Surface terms and dual formulations of gauge theories

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(Received 30 September 1982)

Previously proposed duality transformations for a pure non-Abelian gauge theory are carried out for a Yang-Mills theory with nonvanishing θ parameter. We argue that the theory is no longer self-dual even in the weak-coupling limit, as there is a surface term generated by the duality transformation. This surface term has nonzero Pontryagin index, hence it gets contributions from instanton-type configurations only. However, it does vanish for monopoles and vortices, therefore for this set of configurations self-duality is maintained in the weak-coupling region.

I. INTRODUCTION

Duality transformations have proven to be an indispensable tool in the investigation of phase structures in lattice field theories with Abelian symmetry.[1] Attempts are being made to generalize the concept to theories with non-Abelian symmetry[2] as well as to gauge theories in the continuum.[3–9]

It is hoped that for gauge theories a duality transformation can be found that leads at least in certain regions (like the weak- or strong-coupling region) back to the same theory albeit with inverted coupling constant and interchanged roles for electric and magnetic field strengths. Such a concept of "self-duality" is at the root of the Mandelstam-'t Hooft mechanism for quark confinement (compare Refs. 10 and references therein).

For this mechanism to be possible an essential property to be used is that non-Abelian gauge theories have classical solutions with topological charges. The origin of these objects may be traced back to nontrivial boundary conditions. Conversely, a physically relevant action may contain surface terms (θ terms).

It has been argued by Witten[11] and by 't Hooft[12] that the existence of *CP*-violating surface terms in the action (θ terms) may have a significant physical meaning. In particular, dyons have a minimal charge[11] $g\theta/2\pi$ (g is the coupling constant) and the phase structure of non-Abelian gauge theory gets complicated by the existence of an oblique confinement phase when $\theta\approx\pi$.[12] For this reason the duality transformation of a lattice Z_p gauge theory in the presence of a *CP*-violating θ term has recently been carried out.[13] The emerging picture is quite complicated but basically agrees with 't Hooft's suggestions concerning the phase structure when $\theta\neq 0$.[12]

We therefore find it relevant to investigate anew previously suggested duality transformations for a pure Yang-Mills theory in the continuum,[3–6] now containing a θ term. In doing so, we will also carefully keep track of any surface terms that might be created by the duality transformation itself. Indeed the duality transformation generates topologically relevant surface terms. As a result of this, previous results on the self-duality of the pure Yang-Mills theory in the weak-coupling limit[4] cannot be reproduced. More precisely, we argue that even though the dual of a pure Yang-Mills theory with a θ term is, at least in the weak-coupling limit, again a pure Yang-Mills theory, the θ parameter of the dual theory will in general be large even if it was small in the original theory, unless one identifies the dual θ term with the *CP*-violating part of the dual theory. In this respect, our result differs from the analogous one for the lattice Z_p gauge theories.[11] This difference can be traced back to surface terms generated by the duality transformation due to the non-Abelian structure of the symmetry. We note, however, that apart from these surface terms we can define θ_D and g_D similar to the ones defined in the lattice models, as the duality transformation in the continuum is similar in nature to the one in the Abelian lattice models. These surface terms, however, are instanton type only. They do not contribute for monopoles and vortices. Hence, it is the instanton contributions which spoil the simple self-duality picture in the weak-coupling limit. In a way, this is not surprising since it is precisely these contributions which complicated the phase structure of the system as suggested by 't Hooft.[12] We are as yet lacking tools for the investigation of the phase structure, but we feel that our result may be relevant for the discussion of confinement properties of non-Abelian gauge theories.

We will present two alternative ways to carry out the duality transformation. A common step for both is carried out in Sec. II, namely, the introduction of a set of auxiliary variables in the generating functional as Fourier conjugates of the field strengths. It is this step that leads to an inversion of the coupling constant.[14] At this stage, the exponent in the integrand of the functional integral is at most quadratic in the gauge potentials.

In Sec. III, following Kazama and Savit,[5] we introduce a second set of auxiliary variables, this time Fourier conjugate to the gauge potentials themselves. Doing so means that the duality transformation becomes very similar to those on the lattice.[1] The functional integration over the gauge potentials can now be carried out and produces a δ functional. The presence of this δ functional means that a constraint is to be satisfied by the auxiliary fields. Unfortunately, only one exact solution of this constraint is known. Using it, one can see that the duality transformation itself produces a surface term. Also, we isolate the only term generated by the duality transformation that can possibly be of topological relevance without recourse to any special solution of the constraint.

In Sec. IV, we follow Itabashi's[4] version of the duality transformation. Here the second set of auxiliary fields is not introduced. Rather the Gaussian integration over the gauge potentials is carried out; no constraint appears. We give the general form of the dual theory and discuss in particular its weak-coupling limit. In this limit we see that the duality transformation generates a surface term. However, it is nonzero for instanton contributions only. Hence one can conclude quite safely that for configurations belonging to the trivial maps of $S_3^\infty \rightarrow \mathscr{G}$ ($\mathscr{G}$ is the group space) self-duality in the weak coupling is maintained. This set of configurations includes, in particular, the monopoles and vortices.[15] Section V contains some concluding remarks.

