Brief Reports

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Landau's asymptotic solution of the vertex function in QED

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We consider finite massless QED with the electron mass totally dynamical of origin and the fine-structure constant as the (infinite order) zero of the Callan-Symanzik function. In spacelike regions, the electron propagator (in the generalized Landau gauge) and the photon propagator are then proportional to their free counterparts, and the integral equation for the vertex function Γ_σ may be written solely as a function of Γ_σ itself. Landau's approximation is used to obtain the asymptotics for $\Gamma_\sigma(p,p-k;k)$ for $p^2,k^2\to\infty$, $p^2/k^2\to 0$ in spacelike regions.

The classic papers by Baker and Johnson[1] and Adler[2] on finite QED have stimulated much research (see, e.g., Ref. 3) during the past ten years on the asymptotic anlaysis of the basic functions of QED. We consider finite massless QED with the electron mass totally dynamical of origin, and α, the renormalized fine-structrue constant, as a fixed point in the sense of Ref. 2. That is, we sum all photon self-energy graphs in renormalized QED, fix α as the (infinite order) zero of the Callan-Symanzik[4] function, $\beta(\alpha)=0^\infty$, and take the limit $m\to 0$ for the electron mass, with the anomalous mass dimension[1,2] $\delta(\alpha)>0$. We work in the spacelike region. In this case the (renormalized) photon propagator $D_{\mu\nu}$ may be written[2] as

$$\alpha D_{\mu\nu}(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{q(\alpha)}{k^2}+\alpha G_0\frac{k_\mu k_\nu}{k^4}\ , \tag{1}$$

where (cf. Ref. 5) $(\alpha=e^2/4\pi)$, $q(\alpha)=\alpha-(5/9\pi)\alpha^2+O(\alpha^3)$, and in the generalized Landau gauge[1] $G_0=(3\alpha/8\pi)+O(\alpha^2)$. In the same gauge the (renormalized) electron propagator is[1,2] [up to an overall multiplicative constant of the form $1+O(\alpha)$] given by $S(p)=1/\gamma\cdot p$. This makes it quite convenient to study the asymptotic behavior of the vertex function Γ_σ as the integral equation, for the latter may be solely written in terms of the unknown Γ_σ itself: $\Gamma_\sigma=\Gamma_\sigma(\Gamma)$. It is important to note that if α is not a fixed point [that is, α is not a zero of $\beta(\alpha)$] then electrodynamics with the propagator in (1) is considered as a model for the behavior of skeleton graphs in the theory. We use Landau's approximation[6] to study the asymptotic behavior of $\Gamma_\sigma(p,p-k;k)$ for $p^2,k^2\to\infty$, $p^2/k^2\to 0$ in spacelike regions. We work exclusively in the generalized Landau gauge as, in this case, Γ_σ is both ultraviolet and infrared finite[1] in spacelike regions.

The vertex function $\Gamma_\sigma(p,p-k;k)\equiv\Gamma_\sigma(p,p-k)$ satisfies a well-known integral equation (cf. Ref. 7)

$$\begin{aligned}\Gamma_\sigma(p,p-k)=\gamma_\sigma+(-i\alpha/4\pi^3)\int(dq)D_{\mu\nu}(q)\Gamma^\mu(p,p-q)S(p-q)\\ \times\Gamma_\sigma(p-q,p-q-k)S(p-q-k)\Gamma^\nu(p-q-k,p-k)\\ +(-i\alpha/4\pi^3)^2\int D\Gamma S\Gamma S\Gamma D\Gamma S\Gamma S+(-i\alpha)^3\int\cdots+\cdots\ ,\end{aligned} \tag{2}$$

where all the objects in the integrands denote (full) renormalized quantities and the factor γ_σ is up to an overall finite multiplicative constant of the form $1+O(\alpha)$. We may simply rescale Γ_σ and replace, subsequently, such a constant factor by 1 in our analysis. We consider $\alpha(\sim\frac{1}{137})$ and $|p|/|k|$ as small parameters. To zeroth order in $|p|/|k|$, the Ward identity gives (recall $m=0$) $k^\sigma\Gamma_\sigma(0,-k)=\gamma\cdot k$, or

$$\Gamma_\sigma(0,-k)=\gamma_\sigma+C\left(\gamma_\sigma-\frac{\gamma\cdot kk_\sigma}{k^2}\right)\ , \tag{3}$$

where C is a finite constant and may be determined to any order of accuracy in α by setting $p=0$ in (2) (for $m=0$). By using the four-dimensional angular averages (cf. Ref. 8)

$$\langle 1/(q-k)^2\rangle,\quad \langle q^\mu/(q-k)^2\rangle,\quad \langle q^\mu q^\nu/(q-k)^2\rangle\ , \tag{4}$$

one readily obtains the value $C=-(\alpha/\pi)$ to lowest order in α. Now we move from the region $|p|/|k|=0$ to the region $|p|/|k|\neq 0$. Landau's approximation[6] essentially consists of the following. We rely on the fact that $\alpha(\sim\frac{1}{137})$ is small and hence, if the corrections to γ_σ for Γ_σ in the limit $p^2,k^2\to\infty$, $p^2/q^2\to 0$ vanish rapidly, one expects that the leading contribution to Γ_σ will come from the first integral in (2), which is the only one we retain from now on. The other integrals are expected to give contributions which are at least one order in α smaller than the first one. In our analysis, we will generate corrections to γ_σ for Γ_σ which are of the order $O(1)\times$damping factor for $|p|/|k|\to 0$ and α small in contrast to terms which are of the order $O(\alpha)\times$damping factor which are smaller. The latter corrections are expected to come from the remaining integrals in (2). We consider only the region of integration $p^2 \ll q^2 \ll k^2$ in the contributing first integral in (2), arguing that the other regions would give contributions to Γ_σ which are relatively smaller or, at worst, modify the coefficient of γ_σ to a finite multiplicative factor of the form $1+O(\alpha)$. We now estimate the corrections due to the terms in the first integrand in (2) in the region $p^2\ll q^2\ll k^2$, $p^2,q^2,k^2\to\infty$. The vertex functions $\Gamma_\sigma(p-q,p-q-k)$, $\Gamma^\nu(p-q-k,p-k)$, and $S(p-q-k)$ all depend on the asymptotically leading vector k, for $k^2\to\infty$, and may be safely replaced by γ_σ, γ^ν, and $S(-k)$, respectively. The correctness of the latter is checked self-consistently. Considering, in turn, corrections $\delta S(p-q)$ and $\delta\Gamma^\mu(p,p-q)$, we then obtain the self-consistent approximation due to Landau[6] for $p^2,k^2\to\infty$, $p^2/k^2\to 0$, and α small with $G_0\sim 0$:

