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Linear conformal quantum gravity

B. Binegar, C. Fronsdal, and W. Heidenreich

Department of Physics, University of California, Los Angeles, California 90024

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This paper is an investigation of field theories that satisfy the following two criteria. (1) Among the propagating modes is a pair of massless particles with helicities ±2. (2) The canonical commutation relations are conformally invariant. This study of "linear conformal gravity" is motivated by the belief that conformal invariance may be the key to a future theory of quantum gravity. Our first conclusion is that the fields of linear conformal gravity include a tensor field of rank 3 and mixed symmetry, and a symmetric tensor field of rank 2, tentatively interpreted as a torsion field and a metric field. The free quantum field operator is constructed explicitly, and the propagator is calculated. The Fourier transform is of dimension p^{-4}, which is encouraging for renormalizability. The field inevitably carries along a nonunitary ghost, similar to the one that turns up in linearized Weyl gravity. Our main result is that the ghost can be exorcised by imposing constraints on the external sources and boundary conditions on the physical states.

I. INTRODUCTION

Many people believe that conformal invariance may be the key that will some day solve the problem of quantum gravity. Since the early work of Weyl,[1] conformal field theories have been investigated on the level of classical fields governed by nonlinear differential equations. In order to carry out quantization, and to study renormalizability and related questions,[2,3] one needs a meaningful expansion and a linear approximation. The free quantum field, and the propagator, can then be constructed in terms of a space of solutions of the linear, free-field equations. A certain subset of these free modes becomes the space of one-particle, propagating states. The remaining free modes (including ghosts) have to be prevented from interacting or from propagating, in order to save the unitarity of the theory. The dynamical content of the theory is thus to a large extent fixed at the linear level; for this reason we believe that the linear approximation ought to reflect the conformal invariance of the original, nonlinear, classical field theory. Otherwise, one can hardly expect to recover conformal invariance in the full, nonlinear, quantum field theory.

Experience with gravitation[2] and with other conformal field theories has shown that quantization via the linear approximation often leads to a renormalizable quantum field theory, but the construction does not guarantee or even suggest that it is unitary.

In this paper we look into the unitarity of conformally invariant, linear theories of gravitation. We require conformal invariance, not only of the field equations, but also of the free-field commutation relations. This implies that the conformal group $\mathscr{C}$ acts on the space V of free-field modes and on the space of physical, one-particle states. The question of the unitarity of the theory is intimately related to the properties of this representation $D(V)$ of $\mathscr{C}$. However, in gauge theories this relationship is a subtle one, and one of our main tasks is to determine V and $D(V)$. It turns out that V inevitably contains, besides the physical modes and analogs of the scalar and longitudinal modes of electrodynamics, a set of ghost modes. Scalar and longitudinal modes may be prevented from propagating in the usual way, with the help of conservation laws for the current. Elimination of the ghost is a more difficult problem; it is solved here for the first time, as far as we know.

Our approach is thus the reverse of the usual one; it is designed to make sure of the unitarity of the theory from the outset. We first determine the free-field modes and find the equations that they satisfy. Then we formulate an action principle that describes interactions with external sources. Next necessary and sufficient constraints are found to guarantee that only physical modes propagate. Finally, we hope that the theory is the linear approximation of a complete, necessarily nonlinear, theory of interacting fields.

The physical one-particle states, the states that actually propagate and interact, are described by a subset of the free-field modes. In quantum gravity they must include two massless particles with helicity $+2$ and -2. Each carries a unitary, irreducible representation of the Poincaré group $\mathscr{P}$, and each of these two representations of $\mathscr{P}$ has a unique extension to a unitary, irreducible representation of $\mathscr{C}$.[4]

The conformal group $\mathscr{C}$ is locally isomorphic to SO(4,2) and its compact subgroup is SO(4) $\otimes$ SO(2). The generator of SO(2) is the conformal energy and will be denoted L_{50}. It is normalized so that its eigenvalues have unit spacing. Let $D(E_0,j_1,j_2)$ denote the irreducible, projective representation of $\mathscr{C}$ that is defined (up to equivalence) as follows: the lowest eigenvalue of L_{50} is E_0, and in this eigenspace so(4)$=$su(2)$\oplus$su(2) acts irreducibly by the representation (j_1,j_2) of dimension $(2j_1+1)(2j_2+1)$. Then the representations of $\mathscr{C}$ that extend the massless, helicity ± 2 representations of $\mathscr{P}$ are $D(3,2,0)$ and $D(3,0,2)$. We call these the physical representations. They are very singular, highly degenerate representations that can be realized in field theories only as subquotients of nondecomposable representations; that is, they occur only in gauge theories.

The physical representations are thus expected to be imbedded in nondecomposable representations. It is well known that the irreducible subrepresentations and subquotients of a nondecomposable representation must all have the same values for the Casimir operators. Among the irreducible representations that have energy spectra bounded below there are precisely six that have the same values for the Casimir operators as $D(3,2,0)$, and another set of six that are similarly related to $D(3,0,2)$. Please see Fig. 1. Eight of these representations are needed to define the free quantum field operator.

In Secs. II and III we determine the tensor structure of the fields of linear conformal gravity. The result is surprising: in the Dirac six-cone notation, the only tensor field that carries the physical representations of $\mathscr{C}$ is a tensor field of rank 3 and mixed symmetry. In particular, linear conformal gravity cannot be formulated in terms of a symmetric six-tensor of rank 2. In Sec. IV we find the minimal representation of $\mathscr{C}$ that must be carried by the free field, as well as the equations satisfied by the free modes. The free quantum field operator is defined

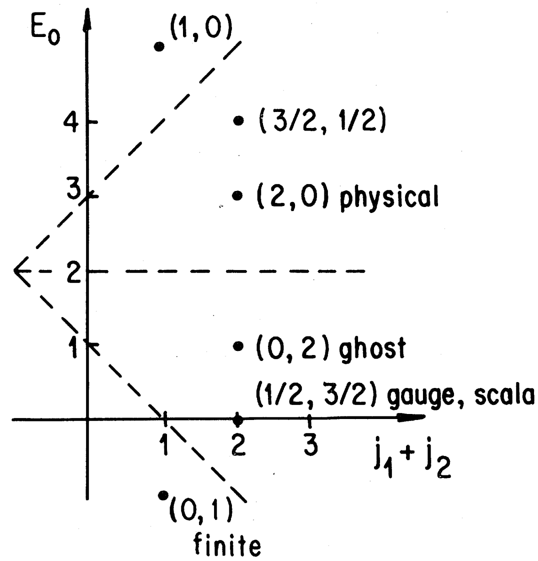

FIG. 1. The six points indicated are the lowest weights of the positive-energy representations that have the same values for the Casimir operators as $D(3,2,0)$. Each point represents a $D(E_0,j_1,j_2)$, with (j_1,j_2) indicated. The broken lines are the intersections between the Weyl reflection planes and the plane of the figure. A similar set of six Weyl equivalent weights is found by helicity conjugation: $(E_0,j_1,j_2)\rightarrow(E_0,j_2,j_1)$.

and the free field homogeneous propagator determined. A reader who cares only for the results may skip all this and go directly to Sec. V, since the rest of the paper is reasonably self-contained.

In Sec. V we formulate an action principle and find the boundary conditions that must be imposed on the physical states, and the constraints that must be satisfied by the external current, in order that the theory be unitary in the linear approximation. Then we verify explicitly that the propagating modes are exactly the same as in Einstein's theory of gravitation (linearized). Finally, in Sec. VI, we speculate on the problem of discovering the full, nonlinear theory.

II. TENSOR STRUCTURE

It is necessary to have a very efficient means of carrying out calculations; we find Dirac's six-cone formalism[5] indispensable. The final result will be reexpressed in familiar Minkowski notation, in Sec. V.

The conformal group does not act globally on Minkowski space, but this defect can be fixed by adding points (a three-cone) at infinity.[6] One gets a compactified Minkowski space that is a homogeneous space for $\mathscr{C}$; it is equivalent (as a homogeneous space) to Dirac's projective cone. In R^6, with coordinates $y_0, \ldots, y_5$, consider the cone

$$y^2 \equiv y_0{}^2 - y_1{}^2 - y_2{}^2 - y_3{}^2 - y_4{}^2 + y_5{}^2 = 0 ,$$

and identify λy with y for $\lambda \neq 0$. This is Dirac's projective cone; the mapping to Minkowski space will be given in Sec. V.