II. DUALITY TRANSFORMATION FOR NONVANISHING θ

In this section we describe the first step in the duality transformation which is common to both formalisms to be employed below.

We consider a non-Abelian gauge theory with a θ term in the Euclidean domain, defined by the Lagrangian

$$\mathscr{L} = -\frac{1}{4g^2}(G^2 - i\tilde{\theta}\tilde{G}G) , \tag{2.1}$$

where the scaled vacuum angle $\tilde{\theta}$ is given by

$$\tilde{\theta} = \frac{g^2}{8\pi^2}\theta . \tag{2.2}$$

The field strength tensor $G = G(V)$ is defined as[16]

$$G^a_{\mu\nu}(V) = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + f^{abc} V^b_\mu V^c_\nu , \tag{2.3}$$

and $\tilde{G}$ is its dual,

$$\tilde{G}^a_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda} G^a_{\rho\lambda} . \tag{2.4}$$

It is convenient to define[14,17]

$$z_\pm = \frac{8\pi^2}{g^2} \pm i\theta \tag{2.5}$$

and

$$G_\pm = \tfrac{1}{2}(G \pm \tilde{G}) . \tag{2.6}$$

Then the Lagrangian (2.1) can be written as[17]

$$\mathscr{L} = -\frac{1}{32\pi^2}(z_- G_+{}^2 + z_+ G_-{}^2) . \tag{2.7}$$

We introduce the generating function $Z[\xi]$ as

$$Z[\xi] = \int DV \exp\left[-\frac{1}{32\pi^2} \int d^4x (z_- G_+{}^2 + z_+ G_-{}^2 + i\xi V) \right] . \tag{2.8}$$

As a first step of a duality transformation one now undertakes a functional Fourier transform

$$\exp\left[\frac{-z_-}{32\pi^2} \int d^4x\, G_+{}^2 \right] = \int DW_+ \exp\left[-\int d^4x \left[\frac{\pi^2}{2z_-} W_+{}^2 + \frac{i}{4}\tilde{W}_+ G_+ \right] \right] , \tag{2.9}$$

and similarly for G_-. Without restriction, the Fourier conjugate variable W_+ (W_-) can be chosen to be self-dual (anti-self-dual), since the integral over an anti-self-dual part from W_+ would be Gaussian and thus produce just a constant factor.[18]

Thus we have

$$Z[\xi]=\int DV DW_+ DW_- \delta^3(W_+-\widetilde{W}_+)\delta^3(W_-+\widetilde{W}_-)$$
$$\times\exp\left[-\int d^4x\left[\frac{\pi^2}{2z_-}W_+{}^2+\frac{\pi^2}{2z_+}W_-{}^2+\frac{i}{4}\widetilde{W}_+G_++\frac{i}{4}\widetilde{W}_-G_-+\frac{i\xi V}{32\pi^2}\right]\right], \tag{2.10}$$

where for generic tensors T and T' the δ functions are implied to mean

$$\delta^3(T-T')=\delta(T_{01}-T'_{23})\delta(T_{02}+T'_{13})\delta(T_{03}-T'_{12})\ . \tag{2.11}$$

Note that the Fourier transformation (2.10) has the effect of inverting the coupling constants, $z_\pm \rightarrow 1/z_\pm$. In order to eliminate the δ functions in (2.10), set

$$W_\pm=\tfrac{1}{2}(K_1\pm K_2)\ . \tag{2.12}$$

The δ functions in (2.10) are then equivalent to

$$\delta^3(K_1-\widetilde{K}_2)\delta^3(K_2-\widetilde{K}_1)=\delta^6(K_1-\widetilde{K}_2)\ . \tag{2.13}$$

We can now carry out the integration with respect to K_2. Setting

$$K_1=2K\ , \tag{2.14}$$

and inserting again g and $\widetilde{\theta}$ instead of $z_\pm$, one finds eventually

$$Z[\xi]=\int DV DK\exp\left\{-\int d^4x\left[\frac{g^2}{4(1+\widetilde{\theta}^2)}(K^2+i\widetilde{\theta}\widetilde{K}K)+\frac{i}{2}\widetilde{K}G+i\hat{\xi}V\right]\right\}, \tag{2.15}$$

with

$$\hat{\xi}=\frac{1}{32\pi^2}\xi\ . \tag{2.16}$$

We now want to eliminate the original variables V from Eq. (2.15). We could do so by just carrying out the integration over V since it is Gaussian.

Another possibility is to first linearize the term quadratic in V in a manner analogous to Eq. (2.9). In this way we introduce yet another field, B, and the V integration will produce a constraint to be satisfied by K and B.

We will present both approaches. The reason for doing so is first that the latter much more resembles the procedure on the lattice. Also, since we do not fix a gauge, the calculations become much simpler. On the other hand, the first method avoids the constraint and thus yields slightly more information.