$$\delta\Gamma_\sigma(p,p-k)\cong(-i\alpha/4\pi^3)\int_{p^2\ll q^2\ll k^2}(dq)D_{\mu\nu}(q)\gamma^\mu\delta S(p-q)\gamma_\sigma S(-k)\gamma^\nu + (-i\alpha/4\pi^3)\int_{p^2\ll q^2\ll k^2}(dq)D_{\mu\nu}(q)\delta\Gamma^\mu(p,p-q)S(-q)\gamma_\sigma S(-k)\gamma^\nu\ . \tag{5}$$

For $p^2\ll q^2$ [see also Eq. (2.6) in Ref. 6],

$$\delta S(p-q)\cong-\frac{1}{\gamma\cdot q}+\left[\frac{\gamma\cdot p}{q^2}-\frac{2\gamma\cdot q\, p\cdot q}{q^4}\right]\ . \tag{6}$$

The first term on the right-hand side of (5) may be then explicitly worked out to yield

$$\frac{\alpha}{8\pi}\left[\frac{\gamma\cdot k\gamma_\sigma\gamma\cdot p}{k^2}-\frac{\gamma_\sigma\gamma\cdot k\gamma\cdot p}{k^2}\right]\ln\left[\frac{k^2}{p^2}\right]\ . \tag{7}$$

Accordingly, we shall seek a correction $\delta\Gamma_\sigma(p,p-k)$ in (5) which solves the latter self-consistently in the form [see also Eqs. (2.7) and (2.8) in Ref. 6]

$$\delta\Gamma^\mu(p,p-q)\cong\frac{\gamma\cdot q\gamma^\mu\gamma\cdot p}{q^2}F_1(p^2,q^2)+\frac{\gamma^\mu\gamma\cdot q\gamma\cdot p}{q^2}F_2(p^2,q^2)\ . \tag{8}$$

Upon substituting the expression (8) in (5) and using (7), we obtain the following elementary integral equations [see also Eq. (2.8) in Ref. 6]:

$$F_1(p^2,k^2)=\frac{\alpha}{8\pi}\int_{p^2}^{k^2}\frac{dq^2}{q^2}[1+F_2(p^2,q^2)-F_1(p^2,q^2)]\ , \tag{9}$$

$$F_2(p^2,k^2)=\frac{\alpha}{8\pi}\int_{p^2}^{k^2}\frac{dq^2}{q^2}[F_1(p^2,q^2)-1-F_2(p^2,q^2)]\ , \tag{10}$$

consistent with the Ward identity. The latter are readily solved by differentiation to yield

$$\Gamma_\sigma(p,p-k)\sim\gamma_\sigma+\frac{-\alpha}{\pi}\left[\gamma_\sigma-\frac{\gamma\cdot k k_\sigma}{k^2}\right]+\frac{1}{2}\left[\frac{\gamma\cdot k\gamma_\sigma\gamma\cdot p}{k^2}-\frac{\gamma_\sigma\gamma\cdot k\gamma\cdot p}{k^2}\right]\left[1-\left[\frac{k^2}{p^2}\right]^{-\alpha/4\pi}\right]\ . \tag{11}$$

Some comments on the solution in (11) follow. We note that for $k\to\infty$, $\Gamma_\sigma(p,p-k)\to\Gamma_\sigma(0,-k)$ consistent with our initial hypothesis made in solving (2) for small α. The solution in (11) is nonperturbative and the coefficient of the third term is $O(1)$ rather than $O(\alpha)$, and this may be traced to be due to the damping factor $(k^2/p^2)^{-\alpha/4\pi}$ which introduces a factor $1/\alpha$ to cancel out the coefficient α in the integral equation for Γ_σ in (2). That is, a term which is formally of order α ends up of order one by a dynamical $1/\alpha$ factor. It is difficult to take the contributions of the remaining integrals in (2) into account. To this end we note, however, that the coefficient C in (3), corresponding to the second p-independent term in (11), may be computed to arbitrary orders in α by setting $p=0$ in (2). Also, since the coefficients of the neglected integrals are formally of order α^2, it is expected that the third term in (11), which is of the form $O(1)\times$damping factor for $|p|/|k|\to 0$ would, in turn, lead to corrections of the form $O(\alpha)\times$damping factor, which are then smaller, and modify the coefficient of the second term in (11) to higher-order corrections in α. We may make contact with perturbation theory by noting that the factor

$$1-\exp[-\alpha\ln(k^2/p^2)]=O\left(\frac{\alpha}{4\pi}\ln(k^2/p^2)\right)$$

in (11) is consistent with the expression in (7). Finally, we note that (11) is consistent with the Ward identity. A similar analysis may be carried out in the asymptotic region $p^2,k^2\to\infty$, $k^2/p^2\to 0$.

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