It will be assumed that the fields of linear conformal gravity are tensor fields. A symmetric tensor field of rank 2 would seem a natural choice, but this does not work, as we shall now show.

A homogeneous tensor field of rank n and degree N is an n-linear function of n vector variables:

$$\psi(y,z_1,\ldots,z_n) = \psi_{\alpha\beta\cdots}(y) z_1^\alpha z_2^\beta \cdots , \tag{2.1}$$

$$\hat{N}\psi = N\psi, \quad \hat{N} \equiv y\cdot\partial = y^\alpha \partial/\partial y^\alpha . \tag{2.2}$$

We consider tensors of specific symmetry type that are transverse, divergence-less, and traceless, that is,

$$y_\alpha \psi_{\alpha\beta\cdots} = 0, \quad \mathrm{Grad}_\alpha \psi_{\alpha\beta\cdots} = 0,$$
$$\psi_{\alpha\alpha\beta\cdots} = 0 , \tag{2.3}$$

$$\mathrm{Grad}_\alpha \equiv y_\alpha \partial^2 - (2\hat{N}+4)\partial_\alpha . \tag{2.4}$$

The operator Grad_α, unlike ∂_α, is intrinsic on the cone; that is, it is a well-defined operator acting on fields that are defined on the cone $y^2 = 0$.[7]

We need an intrinsic wave operator that maps tensor fields on tensor fields with the same rank. Such operators exist only in exceptional cases, namely, for tensor fields of degree $-1, 0, +1, \ldots$:

$$\begin{array}{lcccc} \text{degree of field:} & -1, & 0, & +1, & \ldots, \\ \text{intrinsic wave operator:} & \partial^2, & (\partial^2)^2, & (\partial^2)^3, & \ldots, \end{array} \tag{2.5}$$

Field theories with other (anomalous) dimensions, that do not have differential wave operators,[8] are interesting but will not be considered here. The operators $\partial^2, (\partial^2)^2, \ldots$ will consistently be applied only to tensor fields of the appropriate degree, since they make no sense otherwise. Thus, if ψ is of degree zero, then $\partial^2 \mathrm{Grad}_\alpha \psi$, $(\partial^2)^2\psi$, and $(\partial^2)^3 y_\alpha \psi$ all make sense, but $\partial^2\psi$ does not.

The action of $\mathscr{C}$ takes the form

$$T_\Lambda \psi(y,z) = \psi(\Lambda^{-1}y, \Lambda^{-1}z),$$

with Λ in SO(4,2). The action of the basis elements $(L_{\alpha\beta} = -L_{\beta\alpha})$ $\alpha,\beta = 0,\ldots,5$ of the Lie algebra is given by

$$L_{\alpha\beta}\psi = (M_{\alpha\beta} + S_{\alpha\beta})\psi,$$
$$M_{\alpha\beta} = i(y_\alpha\partial_\beta - y_\beta\partial_\alpha) , \tag{2.6}$$
$$S_{\alpha\beta} = i\sum_k (z_{k\alpha}\partial/\partial z_k^\beta - z_{k\beta}\partial/\partial z_k^\alpha) . \tag{2.7}$$

The second-order Casimir operator is

$$Q = \tfrac{1}{2} L_{\alpha\beta} L_{\alpha\beta} . \tag{2.8}$$

For tensors satisfying (2.3) the value of Q is determined by the rank n, the degree N, and the symmetry type. We find

$$\tfrac{1}{2} M_{\alpha\beta}M_{\alpha\beta} = N(N+4), \quad M_{\alpha\beta}S_{\alpha\beta} = -2n ,$$
$$\tfrac{1}{2} S_{\alpha\beta}S_{\alpha\beta} = n_1(n_1+4) + n_2(n_2+2) + n_3{}^2 .$$

Here $n_1 \geq n_2 \geq n_3 \geq 0$ are integers that label the symmetry type according to the lengths of the rows of the Young diagram. For a symmetric tensor field of rank 2 we have

$$Q = N(N+4) + 8 .$$

The value of Q for the physical representation is 9; therefore it cannot be carried by a symmetric tensor field of rank 2 unless the degree $N = -2 \pm 5^{1/2}$. (Even then we think it is impossible.)

The requirements $Q = 9$ and $N = -1, 0, 1, \ldots$ are satisfied only in the following three cases:

degree of field: $+1$, 0, -1,

symmetry type:  (2.9)

symbol: Λ, Ψ, R .

For typographical reasons, the symbols Λ, Ψ, and R will be used throughout to designate the respective symmetry types, as well as the tensors themselves. We conclude that linear conformal gravity must be formulated in terms of fields of these three types. [Relaxing the subsidiary conditions (2.3) gives nothing substantially new. If the physical representation is lost by imposing, for example, $\psi_{\alpha\alpha\beta\cdots} = 0$, then it is in fact carried by the tensor field $\psi'_{\beta\cdots} = \psi_{\alpha\alpha\beta\cdots}$ of lower rank.]

Absolute ground states

Next, we shall list the absolute ground states of these three tensor fields. An absolute ground state is a field that is an eigenstate of the energy operator L_{50} and that is annihilated by the energy-lowering operators $L_{0i} - iL_{5i}$, $i = 1, \ldots, 4$. The calculation is very simple provided the most efficient notation is used.[7] Tensors will be expressed as polynomials in vector variables, as in (2.1). Instead of $z_1, \ldots, z_4$ we use z, ξ, z, ξ and the abbreviations

$$z_{ij}=z_i\xi_j-z_j\xi_i,\quad z_{ijk}=z_{ij}z_k\ , \tag{2.10}$$
$$z_{ijkl}=z_{ij}z_{kl}\ .$$

These three quantities have symmetry types (2.9), and this is passed on, automatically, to tensor components. In this notation,

$$L_{50}=-y_+\partial_+ + y_-\partial_- + \cdots ,$$
$$y_\pm = y_5 \pm i y_0, \ldots$$
$$L_{0i}-iL_{5i}=y_+\partial_i+2y_i\partial_-+\cdots ,$$
$$-(L_{0i}+iL_{5i})=y_-\partial_i+2y_i\partial_++\cdots ,$$

where ellipses stand for similar terms involving z and ξ. The complete list of absolute ground states is

$$\Lambda_0=\sum y_+ z_{ij} M_{ij}\ , \tag{2.11}$$
$$\Psi_0^{\mathrm{GA}}=y_+{}^{-2}\sum (y_+z_{ijk}y_+ - y_+z_{ij+}y_k)M_{ijk}\ , \tag{2.12}$$
$$R_0=y_+{}^{-3}\sum y_+(z_{ijkl}y_+-2z_{ijk+}y_l)M_{ijkl}\ . \tag{2.13}$$

The sums are over the cyclic permutations of the first three indices following. The M's are numerical polarization tensors; they have the same symmetries as the z's defined in (2.10), and besides they are traceless. These properties of the M's fix the SO(4) representations as $(1,0)\oplus(0,1)$, $(\frac{3}{2},\frac{1}{2})\oplus(\frac{1}{2},\frac{3}{2})$, and $(2,0)\oplus(0,2)$, respectively. The L_{50} eigenvalues are -1,0 and $+1$. The absolute ground states are therefore cyclic vectors for the following representations:

$$\Lambda_0:\ D(-1,1,0)\oplus D(-1,0,1)\ ,$$
$$\Psi_0^{\mathrm{GA}}:\ D(0,\tfrac{3}{2},\tfrac{1}{2})\oplus D(0,\tfrac{1}{2},\tfrac{3}{2})\ ,$$
$$R_0:\ D(1,2,0)\oplus D(1,0,2)\ .$$

None of these is unitary. See Fig. 1.

The ground states of the physical representations, if they exist, can only be relative ground states. An eigenstate ψ of L_{50} is called a relative ground state if there is an invariant subspace V_0 such that the states $(L_{0i}-iL_{5i})\psi$, obtained by pushing down on ψ with the lowering operators, are in V_0, while ψ itself is not in V_0. Such states are ground states of subquotients of nondecomposable representations. In the next section we shall calculate the relative ground states, and among them we shall find those of the physical representations.

