \,.$$

Hence

$$\Delta S_{\rm bh}<1.338N^{1/4}(E_0+E)R\hbar^{-1} \,. \quad (24)$$

The GSL ($\Delta S_{\rm bh}-S\ge 0$) is now seen to place an upper bound on S which, for realistic N, is below bound (1) as applied to the complete system.[12] Hence bound (1) for a complete system is a *necessary* condition for validity of the GSL: despite the role played by buoyancy, violation of bound (1) would imply a violation of the GSL. This conclusion is at variance with that of UW.[2]

We now check whether we have respected condition (20) in our discussion. Solving (22) for χ_0 gives

$$\chi_0=0.0358[Nabc\hbar(E_0+E)^{-1}]^{1/4}M^{-1} \,. \quad (25)$$

Again because the box must be larger than its Compton length, but smaller than the hole ($R<M$), we see that for realistic N, $\chi_0 \ll 1$, i.e., the box floats very near the horizon. In that region $\chi\approx l/4M$ where l is the radial proper distance measured out from the horizon. Thus the box floats at

$$l=l_0\equiv 0.143[Nabc\hbar(E_0+E)^{-1}]^{1/4} \,. \quad (26)$$

We note that $l_0 \ll R$ for realistic N. For a given box, condition (20) is harder to satisfy the smaller l is. Let us thus check it at l_0. With $\chi\approx l/4M$ we find that (20) is equivalent to $b \ll l_0$. If we now make use of (26) we find the requirement

$$b \ll 0.075[Nac\hbar(E_0+E)^{-1}]^{1/3} \,. \quad (27)$$

For realistic N this requires b to be small compared to a or c. We have kept to this assumption throughout.

Another point to check is our implicit assumption that no entropy is produced outside the hole. Because the box is dropped very near to the horizon, it disappears into the hole quickly, and there should be no time for any entropy production processes (i.e., turbulence in the acceleration radiation) to act. In their first version of the problem, UW considered dropping the string with the box. This is, of course, unnecessary. After the box falls, the string may be withdrawn and any energy drawn from lowering it must be repaid. Hence, the string does not enter into the energy bookkeeping. But will it not produce entropy as it relaxes upon being relieved of the box? Not necessarily. If the box is dropped from its floating point where it does not tug on one string, the latter undergoes no sudden change in tension, so no entropy need be produced. It seems, then, that the entropy balance involving only S and $\Delta S_{\rm bh}$ is complete.

We now see that UW's result (19), which implicitly assumes (20), is directly relevant only for a thin box. Yet we know that PUW's entropy principle, as applied to the fields alone, may fail for a thin box. What happens to the principle when applied to box plus contents? Reconsideration of UW's argument as summarized in Eqs. (17)–(19) shows that if the principle were violated for the box plus contents, the GSL would be violated. Hence, to the extent that one today has confidence in the GSL, one can vouch for the applicability of the principle to *complete* systems. This assertion can be extended to the PUW bound in flat spacetime since we always considered the situation in which χ is nearly constant throughout the box. We note, however, that since no independent proof has been produced for the PUW bound, one cannot very well use it to construct a proof of the GSL as UW attempted to do.

IV. OPENING THE BOX NEAR THE BLACK HOLE

The need to consider the mass of the box itself is a drawback. It prevents us from confronting the GSL with the entropy bounds applied to a pure field system. To get around this obstacle we fall back on a strategy proposed by UW.[2] It calls for lowering the box from infinity only to the floating point of the box's contents alone ($E=E_r$), and then opening a port in the box's underside to allow free passage to its contents. The process is completed by hauling the *open* box back to infinity. According to UW, if the box's walls have negligible volume, the open box is not buoyed up, while the buoyant force acting on it when closed can be ascribed to its contents alone. Thus the energy extracted at infinity from lowering the box itself must be repaid upon hauling it back; the box plays no net role in the energy balance. By this reasoning, UW conclude that the minimum increase in $S_{\rm bh}$ is

$$\Delta S_{\rm bh} = S_r(E,V) \tag{28}$$

which is just Eq. (19) with the box's mass excluded (recall, vacuum energy is associated with the box).

