Numerical studies of renormalization in SU(3) gauge theory in four dimensions

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Wilson loops for pure SU(3) gauge theory are calculated by Monte Carlo simulations on an 8^4 lattice. Following renormalization-group arguments, Wilson loops on different length scales are compared. The only observed ultraviolet-stable fixed point lies at vanishing bare coupling. Asymptotic freedom is numerically verified.

In a previous paper[1] by one of the authors, a renormalization procedure for comparing physical ratios of Wilson loops on different length scales was developed and applied to SU(2) lattice gauge theory. The logarithmic dependence of the bare charge on the length scale, as predicted by asymptotic freedom, was verified at the perturbative fixed point. No additional ultraviolet-stable fixed points were found. Nevertheless, the gauge group believed to describe strong interactions is SU(3); thus it would be desirable to extend the previous calculations to this group. In the present paper we present such an analysis, based on Monte Carlo simulations for SU(3) on an 8^4 lattice at 46 values of the coupling constant. We measured all Wilson loops up to 4×4, extending our previous study of loops up to size 3×3 on a 6^4 lattice.[2]

A hypercubical lattice in four Euclidean space-time dimensions[3,4] was used in our calculations. The link joining the nearest-neighbor lattice sites, denoted by i and j, is signified by $\{i,j\}$. On the link $\{i,j\}$ an $N\times N$ unitary-unimodular matrix U_{ij} of the gauge group SU(N) is attached such that

$$U_{ji}=(U_{ij})^{-1} \ .$$

Our partition function is defined by

$$Z(\beta)=\int\left[\prod_{\{i,j\}} dU_{ij}\right]\exp(-\beta S[U]) \ ,$$

where β is the inverse coupling constant squared given by $\beta=2N/g_0^2$ and g_0 is the bare coupling constant. The measure in the above integral is the SU(N) normalized invariant Haar measure. A sum over all unoriented plaquettes $\square$ defines our action S such that

$$S[U]=\sum_{\square} S_{\square}=\sum_{\square}\left[1-\frac{1}{N}\operatorname{Re}\operatorname{Tr}U_{\square}\right] \ .$$

We used periodic boundary conditions throughout our calculations. The method of Metropolis *et al.*[5] was used to bring our lattice to equilibrium. Further details of our calculational procedure can be found in Ref. 6.

Wilson loops[7] are defined by the expectation value

$$W(I,J)=\frac{1}{N}\langle \operatorname{Re}\operatorname{Tr}U_C\rangle \ ,$$

where C is a rectangle of rectangular dimensions I and J and U_C is the product of link variables around C. The leading-order strong-coupling expansion for the Wilson loop is

$$W(I,J)=\left[\frac{\beta}{2N^2}(1+\delta_{2,N})\right]^{IJ}[1+O(\beta^2)] \ , \quad (1)$$

while the leading-order weak-coupling expansion for the average action per plaquette is

$$\langle E\rangle = 1-W(1,1)=\frac{N^2-1}{4\beta}+O(\beta^{-2}) \ . \quad (2)$$

To extract the string tension, we evaluate the logarithmic ratios

$$\chi(I,J)=-\ln\left[\frac{W(I,J)\,W(I-1,J-1)}{W(I,J-1)\,W(I-1,J)}\right] \ .$$

The leading-order strong-coupling expansion for these quantities is

$$\chi(I,J) = -\ln\left[\frac{\beta}{2N^2}(1+\delta_{2,N})\right] + O(\beta^2) \quad . \tag{3}$$

Since non-Abelian gauge theories are asymptotically free, they have an ultraviolet attractive fixed point at $g_F=0$. A perturbative expansion about this point gives the necessary dependence of the coupling on the lattice spacing a for a continuum limit

$$a\frac{d}{da}g_0(a) \equiv \gamma(g_0) = \gamma_0 g_0^3 + \gamma_1 g_0^5 + O(g_0^7) \quad , \tag{4}$$

where we have

$$\gamma_0 = \frac{11}{3}\left[\frac{N}{16\pi^2}\right] \text{ and } \gamma_1 = \frac{34}{3}\left[\frac{N}{16\pi^2}\right]^2 \quad .$$

Integration of Eq. (5) yields

$$\frac{1}{g_0^2(a)} = \gamma_0 \ln\left[\frac{1}{\Lambda_0^2 a^2}\right] + \frac{\gamma_1}{\gamma_0}\ln\left[\ln\left[\frac{1}{\Lambda_0^2 a^2}\right]\right] + O(g_0^2) \quad . \tag{5}$$