III. GAUGE FIELDS AND PHYSICAL STATES

The K structure of a representation of SO(4,2) is its restriction to the compact subgroup

$$K=\mathrm{SO}(4)\otimes\mathrm{SO}(2)\ .$$

The K structure of $D(E_0,j_1,j_2)$ is easy to calculate except at the "reduction points" (defined below). The raising operators transform (under the adjoint action) according to the representation $(\frac{1}{2},\frac{1}{2})$ of

$$\mathrm{so}(4)=\mathrm{su}(2)\oplus\mathrm{su}(2);$$

therefore the representation of so(4) that appears at energy $L_{50}=E_0+n$, $n=0,1,\ldots,$ is in the generic case

$$(j_1,j_2)\otimes\left[\left(\frac{n}{2},\frac{n}{2}\right)\oplus\left(\frac{n}{2}-1,\ \frac{n}{2}-1\right)\oplus\cdots\right]\ . \tag{3.1}$$

In the case of a representation $D(E_0,j_1,j_2)$ at a reduction point, only a subrepresentation of (3.1) occurs. A representation with minimal weight (E_0,j_1,j_2) is said to be at a reduction point if, within the generic K structure, there appears a weight (E_0',j_1',j_2') such that $D(E_0',j_1',j_2')$ and $D(E_0,j_1,j_2)$ are Weyl equivalent, that is, they have the same values for all the Casimir operators. All the irreducible representations of interest to us are at reduction points. We shall obtain information about nondecomposable representations by studying the approach of irreducible representations to the reduction points.

Tensor fields of type Λ

The first case is the simplest and explains our method, though it fails to reveal the physical representation. The fields

$$\Lambda_0(\epsilon)=y_+{}^{-\epsilon}\sum y_+z_{ij}M_{ij}$$

are absolute ground states for

$$D(\epsilon-1,1,0)\oplus D(\epsilon-1,0,1)\ .$$

The K structure is given, when $0<\epsilon<1$, by (3.1). Pushing up with the energy-raising operators one finds, among other states, the following with weights $(\epsilon,\frac{1}{2},\frac{3}{2})$ and $(\epsilon,\frac{3}{2},\frac{1}{2})$:

$$\Lambda_1(\epsilon)=y_+{}^{-1-\epsilon}\sum y_+z_{ij}y_kM_{ijk}\ .$$

The numerical coefficients M_{ijk} have the same properties as in (2.12). However, this happens only as long as $\epsilon\neq0$, for one obtains Λ_1 multiplied by ϵ. In the limit $\epsilon=0$ the states of the finite-dimensional representation

$$D(-1,1,0)\oplus D(-1,0,1)$$

form an invariant subspace V_0. The limit of Λ_1,

$$\Lambda_1=y_+{}^{-1}\sum y_+z_{ij}y_kM_{ijk} \tag{3.2}$$

is not in V_0; it is a relative ground state for

$$D(0,\tfrac{1}{2},\tfrac{3}{2})\oplus(0,\tfrac{3}{2},\tfrac{1}{2}),$$

and it is cyclic for the nondecomposable representation

$$[D(0,\tfrac{3}{2},\tfrac{1}{2})\to D(-1,1,0)] \oplus[D(0,\tfrac{1}{2},\tfrac{3}{2})\to D(-1,0,1)]\ . \quad (3.3)$$

Notation. If A,B are representations in spaces V_A, V_B, then $A\to B$ denotes a nondecomposable representation in $V_A\oplus V_B$, such that B is realized in V_B and A is realized on the quotient $(V_A\oplus V_B)/V_B$. We say that A leaks into B and that Λ_1 leaks down to Λ_0.

The K structure of (3.3) has lower multiplicities than that of

$$D(\epsilon-1,1,0)\oplus D(\epsilon-1,0,1)\ .$$

One may therefore inquire about the remaining states and especially about additional relative ground states. However, it is easy to see that the physical representation cannot occur. *Definition:* the helicity of a representation (j_1,j_2) of SO(4) is the number $h=j_1-j_2$. The ground state (and as a matter of fact all the states) of $D(3,2,0)$ has helicity $+2$, and that of $D(3,0,2)$ has helicity -2. Now (3.1) shows that the highest value of $|h|$ in $D(E_0,j_1,j_2)$ is j_1+j_2, so $D(E_0,1,0)$ has no states with helicity $h=\pm2$. It is also easy to show directly that no mode of a symmetry type Λ tensor field can have helicity higher than 1. To find the physical states we must investigate the other two cases. Before getting to that we make a useful digression.

The representation $D(0,\frac{3}{2},\frac{1}{2})$ has appeared in two ways: first irreducibly on Ψ tensors with the absolute ground state (2.12), and again as a quotient of (3.3) on Λ tensors with the relative ground state (3.2). There is an interesting coboundary operator that maps one to the other.

The coboundary operator d

Definition. The coboundary operator d acts on tensors of symmetry types Λ and Ψ to give tensors of symmetry types Ψ and R, respectively. If $\Lambda_{\alpha\beta}$ is antisymmetric, and $\Psi_{\alpha\beta\gamma}$ has mixed symmetry and is antisymmetric in α,β, then the action of d is defined by (compare Ref. 9)

$$(d\Lambda)_{\alpha\beta\gamma}=\text{Mix}\Lambda_{\alpha\beta,\gamma}\equiv2\Lambda_{\alpha\beta,\gamma}-\Lambda_{\beta\gamma,\alpha}-\Lambda_{\gamma\alpha,\beta}\ , \quad (3.4)$$

$$(d\Lambda)_{\alpha\beta\gamma\delta}=\text{Box}\Psi_{\alpha\beta\gamma,\delta} \equiv\Psi_{\alpha\beta\gamma,\delta}-\Psi_{\alpha\beta\delta,\gamma}+\Psi_{\gamma\delta\alpha,\beta}-\Psi_{\gamma\delta\beta,\alpha}\ . \quad (3.5)$$

The index that follows the comma should be interpreted as in

$$\Lambda_{\alpha\beta,\gamma}\equiv\text{Grad}_\gamma\Lambda_{\alpha\beta}=[y_\gamma\partial^2-(2\hat{N}+4)\partial_\gamma]\Lambda_{\alpha\beta}\ . \quad (3.6)$$

Applying d to the relative ground state (3.2) one obtains the absolute ground state (2.12). The leak is plugged because d annihilates the fields that carry the finite-dimensional representation

$$D(-1,1,0)\oplus(D(-1,0,1)\ .$$

The absolute ground state Ψ_0^{GA} and all the fields generated from it are exact (in the image of d). It is easy to check that

$$d\circ d=0\ . \quad (3.7)$$

We shall find the physcial representations in the cohomology space Ker d/Im d.

Tensor fields of type Ψ

Next, consider the fields

$$\Psi_0(\epsilon)=y_+^{-\epsilon}\Psi_0^{\text{GA}}\ .$$

This is exact only if $\epsilon=0$, and helicities higher than 1 can occur. In fact, this is an absolute ground state for

$$D(\epsilon,\tfrac{3}{2},\tfrac{1}{2})\oplus D(\epsilon,\tfrac{1}{2},\tfrac{3}{2})\ ,$$

in which the highest helicities are ±2 (if $\epsilon\neq0$). Pushing up with the raising operators one finds (among others) the states

$$\Psi_1(\epsilon)=y^{-2-\epsilon}\Sigma y_+z_{ijk}y_lM_{ijkl}\ . \quad (3.8)$$

The numerical tensor M has the same properties as in (2.13) and those properties make this a state with helicities ±2. It appears only as long as $\epsilon\neq0$. In the limit $\epsilon=0$ the states of

$$D(0,\tfrac{1}{2},\tfrac{3}{2})\oplus D(0,\tfrac{3}{2},\tfrac{1}{2})$$

generated from Ψ_0^{GA} form an invariant subspace that does not include the limit of (3.8):

$$\Psi_0^{\text{GH}}=y_+{}^{-2}\Sigma y_+z_{ijk}y_lM_{ijkl}\ . \quad (3.9)$$

This is a relative ground state for

$$D(1,2,0)\oplus D(1,0,2)\ .$$

It leaks down to Ψ_0^{GA} and is cyclic for the representation

$$[D(1,2,0)\to D(0,\tfrac{3}{2},\tfrac{1}{2})] \oplus[D(1,0,2)\to D(0,\tfrac{1}{2},\tfrac{3}{2})]\ . \quad (3.10)$$

The fields of the subrepresentation are exact, so the leak may be plugged by applying the coboundary operator. Applying d to the relative ground state

(3.9) we obtain the absolute ground state (2.13). The field R_0 and all those generated from it are thus exact.

