Analysis of radiative pion decay $\pi^- \to e^- \nu\gamma$

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We give a quantitative account of radiative pion decays. By the chiral structure of the interaction and the fact of the low mass ratio between an electron and a pion, one can suppress the inner-bremsstrahlung contribution and correspondingly enhance the structure-dependent contribution by looking at decays with the emission of a negative-helicity electron. By this observation, we examine a way of determining the value of the structure-dependent parameter γ, giving particular attention to the decay spectra with this electron helicity. We also investigate the effects of a neutrino mass in this process.

I. INTRODUCTION

The radiative pion decay $\pi^- \to e^- \nu\gamma$ has been studied by various authors since the late 1950's.[1–7] One of the purposes in the study of this decay process is to extract a phenomenological value for the parameter γ, the ratio between axial-vector and vector pion form factors. As shown by Depommier *et al.*,[8] because of the quadratic dependence on the parameter γ, two possible values could be obtained through the counting of total decay events under certain selected conditions for the electron and photon energies. Although one may hope to resolve the ambiguity by looking at the detailed energy spectra, their results prove that it will be quite difficult to do so in practice, given the limited sensitivity of the experiment. About one decade later, when a second experiment was performed by Stetz *et al.*,[9] this situation was apparently not improved. The main motivation of the present work is then to search for some other specific spectra which may better serve this purpose.

The basic idea is to look for a kinematic configuration where the structure-dependent (SD) contribution is enhanced relative to the known contribution from inner bremsstrahlung (IB). Fortunately, the chiral structure of the usual charged weak currents gives us a simple way to achieve this objective. Since normal π_{l2} decays are forbidden by chirality selection rules, their amplitudes, and hence the IB amplitudes, are proportional to m_l (for any lepton helicity). However, the chirality argument applied to radiative decay only suppresses the SD amplitude for the emergent charged lepton of positive helicity. Therefore, we concentrate on the spectra for charged leptons of negative helicity.

Our calculation is carried out with a nonvanishing neutrino mass. It was initially thought that this might show up significantly in the lepton energy spectrum. However, essentially because of the chirality arguments, our final results depend rather insensitively on this quantity.

The essentials of our calculation and the numerical results are contained in Sec. II. We conclude in Sec. III with a discussion on the feasibility of such a measurement. The detailed algebraic results may be found in the Appendix.

II. THE DECAY AMPLITUDE WITH A NONVANISHING NEUTRINO MASS AND THE NUMERICAL RESULTS

The amplitude of the radiative pion decay with a massless neutrino can be found in Refs. 6 and 10. We shall follow here the notations in Ref. 6 and write down directly the amplitude for the case when $m_\nu \neq 0$:

$$\langle l^- \nu\gamma, \mathrm{out}|\pi^-\rangle = \frac{iGa}{2{P_0}^{1/2}}\left[\left(\frac{1+\gamma}{2}\right)\mathfrak{F}^-_{\mu\nu} - \left(\frac{1-\gamma}{2}\right)\mathfrak{F}^+_{\mu\nu}\right]P_\nu l_\mu + \frac{ef_\pi(m_l-m_\nu)}{2(2P_0)^{1/2}}\left(\frac{m_l m_\nu}{E_l E_\nu}\right)^{1/2}(\mathfrak{F}^+_{\mu\nu}+\mathfrak{F}^-_{\mu\nu})\bar{u}(l)\left[\frac{p_\mu P_\nu}{(p\cdot k)(P\cdot k)}+\frac{i\sigma_{\mu\nu}}{4p\cdot k}\right](1+d\gamma_5)v(\nu) \;, \tag{2.1}$$

where a new parameter

$$d \equiv \frac{m_l+m_\nu}{m_l-m_\nu}$$

is now introduced.

Due to the chirality argument, Eq. (2.1) gives us an electron spectrum which is infrared-divergence free for the case where the electron has a negative helicity regardless of how the photon is polarized. As mentioned in the Introduction, this particular electron spin state is, at the same time, the one we are interested in when trying to resolve the ambiguity of the parameter γ. Note, however, that when the photon is polarized perpendicular to the product plane, an infrared-divergence-free spectrum is also obtained for the case where the electron has a positive helicity. This could be understood by recalling that the classical $\vec{A}$ field is parallel to the direction of motion of the charged particle. Hence, in the situation here, we may expect that most of the soft photons are polarized along the product plane, and those spectra corresponding to the situation where the photon is polarized perpendicular to the product plane will be infrared convergent.

The explicit differential decay rate with respect to the electron energy and photon energy is given in the Appendix. Here we shall give directly only the numerical result for the situation where the electron has the favored helicity, namely, the negative one. The electron spectrum for this particular case is shown in Fig. 1, where the various values of γ are taken from the data obtained by Stetz *et al.*,[9] which gives

$$\gamma = 0.15 \quad \text{or} \quad \gamma = -2.07 \ ,$$

and the earlier one obtained by Depommier *et al.*,[8] which gives

$$\gamma = 0.26 \quad \text{or} \quad \gamma = -1.98$$

after using the recent value $\tau_{\pi^0} = 0.828 \times 10^{-16}$ sec for the π^0 lifetime.[7] As can be seen in the figure, these two sets of data are in agreement with each other.