III. DUALITY TRANSFORMATION IN THE CONSTRAINT FORMALISM

In this section, arguments run much along the same lines as in the paper by Kazama and Savit,[5] to which we refer the reader for any further details.

Using the definition of the field strength tensor, $G=G(V)$, we find

$$\tfrac{1}{2}\int d^4x\,\widetilde{K}G=-\int d^4x[(\partial\widetilde{K},V)-(\xi_\Sigma,V)-\tfrac{1}{2}(V,TV)]\ . \tag{3.1}$$

Here we use the following shorthand notations:

$$(\partial\widetilde{K},V)=(\partial_\mu\widetilde{K}^{\,a}_{\mu\nu})V^a_\nu\ . \tag{3.2}$$

$$T^{ab}_{\mu\nu}=\widetilde{K}^{\,c}_{\mu\nu}f^{abc} \tag{3.3}$$

and the "surface current" ξ_Σ is introduced as

$$\xi^a_{\Sigma,\nu}=\hat{s}_\mu\widetilde{K}^{\,a}_{\mu\nu}\delta(\Sigma) \tag{3.4}$$

where $\delta(\Sigma)$ is a δ function with support on the surface Σ, and $\hat{s}_\mu$ is the normal vector on this surface. We have the property that

$$\int d^4x(\xi_\Sigma,V)=\int d^4x\,\partial_\mu(\widetilde{K}^a_{\mu\nu}V^a_\nu)\ . \tag{3.5}$$

We now linearize the last term in (3.1) by introducing a new field A, viz.,

$$\exp\left[\frac{-i}{2}\int d^4x(V,TV)\right]=\int DA(\det T)^{-1/2}\exp\left\{\int d^4x\left[\frac{i}{2}(A,T^{-1}A)-i(A,V)\right]\right\}. \tag{3.6}$$

As usual, the Gaussian integration with imaginary exponential is to be understood by analytic continuation. In order that (3.6) be valid it is not necessary to assume that the eigenvalues of T have a definite sign. We do ignore the question of zero eigenvalues of T however. The existence of an inverse T^{-1} will be assumed throughout.

Using (3.8), the V integration in (3.1) produces a δ function. Shifting variables according to

$$A'=A+\hat{\xi}+\xi_\Sigma \tag{3.7}$$

and introducing variables B through

$$A'=TB \tag{3.8}$$

we eventually obtain for the generating functional

$$Z[\xi]=\int DK\,DB(\det T)^{1/2}\delta(\partial\tilde{K}-TB)\exp\left\{\int d^4x\left[\frac{-g^2}{4(1+\tilde{\theta}^2)}(K^2+i\tilde{\theta}\tilde{K}K)-i(B,\hat{\xi}+\xi_\Sigma)+\frac{i}{2}(\hat{\xi}+\xi_\Sigma,T^{-1}(\hat{\xi}+\xi_\Sigma))+\frac{i}{2}(B,TB)\right]\right\}. \tag{3.9}$$

The δ function in Eq. (3.9) implies the constraint $\partial\tilde{K}=TB$, or, written out explicitly

$$\partial_\mu \tilde{K}^a_{\mu\lambda}=f^{abc}B^b_\mu \tilde{K}^c_{\lambda\mu}\ . \tag{3.10}$$

One notices that it takes the form of a Bianchi identity for K.

Using the constraint (3.10) we obtain the identity

$$\frac{i}{2}(B,TB)=i\partial_\nu(B^a_\mu\tilde{K}^a_{\nu\mu})-\frac{i}{2}\tilde{K}^a_{\nu\mu}G^a_{\nu\mu}(B)\ , \tag{3.11}$$

where $G(B)$ is the field strength tensor as defined in (2.3), but with $V\rightarrow B$. We thus obtain instead of (3.9)

$$Z[\xi]=\int DK\,DB(\det T)^{1/2}\delta(\partial\tilde{K}-TB)\exp\left\{\int d^4x\left[\frac{-g^2}{4(1+\tilde{\theta}^2)}(K^2+i\tilde{\theta}\tilde{K}K)-i(B,\hat{\xi})+\frac{i}{2}(\hat{\xi}+\xi_\Sigma,T^{-1}(\hat{\xi}+\xi_\Sigma))-\frac{i}{2}(\tilde{K},G(B))\right]\right\}. \tag{3.12}$$

Since there is still an unsolved constraint, we have not really found the dual theory yet. However, since (3.10) is just a Bianchi identity, there is a special solution,

$$K=G(B)\ . \tag{3.13}$$

Using it, the first term in the exponential of (3.12) becomes

$$\frac{-g^2}{4(1+\tilde{\theta}^2)}[G^2(B)+i\tilde{\theta}\tilde{G}(B)G(B)] \tag{3.14}$$

and the last term becomes

$$\frac{-i}{2}\tilde{G}(B)G(B)\ . \tag{3.15}$$

This indicates that the duality transformation in general produces instanton-type surface terms corresponding to large values of the θ parameter even if none were present in the original theory ($\theta=0$).