The K structure of (3.10) is simpler than that of

$$D(\epsilon,\tfrac{1}{2},\tfrac{3}{2})\oplus D(\epsilon,\tfrac{3}{2},\tfrac{1}{2}) .$$

Among the missing states we shall discover states with weights (3,2,0) and (3,0,2) that are relative ground states of the physical representations.

The physical states

The $\epsilon\to 0$ limit of the K structure of $D(\epsilon,\frac{1}{2},\frac{3}{2})$ is shown in Fig. 2. The states that belong to $D(0,\frac{1}{2},\frac{3}{2})$ are indicated by dots and those of $D(1,0,2)$ by circles. The state of lowest energy that does not belong to either of these is indicated by a cross and has weight (3,2,0). This shows that the physical representations occur here. The ground state is not an absolute ground state, so it must leak down. The states of $D(1,0,2)$ are too far away in helicity, so it must leak to the states of $D(0,\frac{1}{2},\frac{3}{2})$. All those states are exact, while the physical states cannot be exact, as their helicity is too high to be carried by a tensor field of symmetry type Λ. The physical ground state is thus characterized by the fact that $(L_{0i}-iL_{5i})\Psi_0^{\rm PH}$ is exact, while $\Psi_0^{\rm PH}$ itself is not.

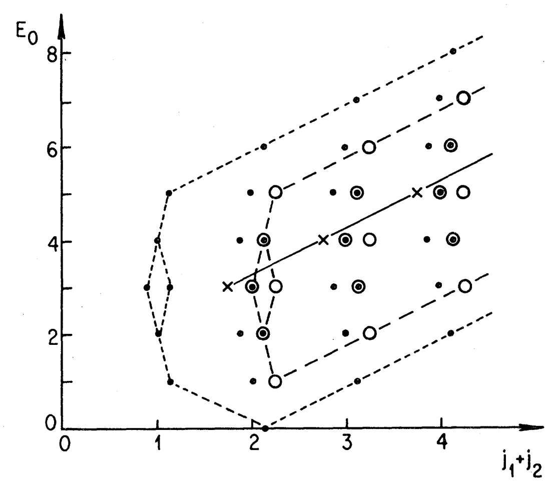

FIG. 2. The limit, as $\epsilon\to 0$, of the K structure of $D(\epsilon,\frac{1}{2},\frac{3}{2})$. The dots belong to $D(0,\frac{1}{2},\frac{3}{2})$, the circles to the ghost D(1,0,2), and the crosses to the physical representation D(3,2,0). Positive helicity $(j_1-j_2>0)$ is indicated by a shift to the left, negative helicity $(j_1-j_2<0)$ by a shift to the right. Thus, at $E_0=3$ and $j_1+j_2\approx 2$ we have the physical ground state (3,2,0), a ghost mode (3,0,2), and two (3,1,1) states of which one is a ghost and the other a gauge mode.

More precisely, the physical ground states may be uniquely characterized as fields with helicities ± 2 such that $(L_{0i}-iL_{5i})\Psi_0^{\rm PH}$ is exact. The result of a very long calculation is that

$$\Psi_0^{\rm PH}=(y_+{}^{-2}\Sigma y_-z_{ijk}y_l+4y_+{}^{-3}y_iy_ly_\alpha z_{j\alpha k})M_{ijkl} . \tag{3.11}$$

The polarization tensor M has the same properties as in (2.13). This formula was confirmed by another method that will be discussed in Sec. IV. We shall also find that $\Psi_0^{\rm PH}$ is, very remarkably, closed.

Tensor fields of type R

Tensor fields of symmetry type R cannot carry the physical representation. The only absolute ground state is R_0, Eq. (2.13), the ground state of the "ghost"

$$D(1,2,0)\oplus D(1,0,2) .$$

The relative ground state of $D(3,2,0)$, were it to exist, could not leak into $D(1,0,2)$ because the difference in helicity is too great to be bridged by the Lie algebra, nor can it leak into $D(1,2,0)$ because the values of the Casimir operators are not the same. There is therefore just one way to realize the physical representations in terms of tensor fields. (Of course, it is possible to write $\Psi^{\rm PH}$ as the trace of a tensor field of rank 5; but this, and other constructions in a similar vein, would not seem to introduce anything substantially different.)

The fact that tensors of symmetry type R cannot carry the physical representations has an important corollary, namely, that the physical modes are closed,

$$d\Psi^{\rm PH}=0 , \tag{3.12}$$

a remarkable fact that was first uncovered by a laborious, direct calculation. The point is that $d\Psi^{\rm PH}$ is of type R and, if not zero, would carry the physical representation. It is only necessary to verify that $d\Psi^{\rm PH}$ satisfies all the subsidiary conditions. To do that we first notice that the following identities hold:

$$\mathrm{Grad}_\alpha\mathrm{Grad}_\beta=\mathrm{Grad}_\beta\mathrm{Grad}_\alpha , \tag{3.13}$$

$$\mathrm{Grad}_\alpha\mathrm{Grad}_\alpha=0 , \tag{3.14}$$

$$y_\alpha\mathrm{Grad}_\alpha=-2\hat N(\hat N+1) ,$$

$$(\hat N+2)y_\alpha\mathrm{Grad}_\beta=(\hat N+1)\mathrm{Grad}_\beta y_\alpha+y_\beta\mathrm{Grad}_\alpha+2(\hat N+1)(\hat N+2)\delta_{\alpha\beta}$$

and remember that $\Psi^{\rm PH}$ satisfies all the subsidiary conditions. With this, the verification becomes nearly trivial; for example,

$$y_\alpha(d\Psi)_{\alpha\beta\gamma\delta}=y_\alpha(\Psi_{\alpha\beta\gamma,\delta}-\Psi_{\alpha\beta\delta,\gamma}+\Psi_{\gamma\delta\alpha,\beta}-\Psi_{\gamma\delta\beta,\alpha})$$
$$=2(\Psi_{\delta\beta\gamma}-\Psi_{\gamma\beta\delta}+\Psi_{\gamma\delta\beta})$$
$$=2\Sigma\Psi_{\beta\gamma\delta}=0\ .$$

IV. THE MINIMAL GUPTA-BLEULER TRIPLET

The physical states have now been found, accompanied by gauge modes in the nondecomposable representation

$$[D(3,2,0)\to D(0,\tfrac{1}{2},\tfrac{3}{2})]\oplus[D(3,0,2)\to D(0,\tfrac{3}{2},\tfrac{1}{2})]\ . \quad (4.1)$$

This space of tensor fields cannot be used to construct a free quantum field operator, since it does not have an invariant, nondegenerate symplectic structure.[7] It is necessary to find an additional copy of the subrepresentation

$$D(0,\tfrac{1}{2},\tfrac{3}{2})\oplus D(0,\tfrac{3}{2},\tfrac{1}{2});$$

the associated "scalar" modes are the field variables that are canonically conjugate to the "gauge" modes. That is, we must extend (4.1) to allow for quantization or, what is the same, the existence of a (homogeneous) propagator or reproducing kernel.

Since the irreducible representations $D(\epsilon,\frac{1}{2},\frac{3}{2})$ and $D(\epsilon,\frac{3}{2},\frac{1}{2})$ have nondegenerate (indefinite) symplectic structures, we expect to be able to extract the required additional modes from them. In fact, the relative ground state of the scalar representation is

$$\Psi_0^{\rm SC}=\lim_{\epsilon\to 0}(1/\epsilon)[\Psi_0(\epsilon)-d\Lambda_1(\epsilon)]\ .$$

Discarding a constant factor and a term proportional to $\Psi_0^{\rm GA}$ we get

$$\Psi_0^{\rm SC}=y_+{}^{-1}\Sigma y_+z_{ijk}M_{ijk}\ . \quad (4.2)$$

This has the correct properties, as we now verify.

Pushing down on (4.2) we find the state

$$\Psi_0^{\rm F}=y_+{}^{-1}\Sigma y_+z_{ij+}M_{ij}\ . \quad (4.3)$$

This is an absolute ground state for the finite-dimensional representation

$$D(-1,1,0)\oplus D(-1,0,1)\ .$$

Pushing up again from this state we find not $\Psi_0^{\rm SC}$, but $\Psi_0^{\rm GA}$. Hence $\Psi_0^{\rm F}$ leaks into the gauge modes, and (4.2) is a relative ground state for another copy of

$$D(0,\tfrac{3}{2},\tfrac{1}{2})\oplus D(0,\tfrac{1}{2},\tfrac{3}{2})\ .$$

[The discovery of new ground states does not contradict our previous results, since they do not satisfy all the subsidiary conditions, about which more below.]