The distinction between the two possible values of γ could be clearly recognized for either set of data quoted above and thus confirms the expectation as stated in the Introduction. We shall come back in the next section to some aspects concerning a more realistic point of view about this experiment. Here we would like to turn our attention to another problem related to this calculation, i.e., the effects of the neutrino mass.

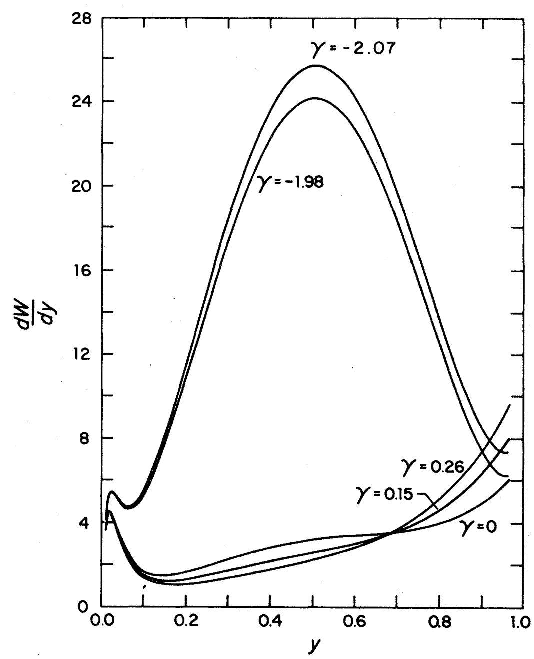

FIG. 1. Electron spectrum for the situation where the photon polarizations are summed up and the electron has a negative helicity. A cutoff in photon energy $x_c = 0.1$ is used.

When we take into account only the lowest-order correction due to the neutrino mass, the differential decay rate Eq. (A1) in the Appendix becomes

$$\left(\frac{d^2W}{dx\,dy}\right)'_{\pm} = \mathrm{IB}'_{\pm} + \mathrm{SD}'_{\pm} + \mathrm{INT}'_{\pm} \tag{2.2}$$

with

$$\mathrm{IB}'_{\pm} = \frac{e^2 f_\pi{}^2 m_l{}^2 \mu}{128\pi^3}\left(1 - 2\frac{m_\nu}{m_l}\right)\frac{2A}{x^2(x-A)^2}\left\{x^2(x-A) - 2B\left(1 - \frac{m_l{}^2}{\mu^2}\right)\right.$$
$$\left. \mp \frac{1}{S}\left[2B\left(y^2 + Ay - 4\frac{m_l{}^2}{\mu^2}\right) + x^2\left(Ay - xy + 2\frac{m_l{}^2}{\mu^2}x\right)\right]\right\} \ , \tag{2.3}$$

$$\mathrm{SD}'_{\pm} = \frac{G^2 a^2 \mu^7}{512\pi^3}\left[(1+\gamma)^2\left(x - A \mp \frac{y(x-A) - 2(m_l{}^2/\mu^2)x}{S}\right)[x(2-x-y) - A]\right.$$
$$\left. + (1-\gamma)^2 A\left(A - x + xy \mp Sx \pm \frac{y(x-A) - 2(m_l{}^2/\mu^2)x}{S}\right)\right] \ , \tag{2.4}$$

$$\mathrm{INT}'_{\pm} = \frac{e f_\pi m_l{}^2 G a \mu^3}{128\sqrt{2}\pi^3}\left(1 - \frac{m_\nu}{m_l}\right)\frac{2A}{xS(x-A)}\{(1+\gamma)B(S \pm 2 \mp 2x \mp y)$$
$$- (1-\gamma)[B(S \pm 2 \mp y) \pm 2x(A - x + \tfrac{1}{2}xy) - Sx^2]\} \ , \tag{2.5}$$

where now

$$A = 1 - y + \frac{m_l{}^2}{\mu^2} \ , \quad B = (1-x)(1-x-y) + \frac{m_l{}^2}{\mu^2} \ .$$

The first-order correction due to the neutrino mass comes from the inner-bremsstrahlung term and accordingly also the interference term. The correction is proportional to m_ν/m_l for the interference term which is very small by itself; hence the correction due to this term is negligible. The IB term gives a correction proportional to $2m_\nu/m_l$. The structure-dependent term contributes no first-order correction. So, contrary to the case in trying to determine the parameter γ, we should look, in principle, at the situation where the charged lepton has a positive helicity to suppress the structure-dependent contribution when trying to see the effects due to the neutrino mass. Taking note of the existing upper bounds for both the electron and muon neutrino masses, we can easily see that the more favorable case is the muon decay channel, which is enhanced by its higher branching ratio. Unfortunately, when we try to measure the end points of the decay spectra, irrespective of whether we perform the difficult measurement on the muon helicity or not, the low counting rate makes the whole idea extremely difficult to carry out. According to our numerical study, about 10^{-8} to 10^{-7} of the total radiative events will fall within the small range relevant in determining the end points. We conclude, therefore, that the effects of the neutrino mass are not yet readily detectable by this particular decay process.[11] There is some hope for the case of a τ neutrino in a similar decay process, but we shall not pursue this particular point any further in this Brief Report, because of the complications arising from competing decay channels.