Two other surface terms which appear in (3.12) are $(\xi_\Sigma,T^{-1}\hat{\xi})$ and $(\xi_\Sigma,T^{-1}\xi_\Sigma)$. The first vanishes if the external source falls fast enough at infinity. We will assume that this is the case. For the second term we have

$$\int d^4x(\xi_\Sigma,T^{-1}\xi_\Sigma)=\int d^3\sigma\hat{s}_\mu\tilde{K}^a_{\mu\nu}T^{-1ab}_{\nu\lambda}\hat{s}_\sigma\tilde{K}^b_{\sigma\lambda}\delta(\Sigma)\ , \tag{3.16}$$

where we use the definition (3.4) of $\xi^a_{\Sigma,\mu}$. This may have a finite contribution but it does not have a topological significance. It vanishes for any field configuration which falls off at infinity faster than r^{-3}. In particular, for instanton configurations we have $K(x)\sim r^{-4}$, hence it is zero. We will argue now that it can be ignored in the weak-coupling limit ($g\rightarrow 0$).

Following Kazama and Savit,[5] we rescale variables according to

$$K=\frac{1}{g}K^{(0)}\ ,\quad B=\frac{1}{g}B^{(0)}\ ,\quad T=\frac{1}{g}T^{(0)}\ . \tag{3.17}$$

Consider now the factor

$$(\det T)^{1/2}\exp\left[\frac{i}{2}\int d^4x(\xi_\Sigma,T^{-1}\xi_\Sigma)\right]=\left[\det\frac{1}{g}T^{(0)}\right]^{1/2}\exp\left[\frac{i}{2g}\int d^4x(\xi_\Sigma^{(0)},T^{(0)-1}\xi_\Sigma^{(0)})\right]\underset{g\to 0}{\rightarrow}\delta(\xi_\Sigma^{(0)}) , \tag{3.18}$$

which uses the following definition of the δ function:

$$\delta(x)=\frac{1}{\sqrt{\pi}}e^{i\pi/4}\lim_{\epsilon\to 0_+}\frac{1}{\sqrt{\epsilon}}e^{-ix^2/\epsilon} . \tag{3.19}$$

Thus one should include in the weak-coupling limit configurations which vanish at infinity faster than r^{-3}. In particular, instanton configurations satisfy this condition.

If in the present formalism we write

$$K=G(B)+\omega , \tag{3.20}$$

we do not obtain much information about ω, except that on the surface Σ, $G(B)=-\omega$ for small g. This situation improves in a formulation of the duality transformation without constraints which we will give below. There we will argue that at least in the weak-coupling theory ω can be ignored. In that case one can identify θ_D and g_D of the dual theory. This will be done in the next section.

IV. DUALITY TRANSFORMATION WITHOUT CONSTRAINT

In this section we will give an alternative version of the previous considerations, as followed also by the authors of Refs. 3, 4, 8, and 9. Two main points will be different.

First, we will carry out directly the V integration in Eq. (2.15) without linearizing the term quadratic in V. This way we lose some of the similarity of our duality transformation with those on the lattice, but we avoid the constraint.

The other important difference is that we will adopt a gauge, the radial gauge

$$x_\mu V_\mu^a(x)=0 . \tag{4.1}$$

This gauge is compatible with regular field configurations only. Its main advantages are that the ghosts decouple and that a simple field strength formulation of non-Abelian gauge theories is possible. The latter property is the reason why it has been used in the present context.[4]

From (4.1) one finds

$$x_\rho G_{\rho\mu}^a(V(x))=(1+x_\rho\partial_\rho)V_\mu^a(x) , \tag{4.2}$$

which may be considered as a differential equation with the formal solution

$$V_\mu^a(x)=\int_0^1 d\alpha[y_\rho G_{\rho\mu}^a(y)]\,|_{y=\alpha x} . \tag{4.3}$$

The inversion (4.3) is unique. In other words, the radial gauge is complete; there is no problem with field strength copies.

The structure of the calculation is the same as in the previous section. Because of the gauge condition, some details will get more complicated, however. The reader interested in any details omitted here is referred to the paper by Itabashi.[4]

The first step is to process the term

$$\exp\left[\frac{i}{2}\int d^4x\,\tilde{G}K\right] .$$

The following identity holds:

$$\frac{1}{2}\int d^4x\,\tilde{G}_{\mu\nu}^a K_{\mu\nu}^a=\int\frac{d^4x}{x_4}(V_i^aX_i^a-\tfrac{1}{2}V_i^aT_{ij}^{ab}V_j^b-V_i^aJ_{\Sigma,i}^a) , \tag{4.4}$$

where

$$X_i^a=\tfrac{1}{2}\epsilon_{ijk}\{(2+x_\lambda\partial_\lambda)K_{jk}^a-[\partial_j(x_\rho K_{\rho k}^a)-\partial_k(x_\rho K_{\rho j}^a)]\} , \tag{4.5}$$

and the "surface current" $J_{\Sigma,i}^a$ is given as

$$J_{\Sigma,i}^a=\tfrac{1}{2}\epsilon_{ijk}\left[\int d^3\sigma_\lambda(\rho)\rho_\lambda K_{jk}^a(\rho)\delta^4(x-\rho)-2\int d\zeta_4\int d^2\sigma_j(\zeta)\zeta_\rho K_{\rho k}^a(\zeta)\delta^4(x-\zeta)\right] . \tag{4.6}$$