Pushing up from (4.2) we find $\Psi_0^{\rm GH}$. Finally, we must push up three times from (4.2) to look for the physical representation. This is tricky, since it is necessary to evade the modes of $D(1,2,0)$ and $D(1,0,2)$, the ghosts. We construct an operator in the enveloping algebra that increases the energy by three units and that maps any $(\frac{3}{2},\frac{1}{2})$ state to a (0,2) state, and any $(\frac{1}{2},\frac{3}{2})$ state to a (2,0) state. Applying this to (4.2) we evade the ghosts and rediscover (or confirm) the physical ground states (3.11). There is an *a priori* possibility that further application of raising operators might reveal leakage into $D(4,\frac{3}{2},\frac{1}{2})$, $D(4,\frac{1}{2},\frac{3}{2})$, $D(5,1,0)$, and $D(5,0,1)$. That this does not in fact happen will be shown later, after we have evaluated the propagator.

We thus finally conclude that the states (4.2) are cyclic for two completed Gupta-Bleuler triplets,

$$D(0,\tfrac{1}{2},\tfrac{3}{2})\to\begin{bmatrix}D(3,2,0)\\D(1,0,2)\\D(-1,0,1)\end{bmatrix}\to D(0,\tfrac{1}{2},\tfrac{3}{2})\ , \quad (4.4)$$

and the helicity conjugate. The five ground states are listed in Table I.

TABLE I. The important ground states.

Mode		Type of ground state	Satisfies
Scalar	$\Psi_0^{\rm SC}=y_+{}^{-1}\Sigma y_+z_{ijk}M_{ijk}$	Relative	(4.5)—(4.8)
Finite	$\Psi_0^{\rm F}=y_+{}^{-1}\Sigma y_+z_{ij+}M_{ij}$	Absolute	(4.5)—(4.8)
Ghost	$\Psi_0^{\rm GH}=y_+{}^{-3}\Sigma y_+(z_{ijkl}y_+ -2z_{ijk+}y_l)M_{ijkl}$	Relative	(4.5)—(4.9)
Physical	$\Psi_0^{\rm PH}=(y_+{}^{-2}\Sigma y_-z_{ijk}y_l +4y_+{}^{-3}y_iy_ly_\alpha z_{j\alpha k})M_{ijkl}$	Relative	(4.5)—(4.10)
Gauge	$\Psi_0^{\rm GA}=(y_+{}^{-1}\Sigma y_+z_{ijk} -y_+{}^{-2}\Sigma y_+z_{ij+}y_k)M_{ijk}$	Absolute	(4.5)—(4.11)

Equations

Covariant equations that hold for a cyclic vector of a representation are satisfied on the whole representation space. Inspection of Table I confirms the following hierarchy.

(1) The entire Gupta-Bleuler triplet satisfies

$$(\partial^2)^2\Psi_{\alpha\beta\gamma}=0\ , \tag{4.5}$$

$$y_\alpha\Psi_{\alpha\beta\gamma}+y_\alpha\Psi_{\alpha\gamma\beta}=0\ , \tag{4.6}$$

$$\Psi_{\alpha\beta\beta}=0\ , \tag{4.7}$$

$$\mathrm{Grad}_\alpha\Psi_{\alpha\beta\gamma}=0\ . \tag{4.8}$$

(2) The subspace of physical states, ghosts and gauge fields satisfy, in addition, the stronger transversality condition

$$y_\alpha\Psi_{\alpha\beta\gamma}=0\ . \tag{4.9}$$

Equations (4.7)–(4.9) are the complete set of subsidiary conditions introduced in (2.3). The fact that the scalar modes cannot satisfy all these conditions was expected by analogy with electrodynamics.[7] Equation (4.9) is an imperfect analog of the Lorentz condition: it eliminates the scalar modes and cannot be satisfied by the quantum field operator, but it holds on a space that contains more than just physical states and gauge fields.

(3) Physical modes and gauge modes are closed,

$$d\Psi=0\ . \tag{4.10}$$

This is analogous to the Lorentz condition in the sense that it defines, within the Gupta-Bleuler triplet, the smallest invariant subspace that contains the physical states.

(4) Gauge modes are exact,

$$\Psi=d\Lambda\ . \tag{4.11}$$

Within the Gupta-Bleuler triplet (and also within some larger spaces) the space of physical states is precisely the cohomology space $\mathrm{Ker}\,d/\mathrm{Im}\,d$; the proof will come later.

2. The propagator and the quantum field operator

All the expressions for the ground states, collected in Table I, suggest that Ψ may usefully be considered as derived from a traceless tensor of rank 4:

$$\Psi=\Phi_{\alpha\beta\gamma\delta}\Sigma y_\alpha z_{\beta\gamma\delta}\ . \tag{4.12}$$

If $\partial^2\Phi=0$, then $(\partial^2)^2\Psi=0$. The homogeneous propagator for the equation $\partial^2\Phi=0$ is well known; it can be written $(y\cdot y')^{-1}$, if this formal expression is interpreted as a distribution with positive-energy Fourier components. This suggests that the propagator for Ψ is

$$K(y,z,\xi;y',z'\xi')=(y\cdot y')^{-1}y_\alpha z_{\beta\gamma\delta}T\Sigma y'_\alpha z'_{\beta\gamma\delta}\ , \tag{4.13}$$

where T is the traceless projection operator. Expanding $(y\cdot y')^{-1}$ as a Fourier series we get after easy rearrangement the Fourier series for K, with energies $-1,0,1,\ldots$:

$$K(\ ;')=\sum_m \sigma_m\Psi_m(\ )\overline{\Psi}_m(')$$
$$=\Psi_0^{\rm F}\overline{\Psi}_0^{\rm F}+\Psi_0^{\rm SC}\overline{\Psi}_0^{\rm GA}+\Psi_0^{\rm GA}\overline{\Psi}_0^{\rm SC}+\cdots\ . \tag{4.14}$$

Here $\sigma_m=\pm1$; both signs appear since K is not a positive operator.

Since $\Psi_0^{\rm SC}$ appears here, so do all the states generated from this mode; that is, all the modes of both Gupta-Bleuler triplets. But this is not all. It is easy to see that (4.13) satisfies

$$(\partial^2)^2K=0,\quad y\cdot\partial_\xi K=0\ ,$$
$$\partial_z\cdot\partial_\xi K=0\ . \tag{4.15}$$

This means that each mode satisfies (4.5)–(4.7), but not all modes satisfy (4.8). It can be verified that the extra modes are not Weyl equivalent to either of the physical representations; therefore (4.13) can be improved by subtracting them off, if convenient. (Strong conditions have to be imposed on the interactions to prevent the propagation of unphysical fields within the triplets, and this will decouple the extra modes as well.)

The distribution $(y\cdot y')^{-1}$ is the propagator for the singleton $D(1,0,0)$. The K structure is highly degenerate, consisting of weights of the form $(2j+1,j,j)$ only. From this it may be inferred that the space of modes of K does not include $D(4,\frac{1}{2},\frac{3}{2})$, $D(5,1,0)$, or their helicity conjugates, and that these representations do not occur in the Gupta-Bleuler triplets.

The free quantum field operator is defined by

$$\hat{\Psi}(y,z,\xi)=\sum_m[\Psi_m(y,z,\xi)a_m+\overline{\Psi}_m(y,z,\xi)a_m^*]\ . \tag{4.16}$$

The operators a_m annihilate the vacuum, $a_m|0\rangle=0$, a_m commutes with a_n, and a_m^* commutes with a_n^*, while

$$[a_m,a_n^*]=\sigma_m\delta_{mn}\ .$$

This is covariant, indefinite-metric, Gupta-Bleuler quantization. Covariance is confirmed by the fact that

$$\langle 0|\hat{\Psi}(y,z,\xi)\hat{\Psi}(y',z',\xi')|0\rangle=K(\ ,')\ . \tag{4.17}$$

V. LINEAR CONFORMAL GRAVITY

In this section we shall first formulate an action principle in Dirac's six-cone notation,[5] then transcribe it to Minkowski notation.