Asymptotic freedom has introduced a scale parameter Λ_0 defined by

$$\Lambda_0 = \lim_{a\to 0}\frac{1}{a}[\gamma_0 g_0^2(a)]^{(-\gamma_1/2\gamma_0^2)}\exp\left[-\frac{1}{2\gamma_0 g_0^2(a)}\right] \quad . \tag{6}$$

Equation (5) implies that if the cutoff is changed by a factor of 2 we have

$$\frac{1}{g_0^2(a/2)} = \frac{1}{g_0^2(a)} + 2\gamma_0 \ln 2 + O(g_0^2) \quad . \tag{7}$$

In Ref. 1 the quantity $P(r,a,\beta)$ was introduced and defined by

$$P(r,a,\beta) = 1 - \frac{W(r/a,r/a)\,W(r/2a,r/2a)}{[W(r/a,r/2a)]^2} \quad .$$

This ratio should be a physical quantity with a finite continuum limit when a and g_0 go to zero together as dictated in Eq. (5). The leading-order weak-coupling expansion gives

$$P(r,a,\beta) = p_1\frac{2N}{\beta} + O(\beta^{-2}) + O\left[\frac{a^2}{r^2\beta}\right] \quad ,$$

where

$$p_1 = \frac{N^2-1}{2N}\frac{1}{4\pi^2}[8\arctan 2 + 2\arctan\tfrac{1}{2} - 2\pi - 4\ln(\tfrac{5}{4})] \quad ,$$

i.e., the U(1) result of Ref. 1 times the ratio of the number of gluons to twice the dimension of the group matrices. Following Ref. 1 we consider P for $r/a=2$ and $r/a=4$ by introducing the ratios

$$F(\beta) = 1 - \frac{W(1,1)\,W(2,2)}{W(2,1)\,W(2,1)} \tag{8}$$

and

$$G(\beta) = 1 - \frac{W(2,2)\,W(4,4)}{W(4,2)\,W(4,2)} \quad . \tag{9}$$

These satisfy the leading-order strong-coupling expansions

$$F(\beta) = 1 - \left[\frac{\beta}{2N^2}(1+\delta_{2,N})\right] + O(\beta^3) \tag{10}$$

and

$$G(\beta) = 1 - \left[\frac{\beta}{2N^2}(1+\delta_{2,N})\right]^4 + O(\beta^6) \quad , \tag{11}$$

and leading-order weak-coupling expansions for SU(3),

$$F(\beta) = G(\beta) + O(\beta^{-2}) = \frac{6p_1}{\beta} + O(\beta^{-2}) + \left[\frac{a^2}{r^2\beta}\right] \quad . \tag{12}$$

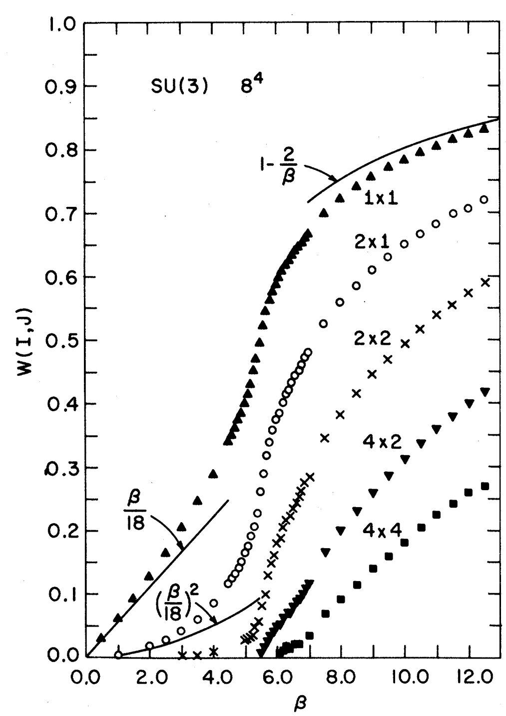

FIG. 1. The Wilson loops $W(I,J)$ for pure SU(3) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β. The solid upward triangles represent $I=J=1$, the open circles represent $I=2$, $J=1$, the crosses represent $I=J=2$, the solid downward triangles represent $I=4$, $J=2$ and the solid squares represent $I=J=4$. The curves represent the leading-order strong- and weak-coupling expansions of Eqs. (1) and (2), respectively.