The matrix T is now

$$T_{ij}^{ab}=x_\rho K_{\rho k}^c\epsilon ijkf^{abc} . \tag{4.7}$$

Here Latin indices run from 1 to 3 and Greek ones from 1 to 4. Equation (4.4) may be proved by repeated use of the gauge condition. The generating functional is now [compare with Eq. (2.15)]

$$Z[\xi]=\int DV\,DK\,\delta(x_\mu V^a_\mu)\exp\left[\int d^4x\left[\frac{-g^2}{4(1+\tilde\theta^2)}(K^2+i\tilde\theta\tilde K K)\right]\right](\det T)^{-1/2}$$
$$\times\exp\left\{-i\int\frac{d^4x}{x_4}[V^a_i(X^a_i+\eta^a_i-J^a_{\Sigma,i})-\tfrac{1}{2}V^a_iT^{ab}_{ij}V^b_j]\right\}, \tag{4.8}$$

where

$$\eta^a_i=x_4\hat\xi^a_i-x_i\hat\xi^a_4 . \tag{4.9}$$

We also use the identity

$$X^a_i=Y^a_i+T^{ab}_{ij}B^b_j , \tag{4.10}$$

where the "dual" field B is introduced through

$$B^a_\mu(x)=\int_0^1 d\alpha[y_\rho K^a_{\rho\mu}(y)]_{y=\alpha x} , \tag{4.11}$$

with

$$Y^a_i=\tfrac{1}{2}(2+x_\lambda\partial_\lambda)\epsilon_{ijk}[K^a_{jk}-G^a_{jk}(B)] , \tag{4.12}$$

to rewrite (4.8) as

$$Z[\xi]=\int DK(\det T)^{-1/2}\exp\left[\frac{-g^2}{4(1+\tilde\theta^2)}\int d^4x(K^2+i\tilde\theta\tilde K K)\right]$$
$$\times\exp\left[-\frac{i}{2}\int\frac{d^4x}{x_4}(Y+TB+\eta-J_\Sigma,T^{-1}(Y+TB+\eta-J_\Sigma))\right] . \tag{4.13}$$

The surface terms that are generated by the duality transformation are those that contain J_Σ and the term (B,TB). The scalar products (,) are to be understood as three dimensional.

Using the identity (4.4) with $V\to B$, the last factor in (4.13) may be rewritten as

$$\exp\left[-\frac{i}{2}\int\frac{d^4x}{x_4}[(Y+\eta-J_\Sigma,T^{-1}(Y+\eta-J_\Sigma))+2(B,\eta)]\right]\exp\left[-\frac{i}{2}\int d^4x\tilde G^a_{\mu\nu}(B)K^a_{\mu\nu}\right] . \tag{4.14}$$

Equation (4.13) is exactly analogous to the corresponding equation in Ref. 4 except for the $\tilde K K$ term and a substitution $Y\to Y-J_\Sigma$. We can therefore just quote the result for the generating functional,

$$Z[\xi]=\int DB^a_\mu\delta(x_\mu B^a_\mu)(\det T)^{-1/2}\exp\left[-\frac{g^2}{4(1+\tilde\theta^2)}\int d^4x(G^2+i\tilde\theta\tilde G G)\right]$$
$$\times\int D\omega^a_{\mu\nu}\delta(x_\rho\omega^a_{\rho k})\exp\left\{-\frac{g^2}{4(1+\tilde\theta^2)}\int d^4x[\omega^2+2G\omega+i\tilde\theta(\tilde\omega\omega+2\tilde G\omega)]\right\}$$
$$\times\exp\left\{-\frac{i}{2}\int\frac{d^4x}{x_4}[(Y+\eta-J_\Sigma,T^{-1}(Y+\eta-J_\Sigma))+2(B,\eta)]\right\}\exp\left[-\frac{i}{2}\int d^4x\tilde G K\right] . \tag{4.15}$$

In this expression

$$K^a_{\mu\nu}=G^a_{\mu\nu}(B)+\omega^a_{\mu\nu} , \tag{4.16}$$

where $\omega^a_{\mu\nu}$ has to satisfy

$$x_\rho\omega^a_{\rho k}=0 . \tag{4.17}$$

Therefore the independent variables of the dual theory, which replace $K^a_{\mu\nu}$, are B^a_i and ω^a_{ik}. Note that because of (4.17) T and T^{-1} do not depend on ω^a_{ik}.[19]