Linear conformal gravity is described by a six-tensor field of rank 3, mixed symmetry and of degree 0:

$$\Psi_{\alpha\beta\gamma}=-\Psi_{\beta\alpha\gamma},\ \sum_{\text{cycl}}\Psi_{\alpha\beta\gamma}=0\ ,\qquad \hat{N}\Psi_{\alpha\beta\gamma}=0\ ,\ \ \hat{N}\equiv y\cdot\partial\ . \tag{5.1}$$

We shall impose tracelessness, $\Psi_{\alpha\beta\beta}=0$, as an *a priori* constraint, but no others. [An alternative action principle, with the additional constraint (4.6), is also possible.]

The simplest choice of action is

$$\int dy[\tfrac{1}{2}\Psi\cdot(\partial^2)^2\Psi-\Psi\cdot j]\ . \tag{5.2}$$

Here

$$\partial^2\equiv(\partial/\partial y_\alpha)^2\ ,$$

and integration over the cone has the usual meaning.[10] The kinetic term has degree -4 as required by invariance. The external current j is a tensor field of rank 3 and degree -4, traceless and with the same symmetry properties as Ψ. The field equation is

$$(\partial^2)^2\Psi_{\alpha\beta\gamma}=j_{\alpha\beta\gamma}\ . \tag{5.3}$$

We have shown, in Sec. IV, that the free-field equation $(\partial^2)^2\Psi=0$ admits solutions that can be associated with the physical states. We must now find a set of constraints and/or boundary conditions, sufficient to ensure that only physical modes propagate.

Constraints

From (5.3) it immediately follows, by contraction with y_α, that

$$y_\alpha(\partial^2)^2\Psi_{\alpha\beta\gamma}=\partial^2\text{Grad}_\alpha\Psi_{\alpha\beta\gamma}=y_\alpha j_{\alpha\beta\gamma}\ ,\qquad \text{Grad}_\alpha\equiv y_\alpha\partial^2-(2\hat{N}+4)\partial_\alpha,\ \ \hat{N}\equiv y_\alpha\partial_\alpha\ . \tag{5.4}$$

Similarly, applying Grad_α,

$$\text{Grad}_\alpha(\partial^2)^2\Psi_{\alpha\beta\gamma}=(\partial^2)^3y_\alpha\Psi_{\alpha\beta\gamma}=\text{Grad}_\alpha j_{\alpha\beta\gamma}\ . \tag{5.5}$$

Finally, multiplying (5.3) by y_δ and symmetrizing, we get

$$\text{Box}\,y_\delta(\partial^2)^2\Psi_{\alpha\beta\gamma}=\partial^2(d\Psi)_{\alpha\beta\gamma\delta}=\text{Box}\,y_\delta j_{\alpha\beta\gamma}\ . \tag{5.6}$$

The coboundary operator d and the Young symmetrizer Box were defined by Eq. (3.5).

The fields $\text{Grad}\cdot\Psi$, $y\cdot\Psi$, and $d\Psi$, appearing in the middle terms of (5.4), (5.5), and (5.6) are unphysical in the sense that all these quantities vanish if Ψ describes a free, propagating physical state. All the components of the Gupta-Bleuler triplet satisfy $\text{Grad}\cdot\Psi=0$, so it is possible to impose this condition as a strong constraint on the field operator. All the modes of the triplet also have the property that the symmetric part of $y\cdot\Psi$ vanishes. Therefore, this can also be imposed as a strong constraint on the quantum field, and in fact the free-field operator constructed in Sec. IV does satisfy this condition. The antisymmetric part of $y\cdot\Psi$ describes the scalar modes (and the 20 modes of the finite representation) of the triplets; they are canonically conjugate to the gauge modes, and analogous to the scalar modes $\partial\cdot A$ of QED. Such modes cannot be excluded from the free quantum field operator; they have to be constrained by imposing boundary conditions on the physical states, as $\partial\cdot A$ is constrained in QED. The most difficult part of the problem is presented by the field $d\Psi$. If $\text{Grad}\cdot\Psi$ and $y\cdot\Psi$ vanish, then the remaining modes described by $d\Psi$ are ghost modes. Therefore, this field also has to be constrained.

A propagating mode is a mode that is carried by the field in an empty region of space-time. To prevent all the unphysical modes, including the ghosts, from propagating, we must impose boundary conditions that ensure the effective vanishing of $y\cdot\Psi$, $\text{Grad}\cdot\Psi$, and $d\Psi$ in empty space.

We shall require that

$$\text{Grad}_\alpha\Psi_{\alpha\beta\gamma}\simeq\omega_{\beta\gamma},\ \ y_\alpha\Psi_{\alpha\beta\gamma}\simeq w_{\beta\gamma}\ , \tag{5.7}$$

$$(d\Psi)_{\alpha\beta\gamma\delta}\simeq s_{\alpha\beta\gamma\delta}\ . \tag{5.8}$$

All these "weak equalities" are to be understood in the same sense, so it is enough to explain one of them. The interpretation of ω, w, and s will be discussed subsequently.

The physical in states will be required to satisfy the following boundary condition,

$$(d\Psi-s)^+\,|\,\text{in}\rangle=0\ , \tag{5.9}$$

where $+$ indicates the positive-frequency part. If $d\Psi-s$ is a free field, then this condition is preserved by the time development, and all matrix elements of $d\Psi-s$, between states that satisfy the boundary conditions, vanish. We express this by (5.8).

As Eqs. (5.4)–(5.6) show, $\text{Grad}\cdot\Psi-\omega$, $y\cdot\Psi-w$, and $d\Psi-s$ are free fields if

$$y_\alpha j_{\alpha\beta\gamma}=\partial^2\omega_{\beta\gamma},\ \ \text{Grad}_\alpha j_{\alpha\beta\gamma}=(\partial^2)^3w_{\beta\gamma}\ , \tag{5.10}$$

$$\text{Box}\,y_\delta j_{\alpha\beta\gamma}=\partial^2 s_{\alpha\beta\gamma\delta}\ , \tag{5.11}$$

for, in that case

$$\partial^2(\text{Grad}\cdot\Psi-\omega)=0\ ,$$

$$(\partial^2)^3(y\cdot\Psi-w)=0\ , \tag{5.12}$$

$$\partial^2(d\Psi-s)=0\ . \tag{5.13}$$

It is therefore necessary that j, ω, w, and s be related as in (5.10) and (5.11). In order that (5.7) and (5.8) have the desired implications, it is necessary that ω, w, and s, rather than j, be interpreted as the external sources. That is, an empty region of space time is one in which $\omega=w=s=0$. In that case the ghost and the other unphysical field components vanish in empty space; that is, they do not propagate. This mechanism for preventing propagation is not completely novel; it operates in several field theories that contain an excessive number of field variables. The most notable example, and perhaps a very relevant analog, is the nonpropagating torsion of Einstein-Cartan theory.[11]

The only free modes that satisfy $\text{Grad}\cdot\Psi=y\cdot\Psi=d\Psi=0$ are the physical and gauge modes. Therefore, if (5.7) and (5.8) hold, then these are the only propagating modes. To eliminate the gauge modes we must finally impose gauge invariance on the interaction term in the action (5.2). A gauge field is a field that is exact,

$$\Psi_{\alpha\beta\gamma}=(d\Lambda)_{\alpha\beta\gamma}=\text{Mix Grad}_\gamma\Lambda_{\alpha\beta},\ \ \Lambda_{\alpha\beta}=-\Lambda_{\beta\gamma}\ . \tag{5.14}$$

The Young symmetrizer Mix was defined by Eq. (3.4). We have

$$\int dy\, j\cdot d\Lambda=-\int dy\,\Lambda\cdot(\text{Grad}\cdot j)\ . \tag{5.15}$$

If this is to vanish for all Λ, then it is necessary that[12]

$$\text{Grad}_\alpha(j_{\alpha\beta\gamma}-j_{\alpha\gamma\beta})=(\partial^2)^3(w_{\beta\gamma}-w_{\gamma\beta})=0\ . \tag{5.16}$$

Finally, it should be mentioned that if the asymptotic in field is the free-field operator constructed in Sec. IV, then the symmetric part of $y\cdot\Psi\simeq w$ must vanish.

Minkowski notation

We shall now translate the theory to Minkowski notation, and show that the dynamical content is the same as in the linear approximation to general relativity.

The mapping between Dirac's projective six-cone and Minkowski space is given by

$$x^\mu=y^\mu/\lambda\cdot y,\ \ \mu=0,1,2,3\ ,$$

$$\lambda\cdot y=y_4+y_5\ .$$

We introduce two additional variables,

$$x^+=\ln\lambda\cdot y,\ \ x^B=y^2(\lambda\cdot y)^{-2}\ .$$

On the cone $x^B=0$ and the variable x^+ drops out.

