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Dirac equation revisited: Charge quantization as a consequence of the Dirac equation

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It is shown that the Dirac equation possesses an internal-symmetry group, which contains the usual electromagnetic gauge group. Well-known group-theoretical results then imply that charge is quantized.

I. THE INTERNAL-SYMMETRY GROUP OF THE DIRAC EQUATION

As is well known,[1-3] the Dirac equation (for flat or curved space-times) can be written in terms of a pair of two-spinors u_A and v_A as

$$\nabla^{A\dot{A}}u_A+\frac{im}{\sqrt{2}}\bar{v}^{\dot{A}}=0 \quad , \tag{1}$$

$$\nabla^{A\dot{A}}v_A+\frac{im}{\sqrt{2}}\bar{u}^{\dot{A}}=0 \quad , \tag{2}$$

where the spinor notations and conventions are those of Ref. 4.

Equations (1) and (2) may be derived from the following Lagrangian:

$$L(u_A,v_A)=i(\bar{u}_{\dot{A}}\nabla^{A\dot{A}}u_A-\bar{v}_{\dot{A}}\nabla^{A\dot{A}}v_A)+\frac{M}{\sqrt{2}}(u^Av_A+\bar{u}^{\dot{A}}\bar{v}_{\dot{A}}) \quad . \tag{3}$$

Equations (1) and (2) have the suggestive appearance of two interacting Weyl neutrino fields [this idea can be taken much further and used to build models for the weak interactions, etc. (see Ref. 5)]. It is thus natural to see if these equations possess an internal-symmetry group acting on the pair (u_A,v_A). To this end, we seek linear transformations of the (u_A,v_A) which preserve the Lagrangian (3). Writing

$$\hat{u}_A=au_A+bv_A \quad ,$$

$$\hat{v}_A=cu_A+dv_A \quad ,$$

with a, b, c, and d complex constants, we find

$$L(\hat{u}_A,\hat{v}_A)=L(u_A,v_A) \quad ,$$

if $c=\bar{b}$ and $d=\bar{a}$ (an overbar indicates complex conjugation). So the Lagrangian (3) is preserved for the following transformations:

$$\begin{pmatrix}u_A\\ v_A\end{pmatrix}\rightarrow\begin{pmatrix}a & b\\ \bar{b} & \bar{a}\end{pmatrix}\begin{pmatrix}u_A\\ v_A\end{pmatrix} \quad . \tag{4}$$

The set of matrices $(\begin{smallmatrix}a & b\\ \bar{b} & \bar{a}\end{smallmatrix})$ give the matrix representation of the group SU(1,1), the group of linear transformations of C^2 preserving the pseudo-Hermitian form $|z_1|^2-|z_2|^2$, for $(z_1,z_2)\in C^2$.

II. CONSERVED CURRENTS AND CHARGE QUANTIZATION

The Dirac equations give rise to three (real) conserved currents j^α, $(s^\alpha+\bar{s}^\alpha)$, and $i(s^\alpha-\bar{s}^\alpha)$, where

$$j^\alpha=\sigma^\alpha{}_{A\dot{A}}(u^A\bar{u}^{\dot{A}}+v^A\bar{v}^{\dot{A}}),\quad s^\alpha=\sigma^\alpha{}_{A\dot{A}}u^A\bar{v}^{\dot{A}} \quad , \tag{5}$$

with $\sigma^\alpha{}_{A\dot{A}}$ the van der Waerden symbols (see Ref. 4).

These three currents are mutually orthogonal, with j^α (the Dirac probability density vector) being timelike, and the real and imaginary parts of s^α (the spin-density vector) being spacelike.

We must expect that these three conserved currents are coupled to three generators of some symmetry group; in fact, they are coupled to the three generators of the SU(1,1) group discussed above. Writing the generators of SU(1,1) as

$$Q=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix},\quad P_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad P_2=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \tag{6}$$

(note that these generators differ by a factor of 2 from the usual choice[6,7]), we see that Q couples to j^α, P_1 to $i(s^\alpha-\bar{s}^\alpha)$, and P_2 to $(s^\alpha+\bar{s}^\alpha)$.

Now the electric charge generator Q (Ref. 8) generates the maximal compact subgroup U(1) of SU(1,1), and as such its eigenvalues can take on only integer values (see Ref. 7, keeping in mind the differences in the definition of the generators). The Casimir invariant of SU(1,1) then gives us the other quantum number necessary in specifying representations of SU(1,1), the eigenvalues of this operator [when written as $4l(l+1)$] give a quantum number l. In the discrete series representations of SU(1,1), Refs. 6 and 7, we see that there is just one lattice point with charge -1, that for which $l=-\frac{1}{2}$, and for charge $+1$ again there is just one lattice point $l=-\frac{1}{2}$. Consequently, the Dirac equation (by itself) cannot be used to build a theory of sequential leptons $e^{\pm}$, $\mu^{\pm}$, $\tau^{\pm}$, . . . [however, if the weak interactions are also considered, then it does appear possible that this can be done (see Ref. 5)].

III. CONCLUSIONS AND DISCUSSION

The Dirac equations thus possess an internal-symmetry group SU(1,1), which provides a very natural explanation of charge quantization. However, this is achieved at the expense of introducing a larger, noncompact internal-symmetry group with the inherent problems of quantization. The situation is, in a sense, analogous to that encountered with the Lorentz group and its associated spin group SL(2,C); this group has as its maximal compact subgroup SU(2). Global conditions on the representations then imply that the quantum number associated with intrinsic spin must be quantized.

It is interesting to note that SU(1,1) is the "spinor group" for SO(1,2), this latter group being realized as the set of rotations of the three vectors j^{α}, $(s^{\alpha}+\bar{s}^{\alpha})$, and $i(s^{\alpha}-\bar{s}^{\alpha})$, i.e., the set of rotations leaving fixed the vector $\sigma^{\alpha}{}_{A\dot{A}}(u^{A}\bar{u}^{\dot{A}}-v^{A}\bar{v}^{\dot{A}})$, giving a principal fiber bundle which may be considered as a sub-bundle of the tangent bundle of the underlying space-time manifold.

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[8]The U(1) subgroup generated by Q is the usual electromagnetic U(1): a U(1) transformation generated by Q takes the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \mathrm{U}(1) \subset \mathrm{SU}(1,1)$$

when acting on the two-spinor pair $\binom{u_A}{v_A}$. That is,

$$\begin{pmatrix} u_A \\ v_A \end{pmatrix} \to \begin{pmatrix} e^{i\theta}u_A \\ e^{-i\theta}v_A \end{pmatrix} .$$

Thus the usual Dirac bispinor $\psi=\binom{u_A}{\bar{v}^{\dot{B}}}$ transforms as

$$\psi \to e^{i\theta}\psi$$

under the U(1) subgroup of SU(1,1). This is precisely the form of the usual electromagnetic U(1) transformations.