Quantization in the temporal gauge

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(Received 7 September 1982)

It is pointed out that the validity of the canonical quantization procedure in the temporal gauge is very questionable. The quantization can be discussed only within the framework of the path-integral formalism. However, in that case, the vacuum is not an energy eigenstate.

The temporal-gauge condition[1]

$$\vec{A}_0 = 0 \tag{1}$$

is frequently used in the investigations[2] of quantum chromodynamics. The advantages of this gauge seem to be as follows:

(1) The theory is ghost free.

(2) The canonical quantization procedure is simple.

(3) There still remains the invariance under the time-independent residual gauge transformation.

If these are the cases, we have the canonically quantized field theory with simple structure and yet with gauge invariance. And we can expect that the analyses[2] in the temporal gauge play important roles in the development of quantum chromodynamics.[2] In this Brief Report, however, we demonstrate that the validity of the canonical quantization procedure itself is very questionable.

For this purpose we begin with a brief review of the prevailing canonical formalism. The system is given by

$$L = -\tfrac{1}{4}\vec{F}_{\mu\nu}{}^2, \quad \vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu \ , \tag{2}$$

$$(\partial_\mu + g\vec{A}_\mu \times)\vec{F}_{\mu\nu} = 0 \ . \tag{3}$$

In the temporal gauge,

$$\vec{\pi}_i = \partial_0 \vec{A}_i \ , \tag{4}$$

$$H = \int dx \left(\tfrac{1}{2}\vec{\pi}_i{}^2 + \tfrac{1}{4}\vec{F}_{ij}{}^2\right) \ , \tag{5}$$

$$[A_i{}^a(x), \pi_j{}^b(y)]_{x_0 = y_0} = i\delta^{ab}\delta_{ij}\delta^3(\vec{x}-\vec{y}) \ , \tag{6}$$

and then

$$\partial_0 \vec{A}_i = i[H, \vec{A}_i] = \vec{\pi}_i \ , \tag{7}$$

$$\partial_0 \vec{\pi}_i = i[H, \vec{\pi}_i] = (\partial_j + g\vec{A}_j \times)\vec{F}_{ji} \ . \tag{8}$$

This system is invariant under the transformation which is generated by

$$\vec{G}_R = (\partial_i + g\vec{A}_i \times)\vec{\pi}_i \ . \tag{9}$$

The Heisenberg equations (7) and (8) do not reproduce all the Euler equations (3) with $\vec{A}_0 = 0$. The Gauss law

$$\vec{G}_R = (\partial_i + g\vec{A}_i \times)\partial_0 \vec{A}_i = 0 \tag{10}$$

is missing. Usually this law is required not as the operator equation but as the constraint

$$\vec{G}_R |\alpha\rangle = 0 \tag{11}$$

on the physical state $|\alpha\rangle$. Since

$$[H, \vec{G}_R] = 0 \ , \tag{12}$$

there is no contradiction between the Gauss law (11) and the Heisenberg equation. The quantization procedure outlined in the above is widely accepted.[1,2]

Nevertheless, we can easily show that the condition (11) leads to a contradiction. Using Eq. (6) we have

$$[G_R{}^a(\vec{x}), A_i{}^b(\vec{y})] = i\delta^{ab}\partial_i\delta^3(\vec{x}-\vec{y}) + igf^{abc}A_i{}^c(\vec{x})\delta^3(\vec{x}-\vec{y}) \ , \tag{13}$$

which makes sure that $\vec{G}_R$ is the generator of the residual gauge transformation. Sandwiching both sides of Eq. (13) by $|\alpha\rangle$ and $|\alpha'\rangle$, we obtain

$$\langle \alpha | \alpha' \rangle = 0 \tag{14}$$

for any isoscalar physical states,[3] since the left-hand side of the resulting equation is zero because of condition (11), and the second term on the right-hand side vanishes for the isoscalar states. Evidently, Eq.

(14) is unphysical and tells us the invalidity of condition (11). In this way it becomes clear that we have no knowledge of the treatment of the Gauss law, one of the fundamental equations. Therefore further discussions on the canonical formalism in this gauge seem to be meaningless.

On the other hand, we have another quantization procedure based on the Feynman path integral. The vacuum expectation values of time-ordered products of field variables can be obtained by the path integral irrespective of the knowledge of the Gauss law. For example, the propagator is

$$\langle 0|T(A_i^a(x),A_j^b(y))|0\rangle = -i\frac{\delta^{ab}}{(2\pi)^4}\int d^4k\, e^{ik\cdot(x-y)}\frac{1}{k^2-i\epsilon}\left[\delta_{ij}-\frac{k_ik_j}{k_0{}^2}\right]+O(g^2) \ . \tag{15}$$

This immediately leads to

$$\langle 0|T(G_R{}^a(x),A_j^b(y))|0\rangle = i\frac{\delta^{ab}}{(2\pi)^4}\int d^4k\, e^{ik\cdot(x-y)}\frac{k_j}{k_0}+O(g^2) \ . \tag{16}$$

Since the right-hand side is obviously nonzero, we see that the condition (11) does not hold, at least for the vacuum state. Therefore the Gauss law which may hold in the path-integral formalism is not in the form (11) but at most in the form

$$\langle \alpha|\vec{G}_R|\alpha'\rangle = 0 \ , \tag{17}$$

which does not contradict Eqs. (13) and (16).

Finally, we demonstrate that the vacuum state is not an eigenstate of the Hamiltonian. First we calculate

$$\langle 0|T(A_i^a(x),A_j^b(y),\tfrac{1}{2}\vec{\pi}_k{}^2(z)+\tfrac{1}{4}\vec{F}_{kl}{}^2(z))|0\rangle \ . \tag{18}$$

The equal-time limit of (18) from $x_0 > y_0 > z_0$ gives us

$$\langle 0|A_i^a(\vec{x})A_j^b(\vec{y})H|0\rangle = -\frac{\delta^{ab}}{4(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{k_ik_j}{\vec{k}^2}+O(g^2) \ , \tag{19}$$

while we have, from Eq. (15),

$$\langle 0|A_i^a(\vec{x})A_j^b(\vec{y})|0\rangle = \frac{\delta^{ab}}{2(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{1}{|\vec{k}|}\left[\delta_{ij}-\frac{k_ik_j}{\vec{k}^2}\right]+O(g^2) \ . \tag{20}$$

The first term on the right-hand side of Eq. (19) has a different functional form from that of Eq. (20). This fact shows us that the vacuum state in the path-integral quantization is not the eigenstate of the total Hamiltonian.

From the above investigations we come to the conclusion that in gauge theories with $\vec{A}_0 = 0$ the validity of the canonical quantization procedure, which is widely accepted and outlined at the beginning of this paper, is very questionable. The quantization can be discussed only within the framework of the path-integral formalism. However, the vacuum state is not an eigenstate of the total Hamiltonian. It should be examined whether such theories can still be understood in the same way as the usual quantum theories based on operators and state vectors.

[1]R. P. Feynman, *Weak and Electromagnetic Interaction at High Energy* (North-Holland, Amsterdam, 1977), p. 121.

[2]C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Lett. 63B, 334 (1976); W. Marciano and H. Pagels, Phys. Rep. 36C, 137 (1978), and references therein.

[3]Also, in the axial gauge we are in a similar situation, as is seen in A. Hosoya, Y. Kakudo, Y. Taguchi, A. Tanaka, and K. Yamamoto, Nuovo Cimento 68A, 150 (1982), although there is no serious problem.