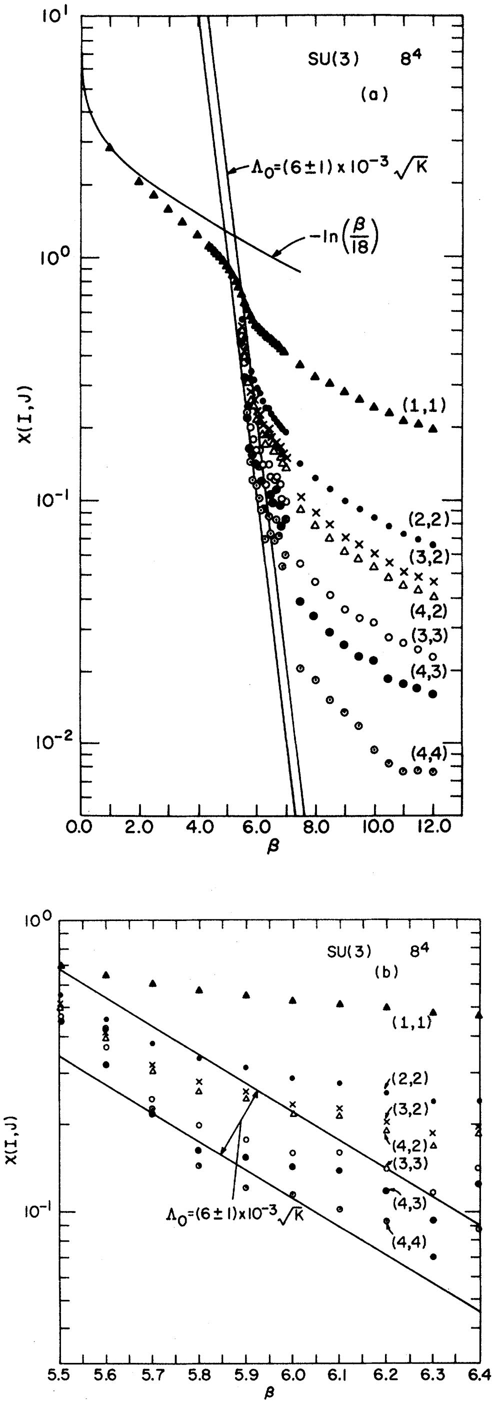

FIG. 2. (a),(b) The string tension $\chi(I,J)$ for pure SU(3) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β. The solid upward triangles represent $I=J=1$, the solid circles represent $I=J=2$, the crosses represent $I=3$, $J=2$, the open triangles represent $I=4$, $J=2$, the open circles represent $I=J=3$, the circles with crosses represent $I=4$, $J=3$, and the circles with dots represent $I=J=4$. The leading-order strong-coupling expansion of Eq. (3) is also shown.

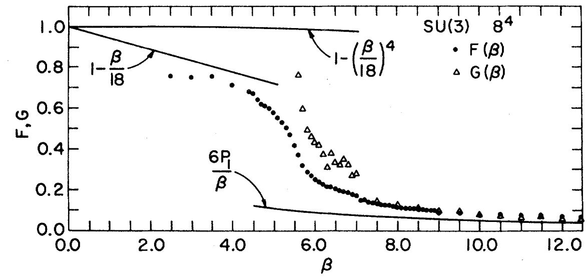

FIG. 3. The quantities $F(\beta)$ and $G(\beta)$ for pure SU(3) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β. The curves represent the leading-order strong- and weak-coupling expansions of Eqs. (10), (11), and (12).

The quantities F and G represent the same physical ratio measured at two different lengths in lattice units. Asymptotic freedom, as manifested in Eq. (7), says that at weak coupling these functions should match if the coupling is appropriately shifted,

$$F(\beta)=G\left[\beta+\frac{33}{4\pi^2}\ln 2\right]+O(\beta^{-3}) \; . \qquad (13)$$

To perform our calculations, we first carried out 20 iterations through the 8^4 lattice with 9 Monte Carlo upgrades per link without any calculation of Wilson loops. As a result of this the space-time lattice was brought to equilibrium reasonably well. The Wilson loops were averaged over the next 260 iterations ignoring every second iteration, in order to reduce correlations between events (i.e., 130 lattices were included in the average). Disordered starting lattices were used for $\beta \leq 4.0$, mixed-phase[4] starting lattices were used for $4.4 \leq \beta \leq 7.0$, and ordered starting lattices were used for $\beta > 7.0$.