The weak-coupling limit now yields

$$(\det T)^{-1/2}\exp\left[-\frac{i}{2}\int\frac{d^4x}{x_4}(Y-J_\Sigma,T^{-1}(Y-J_\Sigma))\right]\underset{\text{small}\,g}{\sim}\delta(Y-J_\Sigma)\,. \tag{4.18}$$

From the definition of Y [Eq. (4.12)] we obtain the differential equation

$$D\omega_{ij}^a=\epsilon_{ijk}Y_k^a\,, \tag{4.19}$$

with $D=(2+x_\lambda\partial_\lambda)$. Because of (4.18) we may replace this in the weak-coupling limit by

$$D\omega_{ij}^a=\epsilon_{ijk}J_{\Sigma,k}^a\,. \tag{4.20}$$

This differential equation has the formal solution

$$\omega_{ij}^a(x)=\epsilon_{ijk}\int_0^1 d\alpha\,\alpha J_{\Sigma,k}^a(\alpha x)\,, \tag{4.21}$$

which, using the explicit form (4.6) for the surface current, may be written as

$$\omega_{ij}^a(x)=\int_0^1 d\alpha\,\alpha\left\{\int_{\Sigma_3}d^3\sigma_\lambda\rho_\lambda K_{ij}^a(\rho)\delta^4(\alpha x-\rho)\right.$$
$$\left.-\int d\zeta_4\int_{\Sigma_2}[d^2\sigma_i\zeta_\rho K_{\rho j}^a(\zeta)-d^2\sigma_j\zeta_\rho K_{\rho i}^a(\zeta)]\delta^4(\alpha x-\zeta)\right\}. \tag{4.22}$$

Here Σ_2 and Σ_3 specify two- and three-dimensional surfaces, respectively. Even though this solution is formal in the sense that ω_{ij}^a appears on both sides, we can still draw some information from it.

In fact, let Σ_2 and Σ_3 be the spheres with radius r and R, respectively. Because of the δ function in (4.22) and the range of the α integration, $\omega_{ij}^a(x)$ is nonvanishing and gets contributions from K only from outside these surfaces. Now for finite-action configurations, K tends to zero for large enough r and R more rapidly than r^{-2} (R^{-2}).

We therefore conclude, in particular, that if we choose the surfaces Σ_2 and Σ_3 sufficiently far out, then we can neglect the integrals $\int\omega G$, $\int\tilde{\omega}G$, $\int\omega\omega$, and $\int\tilde{\omega}\omega$ as compared to the integrals $\int GG$ and $\int G\tilde{G}$, at least for finite-action configurations.

We also note in passing, that on purely algebraic grounds, $\tilde{\omega}\omega=0$ if we set, formally,[20]

$$\omega_{ij}^a=\epsilon_{ijk}D^{-1}J_{\Sigma,k}^a\,. \tag{4.23}$$

We therefore obtain as the weak-coupling limit of (4.15)

$$Z[\xi]\underset{\text{small}\,g}{\approx}\int DB\,\delta(x_\mu B_\mu^a)\exp\left\{-\int d^4x\left[\frac{g^2}{4(1+\tilde{\theta}^2)}(G^2+i\tilde{\theta}\tilde{G}G)+iB\hat{\xi}+\frac{i}{2x_4}(B,TB)\right]\right\} \tag{4.24}$$

with

$$G=G(B)\,.$$

In (4.24) we use

$$\int\tilde{G}_{\mu\nu}^a(B)K_{\mu\nu}^a\,d^4x=\int\frac{d^4x}{x_4}(B,TB)\,, \tag{4.25}$$

which is valid in the weak-coupling limit. To prove (4.25) one has to insert (4.10) and (4.18) in (4.4). Obviously this result is more specific than the corresponding one in the previous section. Again we note the extra surface term originating in the duality transformation. Yet it does not emerge as a CP-violating surface term.

To understand this point consider the case $\theta=0$. The original theory is CP conserving. That is,

$$\exp\left[-\frac{1}{4g^2}\int d^4x\,G^2(V)\right]$$

is real [when $\theta=0$ the decomposition into $G_\pm(V)$ is not needed], so in the duality transformation in (2.9) we could use

$$\cos\left[\frac{1}{4}\int d^4x\,\tilde{W}G\right]$$

rather than

$$\exp\left[-\frac{i}{4}\int d^4x\,\tilde{W}G\right].$$

The final result will be *CP* conserving, even though we would still have the right-hand side of (4.25) (in the weak-coupling region), this time in the cos rather than in the exponential. Of course the reason for the invariance is that the cos is an even function of its argument whereas the exponential is not. So the fact that (4.25) is odd under *CP* does not affect the generating functional. This is true as long as $\theta=0$. When $\theta\neq 0$ the duality transformation has to be done as in (2.9); however, (4.25), which is generated by the duality transformation, should not be mixed with the *CP*-violating surface terms. This observation is important when the identification of the parameters of the dual theory is done.