The paradox is clear. If, as we have good reason to believe (Sec. II), it is possible to have $S > S_r(E,V)$ for a thin box's contents, then (28) implies that one can violate the GSL by carrying out UW's prescription with a thin box. Is this conclusion correct? We shall now show it is not; due to a subtlety in the buoyancy of a thin box, Eq. (28) is incorrect whenever the PUW bound breaks down. A detailed argument shows the GSL is respected.

The first point to check is whether the violations of the PUW bound in flat spacetime (Sec. II) extend to the present context. Again, our model is a thin box enclosing noninteracting thermal massless fields. Assuming χ is nearly constant inside the box [condition (20)], we realize that the interior temperature as measured by local observers, β^{-1}, must be related to the *proper* thermal energy E in the box just as in flat spacetime. Specifically, if β^{-1} satisfies conditions (9) and (10), we expect results (11)–(13) to hold for our box near a black hole. For the ambient radiation at local temperature $T_{\rm bh}\chi^{-1}$, the energy and entropy densities should be given by the Boltzmann formulas (for a short discussion see Ref. 4). Thus (4) gives the entropy of the displaced radiation. Then the reasoning following Eq. (14) tells us that it is possible to have $S > S_r(E,V)$, i.e., the PUW bound can be violated in our black-hole context as well.

The second point to check is whether, as UW conclude, the open box experiences no buoyancy. This is certainly correct for a box which is *not* thin in the sense of (10). For it the interior pressure is distributed just as the exterior one, and thus the outside pressure on each wall is compensated for: the box is not buoyed up. For a thin box whose interior is in contact with the ambient radiation through a *small* opening this is not true.[13] Provided that condition (10) is observed, the proper energy in the box does not depend on b for fixed entropy [see (13)]. By the principle of virtual work, this implies that the fields do not exert pressure in the b direction: the fields have become two dimensional and "do not care" about the thin dimension. Thus the external pressure difference between upper and lower walls cannot be balanced by interior pressure. (There is, of course, interior *vacuum* pressure, but this is unaffected by whether the box is open or not; hence its contribution to the energy balance cancels between the trip down and the return trip.) We conclude that for a thin box satisfying (10), full buoyancy is felt by the *open* box as it is hauled back up. Thus, the simple formula (28) cannot hold.

To derive the appropriate result we first check that the fields in the box continue to satisfy condition (10) *after* the box is opened. Initially, by assumption, $\beta\hbar\pi/b \gg 1$. Once the box is opened the fields reach the ambient temperature $T_{bh}\chi_1^{-1}$ where χ_1 corresponds to the point where $E=E_r$ *before* the box is opened. Writing this condition out using (2) and (11) (we assume the same species inside and out) we get

$$\tilde{N}\zeta_R(3)\beta^{-3}ac\hbar^{-2}/\pi = N\pi^2 T_{bh}{}^4\chi_1{}^{-4}abc\hbar^{-3}/15 , \tag{29}$$

or, equivalently,

$$\chi_1\hbar\pi/bT_{bh} = 1.813(N/\tilde{N})^{1/4}(\beta\hbar\pi/b)^{3/4} \gg 1 . \tag{30}$$

Thus the analog of (10) is satisfied after the box is opened which means (11)—(13) may be used with the replacement $\beta^{-1}\to T_{bh}\chi_1{}^{-1}$.

Let us now denote by $(\tilde{E},\tilde{S},\tilde{\beta})$ and (E_1,S_1,β_1) the values of (E,S,β) for the box when still closed, and immediately after being opened, respectively.[14] Evidently, the overall change ΔS in exterior entropy is just $S\ (\beta=T_{bh}{}^{-1})-\tilde{S}$; the first quantity is just the entropy still remaining in the box when it arrives at infinity. By using (12) we may write

$$\Delta S = 3\beta_1 E_1/2 - 3\tilde{\beta}\tilde{E}/2 + \int_{\beta=\beta_1}^{T_{bh}{}^{-1}} dS(\beta) . \tag{31}$$

Since the effects of buoyancy cancel out for the round trip, one may identify the overall change in black-hole mass, ΔM, with the negative of the sum of changes in E, each weighted by the appropriate χ (recall E is *proper* energy):

$$\Delta M = -(E_1-\tilde{E})\chi_1 - \int_{\beta=\beta_1}^{T_{bh}{}^{-1}} \chi\, dE(\beta) . \tag{32}$$

Of course $\Delta S_{bh}=\Delta M/T_{bh}$.