III. DISCUSSION

We discuss here the feasibility of measuring γ, using the kind of spectra considered here. One important question immediately raised would be the problem connected with the statistics. After taking into account the events expected from the dominant positive-electron-helicity decay mode, we find that about 10% of the total events will come from the situation where the electron has a negative helicity. However, this counting ratio still depends on what the value of γ is. If γ equals one of the negative values, e.g., -2.07 as implied by the results in total radiative decay rates, we could hope for a little more than this, while in the other case, i.e., $\gamma=0.15$, the ratio would be reduced somewhat. On the average, for 500 events, which is not an unrealistic figure, about 50 events are expected for the favored helicity case. There is one point worth noticing and that is that not only does the general shape of the spectrum depend on which of the two possible γ values is valid, but so does the counting rate. As could be estimated from Fig. 1, when γ equals the negative one of the two possible values, i.e., $\gamma=-2.07$, the events expected would be approximately three times as many as those where γ equals the positive value, i.e., $\gamma=0.15$. This result appears to relax the difficulty of the experiment to a certain extent. The general features discussed above are basically maintained when we use higher cutoffs in the electron and photon energies. This can be done without much effort from the result (A1). Because of the sensitive dependence on γ, it appears to us that a measurement on this quantity is not entirely out of the question.

As a final remark, we notice also that no distinction could be made regarding whether the neutrino is Majorana or Dirac in this tree-level calculation.

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APPENDIX: THE DIFFERENTIAL DECAY RATES WITH RESPECT TO THE ELECTRON AND PHOTON ENERGY

Though we have calculated the differential decay rate for fixed photon polarization and electron helicity, we shall give here only the result for the case where the photon polarizations are summed up:

$$\left(\frac{d^2W}{dx\,dy}\right)_\pm = \mathrm{IB}_\pm + \mathrm{SD}_\pm + \mathrm{INT}_\pm \ . \qquad \text{(A1)}$$

These three terms above correspond, respectively, to the inner-bremsstrahlung, structure-dependent, and interference contributions. They are, separately,

$$\mathrm{IB}_\pm = \frac{e^2 f_\pi^2 (m_l - m_\nu)^2 \mu}{128\pi^3} \frac{A}{x^2(x-A)^2} \left\{ (1+d^2)\left[x^2(x-A) - 2B\left(1 - \frac{m_l^2}{\mu^2} - \frac{m_\nu^2}{\mu^2}\right)\right] + (1-d^2)\frac{4m_l m_\nu}{\mu^2}B \right.$$
$$\left. \mp \frac{2d}{S}\left[2B\left(y^2 + Ay - 4\frac{m_l^2}{\mu^2}\right) + x^2\left(Ay - xy + 2\frac{m_l^2}{\mu^2}x\right)\right]\right\} , \qquad \text{(A2)}$$

$$\mathrm{SD}_{\pm}=\frac{G^2a^2\mu^7}{512\pi^3}\left[(1+\gamma)^2\left\{x-A\mp\frac{y(x-A)-2(m_l^2/\mu^2)x}{S}\right\}[x(2-x-y)-A]\right.$$
$$\left.+(1-\gamma)^2A\left\{A-x+xy\mp Sx\pm\frac{y(x-A)-2(m_l^2/\mu^2)x}{S}\right\}\right]\,, \tag{A3}$$

$$\mathrm{INT}_{\pm}=\frac{ef_\pi(m_l-m_\nu)Ga\mu^3}{128\sqrt{2}\pi^3}\frac{A}{xS(x-A)}\left\{(1+d)(1+\gamma)m_lB(S\pm2\mp2x\mp y)\right.$$
$$-(1+d)(1-\gamma)m_l[B(S\pm2\mp y)\pm2x(A-x+\tfrac{1}{2}xy)-Sx^2]$$
$$+(1-d)(1+\gamma)m_\nu\left[B(S\mp y)\mp\frac{x^2}{A}\left\{Ay-xy+2\frac{m_l^2}{\mu^2}x\right\}-\frac{x^2S(x-A)}{A}\right]$$
$$\left.-(1-d)(1-\gamma)m_\nu B(S\mp y)\right\}\,, \tag{A4}$$

where the subscript $\pm$ refers to the positive and negative electron helicities, respectively. Whenever there is a double sign in the expression, the upper one refers to the case of positive helicity, while the lower one refers to the other.

The various quantities above are defined as follows:

$$x=\frac{2k_0}{\mu}\,,\quad y=\frac{2E_l}{\mu}\,,\quad A=1-y+\frac{m_l^2}{\mu^2}-\frac{m_\nu^2}{\mu^2}\,,$$

$$B=(1-x)(1-x-y)+\frac{m_l^2}{\mu^2}-\frac{m_\nu^2}{\mu^2}+\frac{(m_\nu^2/\mu^2)x^2}{A}\,,\qquad S=\left\{y^2-4\frac{m_l^2}{\mu^2}\right\}^{1/2}.$$

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