The θ term in the original theory represents a *CP*-violating surface term. The same should be the case in the dual theory. Hence, we define the dual coupling constant g_D and the dual θ angle θ_D as

$$g_D{}^2=\frac{g^2}{64\pi^2}\left[\left[\frac{8\pi^2}{g^2}\right]^2+\theta^2\right], \tag{4.26a}$$

$$\theta_D=-\frac{64\pi^4\theta}{(8\pi^2/g^2)^2+\theta^2}. \tag{4.26b}$$

Apart from some constant factors they coincide with the dual parameters defined in Ref. 13 for a Z_p gauge theory on a lattice. With these definitions we have for the dual Lagrangian in the weak-coupling limit

$$\mathscr{L}_D=-\frac{1}{4g_D{}^2}(GG-i\tilde{\theta}_D\tilde{G}G)-\frac{i}{2x_4}(B,TB) \tag{4.27}$$

and $\tilde{\theta}_D$ is defined as $\tilde{\theta}$ in (2.2) but with θ_D replacing θ.

It is only when $\theta=0$ or in the weak-coupling region when $\theta\neq 0$ that we have the inversion of the coupling constant in the dual theory,

$$g_D \underset{\text{small}\,g}{\sim} \frac{1}{g}, \tag{4.28}$$

In particular, the coupling constant appearing in the unscaled dual field strengths is $1/g$ and not g_D. From the scaling transformation of the dual variables in (3.17) we find that the unscaled field strength tensor is

$$G_{\mu\nu}^{(0)a}=\partial_\mu B_\nu^{(0)a}-\partial_\nu B_\mu^{(0)a}+\frac{1}{g}f^{abc}B_\mu^{(0)b}B_\nu^{(0)c}. \tag{4.29}$$

In other words, for $\theta\neq 0$ the effective coupling constant of the dual theory is no longer $1/g$, unless g is small enough.

The dual Lagrangian in the weak-coupling limit can be given the form (2.7) if we use[21] (4.25) and define

$$z_\pm^D=\frac{8\pi^2}{g_D{}^2}\pm i(\tilde{\theta}_D-2g_D{}^2). \tag{4.30}$$

In that case we have

$$\mathscr{L}_D(x)=-\frac{1}{32\pi^2}(z_-^D G_+{}^2+z_+^D G_-{}^2).$$

This form of $\mathscr{L}_D(x)$ and the fact that the dual fields satisfy the radial-gauge condition suggest that, at least for $\theta\neq 0$, the theory is self-dual in the weak-coupling region with $g_D\approx 1/g$. For $\theta=0$ we have $z_+=z_-$ in the original Lagrangian (2.7). That is, there are equal contributions of self-dual (or anti-self-dual) field configurations ($G=\pm\tilde{G}$, respectively). In the dual theory these contributions to $\mathscr{L}_D$ are weighted by z_-^D and z_+^D, respectively. But in spite of the similarity in the form of $\mathscr{L}_D(x)$ and $\mathscr{L}(x)$ for $\theta=0$, it is clear that self-duality is lost in the weak-coupling region because $\tilde{G}G$ does not contribute to $\mathscr{L}(x)$. The configurations which spoil this property are the instantons.

From (4.25) we know that

$$\int \frac{d^4x}{x_4}(B,TB)$$

is exactly the Pontryagin index.[21] That is, it characterizes the maps of $\pi_3(\mathscr{G})$. Configurations belonging to trivial maps of $S_3^\infty\to\mathscr{G}$ give zero contribution. In particular monopoles and vortices[15] belong to the trivial maps of $\pi_3(\mathscr{G})$; hence for these configurations the last term of (4.27) vanishes. That is, self-duality of the pure Yang-Mills theory holds in the weak-coupling region only for configurations with zero Pontryagin index (unless $\theta\neq 0$). This result is weaker than the one found in Ref. 4, and probably reflects the complications in the phase structure introduced by the instanton contributions.[12,13] We have been able to uncover it by a careful treatment of all surface terms produced by the duality transformation itself.

V. CONCLUDING REMARKS

The duality transformation of a pure Yang-Mills theory in the presence of a *CP*-violating surface term has been performed. We presented it in two different ways. In the first we found the constraint satisfied by the dual variables, thus making it similar to the duality transformation performed for Abelian lattice models. In the second way we picked up the radial gauge and found the specific form of the dual Lagrangian.

In both ways we found, by keeping track of all surface terms, that the duality transformation generates surface terms of a topological nature. They emerge because of the non-Abelian property of our model. In the weak-coupling region (small g) the dual theory resembles the original one with a modified parameter; however, when $\theta=0$, self-duality is lost as the instantons give nonzero contributions to the surface term generated by the duality transformation. For $\theta\neq0$ such a surface term appears in the original Lagrangian; hence the dual Lagrangian is similar in form to the original one.