Tensors of degree N on the cone are related to tensors on Minkowski space by ($\tilde\psi$ depends on $x^0,\ldots,x^3$ only)

$$\begin{aligned}&\psi_{\alpha\beta}\ldots(y)=(\lambda\cdot y)^N x_\alpha^a x_\beta^b\cdots\tilde\psi_{ab}\ldots(x)\ ,\\ &x_\alpha^a=\lambda\cdot y(\partial x^a/\partial y^\alpha),\ \ a=0,1,2,3,+,B\ .\end{aligned} \tag{5.17}$$

Explicitly, with $\mu=0,\ldots,3$ and $\alpha=0,\ldots,5$,

$$x_\alpha^\mu=\delta_\alpha^\mu-x^\mu\lambda_\alpha,\ \ x_\alpha^+=\lambda_\alpha\ ,$$

$$x_\alpha^B=2y_\alpha/\lambda\cdot y-2\lambda_\alpha x^B\ .$$

A "covariant derivative" may be defined by

$$\text{Grad}_\alpha\psi_\beta\ldots(y)=-2(N+1)(\lambda\cdot y)^{N-1}\times x_\alpha^a x_\beta^b\cdots\nabla_a\tilde\psi_b\ldots\ . \tag{5.18}$$

Working this out one finds

$$\begin{aligned}&\nabla_\mu\tilde\psi_a\ldots=\partial_\mu\tilde\psi_a\ldots+\Gamma^b_{\mu a}\tilde\psi_b\ldots+\cdots\ ,\\ &\nabla_+\tilde\psi_a\ldots=N\tilde\psi_a\ldots\ ,\\ &\nabla_B\tilde\psi_a\ldots=\frac{-1}{4N+4}\nabla_\mu\nabla_\mu\tilde\psi_a\ldots\ .\end{aligned} \tag{5.19}$$

The only non-zero components of the "connection" are

$$\Gamma^\nu_{\mu+}=-\delta^\nu_\mu,\ \ \Gamma^B_{\mu\nu}=2\delta_{\mu\nu}\ ,\ \ \mu,\nu=0,\ldots,3\ . \tag{5.20}$$

For the sources j, ω, w, and s we shall use capital letters J, Ω, W, and S to denote the corresponding Minkowski tensors, and for the field Ψ and the gauge parameter Λ we use the same letters, thus dispensing with the tilde, as the risk of confusion is negligible.

The main formulas now appear as follows. The action (5.2) is

$$\int d^4x[\tfrac{1}{2}\Psi\cdot(\nabla_\mu\nabla_\mu)^2\Psi-\Psi\cdot J]\ . \tag{5.21}$$

The field equation (5.3) is

$$(\nabla_\mu\nabla_\mu)^2\Psi_{abc}=J_{abc}\ ,\ \ a,b,c=0,\ldots,3,+,B\ . \tag{5.22}$$

The constraints (5.7) and (5.8) are

$$\nabla_a\Psi_{abc}\simeq\Omega_{bc},\ \ \Psi_{+bc}\simeq W_{bc}\ , \tag{5.23}$$

$$(d\Psi)_{abcd}\equiv\text{Box}\nabla_d\Psi_{abc}\simeq S_{abcd}\ . \tag{5.24}$$

The expressions for $d\Lambda$ and $d\Psi$ are exactly as in Eqs. (3.4) and (3.5), with Latin indices replacing

Greek indices and the comma denoting covariant differentiation with ∇.

To extract the dynamical content of these equations it is necessary to write them out in full detail. As they become very long in the general case, and as we are primarily interested in propagation in empty regions of space-time, we shall compromise by limiting ourselves to regions in which Ω and W vanish. Thus

$$\Omega = W = 0 \,, \tag{5.25}$$

$$\nabla_a \Psi_{abc} \simeq 0, \quad \Psi_{+bc} \simeq 0 \,. \tag{5.26}$$

This implies the following constraints on S:

$$S_{+abc} = 0, \quad S_{\mu b \mu d} = 0 \,, \tag{5.27}$$

$$\nabla_d S_{dabc} = \partial_\mu S_{\mu abc} + (4-2n) S_{Babc} = 0 \,, \tag{5.28}$$

where n is the number of B indices among a,b,c. We emphasize that (5.26)–(5.28) are valid only in those regions of space-time in which Ω and W happen to vanish.

The trace $\psi_{\alpha\alpha\beta}\cdots$ of any six-tensor is reexpressed in Minkowski notation by

$$\widetilde{\psi}_{aab}\cdots \equiv \widetilde{\psi}_{\mu\mu b}\cdots + 2\widetilde{\psi}_{+Bb}\cdots + 2\widetilde{\psi}_{B+b}\cdots \,.$$

Thus, in particular,

$$\nabla_a \psi_{abc} \equiv \nabla_\mu \psi_{\mu bc} + 2\nabla_+ \psi_{Bbc} + 2\nabla_B \psi_{+bc} \,.$$

Dynamical content, constraints

If Ψ is exact, $\Psi = d\Lambda$, then, in view of (5.26), Λ_{+b} and $\nabla_a \Lambda_{ab}$ must vanish. In this case the gauge parameter is a complex consisting of a skew tensor ($\Lambda_{\mu\nu}$) and a vector field ($\Lambda_\mu \equiv \Lambda_{B\mu}$), with

$$\partial_\mu \Lambda_{\mu\nu} + 6\Lambda_\nu = 0 \,. \tag{5.29}$$

In this case the nonzero components of $d\Lambda$ are

$$(d\Lambda)_{\mu\nu\lambda} = \mathrm{Mix}\,\partial_\lambda \Lambda_{\mu\nu} + 6(\delta_{\mu\lambda}\Lambda_\nu - \delta_{\nu\lambda}\Lambda_\mu) \,, \tag{5.30}$$

$$(d\Lambda)_{B\mu\nu} = \tfrac{1}{8}\partial^2 \Lambda_{\mu\nu} + \tfrac{3}{2}(\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu) \,, \tag{5.31}$$

$$(d\Lambda)_{\mu BB} = \tfrac{1}{8}\partial^2 \Lambda_\mu \,. \tag{5.32}$$

In view of (5.26), the field Ψ is a complex consisting of four tensors:

$$\gamma_{\mu\nu\lambda} \equiv \Psi_{\mu\nu\lambda}, \quad h_{\mu\nu} \equiv \Psi_{B\mu\nu} + \Psi_{B\nu\mu} \,, \tag{5.33}$$

$$f_{\mu\nu} \equiv \Psi_{B\nu\mu} - \Psi_{B\mu\nu} = \Psi_{\mu\nu B}, \quad a_\mu \equiv \Psi_{\mu BB} \,. \tag{5.34}$$

This complex is a gauge field (that is, Ψ is exact), if there are gauge parameters ($\Lambda_{\mu\nu}, \Lambda_\mu$) such that

$$\gamma_{\mu\nu\lambda} = \tfrac{1}{3} T\,\mathrm{Mix}\,\partial_\lambda \Lambda_{\mu\nu}, \quad h_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \,, \tag{5.35}$$

$$f_{\mu\nu} = -\tfrac{1}{12}\partial^2 \Lambda_{\mu\nu}, \quad a_\mu = \tfrac{1}{24}\partial^2 \Lambda_\mu \,. \tag{5.36}$$

Here T is the traceless projection operator.

Equations (5.26) are expressed by

$$\partial_\sigma(\gamma_{\sigma\mu\nu} + \gamma_{\sigma\nu\mu}) + 4h_{\mu\nu} = 0 \,, \tag{5.37}$$

$$\partial_\sigma \gamma_{\mu\nu\sigma} + 8f_{\mu\nu} = 0, \quad \partial_\sigma h_{\sigma\mu} - 8a_\mu = 0 \,. \tag{5.38}$$

The remaining content of (5.24) is

$$S_{\mu\nu\lambda\rho} = \mathrm{Box}(\partial_\rho \gamma_{\mu\nu\lambda} + \delta_{\mu\rho} h_{\nu\lambda} - \delta_{\nu\rho} h_{\mu\lambda}) \,, \tag{5.39}$$

$$S_{B\nu\lambda\rho} = \tfrac{1}{4}\partial^2 \gamma_{\lambda\rho\nu} + \mathrm{Mix}\,\partial_\nu f_{\lambda\rho} + 4(\delta_{\nu\rho} a_\lambda - \delta_{\nu\lambda} a_\rho) \,, \tag{5.40}$$

$$S_{B\nu B\rho} = \tfrac{1}{4}\partial^2 h_{\nu\rho} - 2(\partial_\nu a_\rho + \partial_\rho a_\nu) \,. \tag{5.41}$$

As a good check one recovers (5.28) from these equations.