Recalling that for the open box $\beta=\chi T_{bh}{}^{-1}$, and that at $\chi=\chi_1$ we may use (11), we obtain

$$\Delta S+\Delta S_{bh} = \tilde{E}\tilde{\beta}(Z^2/2+Z^{-1}-\tfrac{3}{2}) + \int_{\beta=\beta_1}^{T_{bh}{}^{-1}} [dS(\beta)-\beta dE(\beta)] \tag{33}$$

with $Z\equiv\tilde{\beta}/\beta$.

Now, on thermodynamic grounds $dS/dE=\beta$ so the integral vanishes. Further, for real Z the form $Z^2/2+Z^{-1}-\frac{3}{2}$ is non-negative. Hence $\Delta S_{bh}+\Delta S\geq 0$, and the GSL is satisfied despite the failure of the PUW bound for the thin box. In the present example the bound and buoyancy are irrelevant.

How do we reconcile the failure of bound (1) for a pure field system at low energies with the argument[1] showing bound (1) to be a necessary condition for the GSL? First we note that the argument assumes the system in question can be dropped into the hole from a point adjacent to the horizon. However, this cannot be done here.

Let us assume that the box originally has E so low that it violates bound (1)—Table I presents a case in point. Because of buoyancy one can lower the box with the fields only down to the floating point where $\chi=\chi_0$ (see Sec. III). In an attempt to dump the fields only into the hole one then opens the port. The box's interior then acquires a local temperature $T_{bh}\chi_0{}^{-1}$. From (25) we now find

$$aT_{bh}\chi_0{}^{-1}\hbar^{-1} = 1.11a^{3/4}[Nbc\hbar(E_0+E)^{-1}]^{-1/4} , \tag{34}$$

where E refers to the thermal energy *before* the box was opened. Because $c\approx a$, $\hbar(E_0+E)^{-1}<b$, and $b \ll a$ we see that for N not too large,

$$aT_{bh}\chi_0{}^{-1}\hbar^{-1} > 1 .$$

Table I makes it clear that bound (1) will be satisfied in such a case and that E and S are now large compared to the values for the closed box (say, the first entry in Table I). Hence entropy has flown from hole to box, rather than in the opposite direction as required by the argument. Therefore, it cannot be applied, and no contradiction arises.

V. THE PROBLEM OF MANY SPECIES

Bound (1) has been criticized[2,3] because it fails if a large number of particle species are present. The number required to evade it has, however, been seriously underestimated.[2] An obvious temptation is to compare bound (1) with the Boltzmann formula (4) for entropy of thermal radiation. If one does this, one finds bound (1) to be violated when

$$N > 749ER^4\hbar^{-1}V^{-1} . \tag{35}$$

However, for such N Boltzmann's formula (2) indicates that

$$\beta\hbar\pi/R > 14.80 . \tag{36}$$

Yet the condition for applicability of the Boltzmann formulas is rather $\beta\hbar\pi/R \ll 1$. Hence the result (35) involves an inconsistency. One must go back to first principles, i.e., to (7) and (8).

An example of a precise calculation is Table I. For one species we found that bound (1) fails for energies below $10^{-9}\hbar a^{-1}$ or equivalently, below $7\times10^{-10}\hbar R^{-1}$. Now, for N species a given S/E is reached for a *total* thermal energy a factor N larger than for one species. Hence, only if $N > 1.4 \times 10^9$ will bound (1) be violated for energies $> \hbar R^{-1}$. If we insist on applying the bound only to complete systems (box plus fields) we know that the energy scale begins at some $E_0 > \hbar R^{-1}$. Hence bound (1) cannot be violated with fewer than a billion species. This is well above UW's estimate of $N = 10^2$ for a violation.[2]