Figure 1 shows, as a function of the inverse coupling constant squared β, the calculated values of the Wilson loops needed for our renormalization comparison. Also plotted are the leading-order strong- and weak-coupling expansions of Eqs. (1) and (2), respectively.

The logarithmic ratios $\chi(I,J)$, for $(I,J)=(1,1)$,

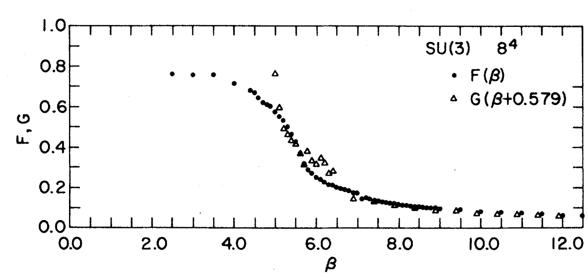

FIG. 4. The quantities $F(\beta)$ and $G(\beta+0.579)$ for pure SU(3) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β.

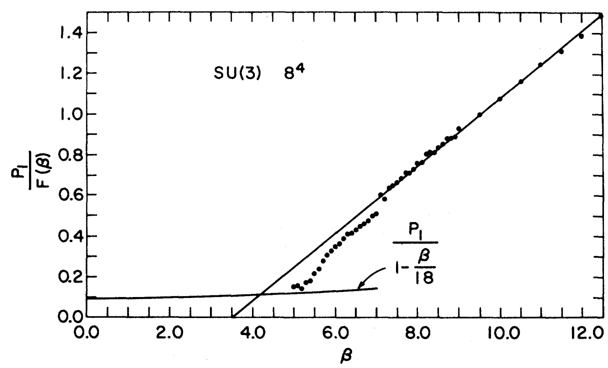

FIG. 5. The inverse renormalized charge squared at $r=2a$ for pure SU(3) gauge theory on an 8^4 lattice as a function of the inverse coupling constant squared β. Also shown in the diagram is Eq. (7) as well as the leading-order strong-coupling expansion of Eq. (10).

(2,2), (3,2), (4,2), (3,3), (4,3), and (4,4), are shown as a function of the inverse coupling constant squared β in Fig. 2. Figure 2(a) also displays the leading-order strong-coupling expansion of Eq. (3). A band corresponding to the functional form given by Eq. (6) is shown in Fig. 2 with

$$\Lambda_0=(6\pm1)\times10^{-3}\sqrt{K} ,$$

where K is the string tension. This confirms our analysis in Ref. 2.

The quantities F and G of Eqs. (8) and (9) for SU(3) are shown as functions of the inverse coupling constant squared β in Fig. 3. The leading-order strong-coupling expansions of Eqs. (10) and (11) and the weak-coupling expansion of Eq. (12) are also indicated in Fig. 3. Quite clearly $F(\beta)$ always lies below $G(\beta)$ which looks like Fig. 1 of Ref. 1. Figure 3 shows no evidence for any fixed point apart from $\beta=\infty$.

In Fig. 4 we show $F(\beta)$ and

$$G\left[\beta+\frac{33}{4\pi^2}\ln 2\right] ,$$

as a function of the inverse coupling constant squared β. For $\beta>5$ we can see excellent agreement with the asymptotic-freedom prediction of Eq. (13).

As in Ref. 1, we can define a renormalized charge such that Eq. (12) has no higher-order corrections. Thus, from $F(\beta)$ we obtain the renormalized $g(r=2a)$. We show the inverse renormalized charge at $2a$ as a function of the inverse coupling constant squared in Fig. 5. The leading-order strong-coupling expansion is also indicated. In the weak-coupling region, the curve has a slope of $\frac{1}{6}$th in β. This shows that a^2/r^2 corrections in Eq. (12) are remarkably small. From the intercept of Fig. 5 we find

$$2\gamma_0\ln\left[\frac{2\Lambda}{\Lambda_0}\right]\simeq0.60 ,$$

or

$$\Lambda=37\Lambda_0 .$$

This number is, in principle, calculable perturbatively. With Eq. (14) we obtain

$$\Lambda=0.22\sqrt{K} .$$

This value is more in line with other physical definitions of the asymptotic-freedom scale parameter than is the bare lattice parameter Λ_0.

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