The propagating states of the theory are described by the space of solutions of (5.37)–(5.41), with $S=0$, modulo the subspace of exact solutions defined by (5.35) and (5.36). We shall prove that this space is precisely the space of physical, massless states with helicity ± 2 of ordinary, linearized gravity.

From (5.38) and (5.41) we get

$$4S_{B\nu B\rho} = \partial^2 h_{\nu\rho} - \partial_\nu \partial_\sigma h_{\sigma\rho} - \partial_\rho \partial_\sigma h_{\sigma\nu} + \partial_\nu \partial_\rho h_{\sigma\sigma} \equiv (d'h)_{\nu\rho} \,. \tag{5.42}$$

This is just the linear approximation to Einstein's field equation, if the left side can be interpreted as the energy momentum tensor of the source. This equation is invariant under gauge transformations of the type

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \equiv h_{\mu\nu} + (d'\xi)_{\mu\nu} \,. \tag{5.43}$$

Equations (5.42) and (5.43) define the coboundary operator d', and gauge invariance of (5.42) is just the statement

$$d' \circ d' = 0 \,.$$

Every solution of (5.37)–(5.41) thus determines a (traceless) solution of (5.42), and it is also easy to see that every traceless solution of (5.42) determines, up to an exact field, a solution of (5.37)–(5.41). (The original six-tensor Ψ is traceless, and when Ψ_{+ab} vanishes then this implies that γ and h are traceless, too.) If Ψ is exact, then (5.35) shows that h is exact. The inverse is also true, as can easily be verified; that is, a solution of (5.37)–(5.41) is exact if h is exact. *The space of solutions of this set of equations, modulo the subspace of exact solutions, is therefore precisely the space* $\mathrm{Ker}\,d'/\mathrm{Im}\,d'$ *of physical states of linearized Einstein gravity.*

Dynamical content, wave equation

The principal constraint Eq. (5.8) is much stronger than the wave equation (5.3), but we do well to emphasize that one cannot simply postulate the former and forget about the latter. The free modes allowed by the constraints are not sufficient to define a free quantum field satisfying nontrivial, invariant commutation relations. The free-field operator does not satisfy the constraints. Calculations of the S-matrix have to be based on the wave equation or on the action. The constraints are to be applied to the asymptotic in states; they allow an arbitrary configuration of incoming gravitons while freezing the unphysical modes. The best way to deal with the constraints may be to use Lagrangian multipliers.[13]

Nevertheless, what we want to examine here is whether the wave equation imposes any additional conditions on Ψ, beyond those fixed by the constraints. Equation (5.6) is, of course, implied by the constraints, so the question is what is lost by the mapping

$$j_{\alpha\beta\gamma} \rightarrow \mathrm{Boxy}_{\delta} j_{\alpha\beta\gamma} \equiv k_{\alpha\beta\gamma\delta} \ .$$

The associated Minkowski tensors are related by

$$K_{\mu\nu\lambda\rho}=0, \quad K_{B\mu\nu\lambda}=J_{\lambda\nu\mu} \ ,$$

$$K_{B\mu B\lambda}=-J_{B\mu\lambda}-J_{B\lambda\mu} \ .$$

The kernel of this mapping is therefore the antisymmetric part of $J_{B\mu\nu}$, and $J_{\mu BB}$. The information that is lost from (5.3) to (5.6) is precisely the corresponding components of (5.3). These are fourth-order equations for $f_{\mu\nu}$ and for a_μ and they merely limit the choice of gauge.

The dynamical content of the wave equation may be found by applying ∂^2 to (5.39)–(5.41). Thus, in particular,

$$(\partial^2)^2 h_{\mu\nu} - \partial^2(\partial_\mu \partial_\sigma h_{\sigma\nu} + \partial_\nu \partial_\sigma h_{\sigma\mu}) = 0$$

in empty space. This equation is consistent with the wave equation of linearized Weyl gravity,[2] since $h_{\sigma\sigma}=0$ and $\partial_\sigma \partial_\tau h_{\sigma\tau}=0$.

VI. SUMMARY AND OUTLOOK

The main input into this construction of linear conformal gravity was to insist that the propagating modes must be a pair of massless particles with helicities ± 2. Our first conclusion is that there is just one possible choice of tensor structure; the conformal field tensor (in Dirac's six-cone notation) can only be a tensor of rank 3 with mixed symmetry and with degree zero. The wave operator is a fourth-order differential operator. A free quantum field operator that satisfies conformally invariant commutation relations was constructed explicitly, and the propagator was calculated. The Fourier transform is of order p^{-4}, which hints at better renormalizability of our theory as compared to usual linear gravity.

It turned out that the field operator inevitably carries along a nonunitary ghost (besides the expected gauge and "scalar" modes). This is quite similar to the ghost that was turned up by Stelle[2] in his investigation of higher-order renormalizable theories of gravity. Our main result is that the ghost can be exorcised by imposing suitable constraints on the external current, and boundary conditions on the physical states. The constraints that are specifically directed against the ghost are Eqs. (5.9) and (5.11).

What should be the next step? It would certainly be valuable to reach a better understanding of this linear theory. First of all, the geometric interpretation is not entirely clear. The basic fields (in Minkowski notation) are a symmetric tensor of rank 2 that may be a metric, and a mixed-symmetry tensor of rank 3 that may be a torsion field or a connection.

It should also be very instructive to study a concrete model of the external sources. The final equation (5.42) obtained for the "metric" field is just Einstein's linearized field equation, and the source term must be related to an energy-momentum tensor. This source term is a part of a larger complex that in Dirac's notation is a six-tensor of rank 4, subject to the constraint (5.11). A distinct obstruction is the fact that the source term in (5.42) has conformal degree -3, while the energy-momentum tensor of any conformally invariant field theory has conformal degree -4. This discrepancy was anticipated, since Einstein's theory has a dimensional coupling constant. It is a strong indication that all interactions with matter violate conformal invariance. Only self-interacting gravity can be conformally invariant.

The search for a genuine interacting field theory that respects the constraints must eventually lead to a nonlinear theory with gravitational self-interaction. It is strongly suggested by our experience with the analogous problem in ordinary gravity that the nonlinear theory must have a non-Abelian gauge group; the discovery of this group would be a key to the theory. The gauge parameters of the linear theory is a space of antisymmetric, Dirac six-tensor fields,

$$\Lambda_{\alpha\beta} = -\Lambda_{\beta\alpha}, \quad \alpha,\beta=0,\ldots,5 \ .$$

The finite-dimensional representation of the conformal group that is associated with such fields is just the adjoint representation, which suggests that the non-Abelian gauge group may be a local extension of the conformal group. Such an extension is defined infinitesimally by the Lie algebra of differential operators of the form

$$\Lambda = \Lambda_{\alpha\beta}(y) L_{\alpha\beta} \; ,$$

where $(L_{\alpha\beta})$ are the operators of any representation of so(4,2). However, this identification of Λ with the Lie algebra of local conformal transformations is obstructed by the fact that the degree of homogeneity (essentially, the conformal degree) of Λ is $+1$, rather than zero. If Λ and Λ' have degree N, then the commutator $[\Lambda,\Lambda']$ has degree $2N$. Hence, either $N=0$, or we have a Kac-Moody extension of the local conformal algebra. The question of reconciling the degrees thus intrudes once more, this time on a much more fundamental level.

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[12]We do not exclude the possibility that this condition could be dispensed with. The reason is that if the gauge field (5.14) is a free field, then $(\partial^2)^3\Lambda=0$. In this case (5.10), $\text{Grad}\cdot j=(\partial^2)^3 w$, is sufficient to annul the integral (5.15).

[13]The fact that a large part of the dynamics is expressed by the constraints is unusual but not without precedent. See the model proposed by Y. Nambu, Phys. Rev. D 7, 2405 (1973), and the analysis of it by F. Bayen and M. Flato, *ibid.* 11, 3049 (1975).