To see that the above large N is not atypical, we turn to a system of N fields of various types confined to a box of arbitrary shape. Let us divide up the box's energy (including vacuum energies) E_0 equally among the N fields. At this point we adapt a result of Ref. 1. For q an arbitrary real number larger than 4, we have, *rigorously,*

$$S/(E+E_0) < \max[N\Gamma(q)\zeta_R(q)\zeta_k(q-1)E_0{}^{-1}]^{1/q} , \tag{37}$$

where Γ and ζ_R are the gamma and Riemann zeta functions, k labels the field species, ζ_k denotes the kth field's zeta function

$$\zeta_k(p) = \sum_{i=1}^{\infty} g_{ki}\epsilon_{ki}^{-p} \tag{38}$$

(notation of Sec. II), and the maximum is taken over the field species. Let us focus on $q > 10$. Then, since $\zeta_R(10) = 1.001$, and ζ_R is monotonically decreasing in its argument, we approximate ζ_R by unity. Likewise, we approximate $\zeta_k(p)$ by $g_{k1}\epsilon_{k1}^{-p}$. For example, for the electromagnetic field in a cube this entails an error on 1.8% for $q = 15$. Finally we express E_0 and ϵ_1 in terms of the natural energy scale: $E_0 = \alpha\hbar R^{-1}$ and $\epsilon_1 = \gamma\hbar R^{-1}$. Then (37) reduces to

$$S/(E_0+E) < [N\Gamma(q)\gamma g_1/\alpha]^{1/q} R\gamma^{-1}\hbar^{-1} , \tag{39}$$

where γ, α, and g_1 already refer to the k which maximizes the expression in (37).

We now look at the example of N scalar fields. A general theorem[1] tells us that $\gamma \geq \pi$. Unless there is an accidental degeneracy $g_1 = 1$. Because the box must be as large as its Compton length $\alpha > 1$. Taking $q = 15$ we get

$$S/(E_0+E) < 1.84 N^{1/15} R\hbar^{-1} \tag{40}$$

which shows that bound (1) is satisfied even for $N = 10^8$. Thus the bound holds for any N relevant in nature, and probably also for N large enough to be relevant in the subject of $1/N$ expansions.

The point can also be made that bound (1) on $S/(E+E_0)$ probably holds for *arbitrarily* large N. Consider N fields of the same type (i.e., all vector fields) confined to a box with effective radius R. If the vacuum energy of one field is negative, we expect the mass of the box itself to compensate for it in such a way that the total E_0 is non-negative.[4] Barring the case where E_0 exactly vanishes for one field, we thus expect $E_0 \propto N$ for N fields. On dimensional grounds we expect E_0/N to be of order $\hbar/R$. In fact, all available calculations show the proportionality coefficient not to be very small.[10,11] This means that in (39) N/α is independent of N, and not too large. Since γ is never smaller than 2,[1] one immediately sees (by taking, say, $q = 15$) that $S/(E_0+E)$ complies with bound (1). The argument is evidently simpler if each field has positive vacuum energy. One can also extend it to situations where one does not know, *a priori*, which fields are present.[4]

VI. SUMMARY AND CONCLUSIONS

All our results are consistent with the belief that the GSL is generally valid. But one is some distance from UW's goal of showing the GSL to be a straightforward consequence of buoyancy, even for the comparatively simple case of a thin box lowered towards a black hole. One finds the PUW entropy bound, the pivotal part of their argument, to break down if applied only to thermal massless fields—which are possible contents of the box. Interestingly, this violation has no deleterious effect on the GSL when the box is opened near the hole: the law avoids violation because of the way the entropy and energy *changes* of the thin box are related. Buoyancy does *not* here play a central role.

Since the PUW bound can fail for the box's contents alone, and since no direct argument has been given for its validity when applied to box plus contents, one has at present little confidence in the argument[2] which uses the bound to show the GSL is satisfied when the box drops bodily into the hole. One is in the uncomfortable position of having to *assume* the GSL always holds in order to establish the bound for a complete system.