However, there is a subset of field configurations, which belong to the trivial maps of $\pi_3(\mathcal{G})$, for which the extra surface term vanishes. Thus, for these configurations self-duality is maintained in the weak-coupling region. This set of configurations includes, in particular, the monopoles and vortices of the theory.

Self-duality of the theory (to be distinguished from the self-duality of the phase[22]) is an important ingredient in understanding confinement in non-Abelian gauge theory. Thus, the self-duality of the theory can facilitate the analysis of the model and can help one to understand the existence of electric vortices (confinement phase) as due to condensation of magnetic monopoles,[10,12] which is analogous to understanding the existence of magnetic vortices (Higgs phase) due to the condensation of electric charges. Here we found that for $\theta=0$ this picture is not spoiled, because for the subset of fields which are responsible for confinement[10,12] self-duality is maintained in the weak-coupling region. The situation is more complicated when $\theta\neq0$. We saw that the surface contributions (in particular the instanton's) are essential in rendering the theory self-dual (in the small-g region). Moreover when $\theta\neq0$ the monopole's charge is modified by a term proportional to θ.[11] This led 't Hooft to suggest the existence of an oblique confinement phase when $\theta\approx\pi$. Hence the extra surface term may be essential in understanding this phase, and the general phase structure when $\theta\neq0$. Unfortunately, we do not have yet a clear picture of the phase structure when $\theta\neq0$, so we do not know yet how important this extra surface term is going to be.

The effective parameter of the dual theory can be written in terms of the coupling constant g and the vacuum angle θ. Apart from some constant factors they are similar to the ones found in the dual model of a Z_p gauge theory on a lattice with a nonvanishing θ parameter. However, there is no clear definition of the dual coupling constant. In the dual unscaled field strengths it appears as $1/g$, whereas in the dual Lagrangian there is another parameter the effective g, though, $\tilde{g}_D(g,\theta)\approx1/g$. So only for these values of g do we have a simple inversion of the coupling constant in the dual theory. However, for $\theta=0$, g_D is exactly $1/g$. The dual θ angle (θ_D) has been identified with the CP-violating part of the dual theory. This led to the agreement with the results of the Z_p gauge model on the lattice.

ACKNOWLEDGMENTS

Thanks are due to Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for the hospitality extended to us at the International Centre for Theoretical Physics, Trieste.

One of us (W.M.) is also indebted in a similar way to Professor P. Budinich and Professor L. Fonda, and the International School for Advanced Studies, Trieste.

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[14]Following 't Hooft (Ref. 17) we find it convenient to define a complex variable $z(g,\theta)$. This generalized coupling constant gets inverted by the duality transformation.

[15]Actually to analyze this sector precisely one has to introduce scalar fields which break down the symmetry. The reason is that in a pure Yang-Mills theory monopoles and vortices are singular field configurations which are not attainable to the radial gauge used in Ref. 4. When this is done (Ref. 8) one finds that self-duality in the weak-coupling region is valid in directions of the algebra space which corresponds to unbroken symmetries or in regions where monopoles and vortices are concentrated. The inclusion of the θ term in the original theory does not spoil this property since the surface term produced by the duality transformation vanishes for monopoles and vortices.

[16]We use here the scaled fields, that is,

$$V_\mu^a = gV_\mu^{(0)a}\ ,\quad G_{\mu\nu}^{(0)a} = gG_{\mu\nu}^a$$

and

$$G_{\mu\nu}^{(0)a} = \partial_\mu V_\nu^{(0)a} - \partial_\nu V_\mu^{(0)a} + gf^{abc}V_\mu^{(0)b}V_\nu^{(0)c}\ .$$

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[18]It is trivial to see that

$$\widetilde{W}_+ G_- = \widetilde{W}_- G_+ = 0\ .$$

[19]Note that from (4.11), (4.16), and (4.17) we can write

$$B_\mu^a(x) = \int_0^1 d\alpha (\alpha x_\rho) G_{\rho\mu}^a(B(\alpha x))\ .$$

[20]To prove this one has to use the gauge condition (4.17).

[21]By using the radial-gauge condition, one can show that the right-hand side of (4.25) is a surface term. We have calculated both sides of (4.25) for the Belavin-Polyakov-Schwartz-Tyupkin instanton after setting $K \to G$ on the left-hand side. Both sides then agree. Therefore, for this particular configuration we have indeed

$$\int \widetilde{G}\omega\, d^4x = \int G\omega\, d^4x = 0\ .$$

[22]One should distinguish between self-duality of the theory and self-duality of the phase. It is possible to have a self-dual theory with a few phases dual to each other (like the confinement and Higgs phases of Yang-Mills theory) and no self-dual phase. A self-dual phase, however, may appear in some self-dual models, e.g., the Georgi-Glashow model has such a phase. In a self-dual theory the dynamics of the dual variables (the Lagrangian in field theory, or the Hamiltonian in a statistical system), is the same as that of the original ones. However, the original and the dual variables may not appear simultaneously as physical excitations in a given phase even though the system is self-dual. This happens only if the phase is self-dual.