The status of bound (1) is different. It is violated only at very low energies when applied to thermal fields. However, a statistical argument shows it is valid for all energies when applied to a complete system. For a thin box one can check that if it did not comply with the bound, the GSL would be violated when the box was dropped into a black hole, even if the effects of buoyancy were allowed for.

Evidently, entropy bounds are not the exclusive reasons for the GSL's validity. Thus any program seeking to demonstrate the law's validity by starting from a single phenomenon, such as buoyancy, and relying on an entropy bound, can only have partial success. The GSL, like any fundamental physical law, is not really susceptible to general proof. At present it seems more fruitful to assume the GSL's validity in a particular situation to derive an entropy bound for material systems. The original "derivation" of bound (1) for a complete system[1] is an example of this procedure; we have seen that consideration of buoyancy does not alter the original conclusion, at least for a thin box. Another example is the derivation of the PUW bound for a complete thin system (Sec. III).

It would be of great interest to see how *thick* boxes fit into the scheme. It seems likely on statistical grounds that fields in such boxes do obey the PUW bound. However, UW's result (19) for the minimal change in black-hole entropy when the box is dropped is not applicable for thick boxes.[4] One is forced to modify it, and the modified formula[4,5] requires knowledge of the energy distribution inside the box. Thus the problem is more complicated than those considered here.

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APPENDIX

Consider a thermal massless field confined to a box of dimensions $a\times b\times c$ with $b \ll a$ and $b \ll c$. We assume conditions (9) and (10) apply. Then eigenvalues, given by (6), for which $n\neq 0$ are strongly suppressed in the thermal sums (7) and (8). Further, the values of $\beta\epsilon$ for adjacent eigenvalues are close together so that (7) may be approximated by

$$E=g\int_0^\infty\int_0^\infty \epsilon(e^{\beta\epsilon}\mp 1)^{-1}dk\,dm\,,\quad n=0\,,\qquad \text{(A1)}$$

where k and m are now regarded as continuous variables and g is assumed constant for all modes. Setting $x\equiv k/a$ and $y=m/b$ and going over to polar coordinates with radial coordinate

$$r\equiv\beta\pi\hbar(x^2+y^2)^{1/2}\,,$$

we have

$$E=(gac\beta^{-3}\hbar^{-2}/2\pi)\int_0^\infty r^2(e^r\mp 1)^{-1}dr\,.\qquad \text{(A2)}$$

The integral is $2\zeta_R(3)$ for the boson case, and $3\zeta_R(3)/2$ for the fermion one.[15]

For an electromagnetic mode with one of the quantum numbers k, n, and m vanishing, $g=1$. Modes with $g=2$ ($knm\neq 0$) are thermally suppressed, and modes with two vanishing quantum numbers are forbidden. Hence the electromagnetic field contributes unity towards $\tilde{N}$ in (11). A scalar field mode obeying Dirichlet boundary conditions[16] must have $knm\neq 0$. Hence scalar fields are thermally suppressed and they contribute little towards $\tilde{N}$. A left-handed neutrino field has $g=1$. Because its thermal integral (A2) is $\frac{3}{4}$ of the boson one, it contributes $\frac{3}{4}$ towards $\tilde{N}$.

The sum for S can be reduced to an integral in like manner. An integration by parts leads to the relation $S=3\beta E/2$.

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[9]This is obtained from the wave equation; one assumes the fields vanish on the boundary. For nonscalar fields, the boundary condition may apply only to one component, so some of k, n, or m are allowed to vanish (as for the electromagnetic case).

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[12]When the condition $b \ll a$ is used, bound (24) may be tightened. Thus our argument is still valid for fairly large N (in excess of 10^3).

[13]When the opening is not small there is no clear distinction between interior and exterior field eigenmodes. Thus the task of computing the thermal energy and entropy to be ascribed to the box itself becomes very complicated, if at all meaningful.

[14]For a thin box internal rearrangement cannot take place as the box is lowered closed, so $\tilde{E}$, $\tilde{S}$, and $\tilde{\beta}$ should be the original values. For a thick box $\tilde{E}$ may change as the box is lowered (see Ref. 4).

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