PHYSICAL REVIEW D VOLUME 27, NUMBER 8 15 APRIL 1983

Class of very simple gauge theories which remain renormalizable even in the limit of infinite gauge coupling constant

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(Received 23 November 1982)

A class of local gauge theories based on compact semisimple Lie groups is studied in the limit of infinite gauge coupling constant ($g=\infty$). In general, in this limit, the gauge fields become auxiliary in all gauge theories, and the system develops a richer structure of constraints. Unfortunately for most gauge theories, this limit turns out to be too singular to quantize and the theory ceases to be renormalizable. For a special class of gauge theories, however, where there are no fermions and there is only one multiplet of scalars in the adjoint representation, we prove that a consistent renormalizable quantum theory exists even in this very singular limit. We trace this exceptional behavior to a new local translationlike symmetry in the functional space that this class of gauge models possesses in the limit of infinite gauge coupling constant. By carrying out the constraint analysis, evaluating the Faddeev-Popov-Senjanovic determinant, and doing the functional integrations over the canonical momenta, the gauge fields, and most of the components of the scalar fields, we obtain an extremely simple result with no non-Abelian structure left in it. For example, for group SU(2), the final answer reduces to the theory of a one-component self-interacting real ϕ^4 scalar field theory. Throughout this paper, we use functional methods and make no approximations; our results are nonperturbative and exact. We also discuss some of the possible implications of our results.

I. INTRODUCTION

The study of quantized gauge field theories in the strong-coupling-constant limit has always presented extreme difficulties in calculations, if not in the formalism. Since perturbative calculations become completely unreliable, physicists have sought alternative means of handling this case by the use of lattice calculations and Monte Carlo simulations. In this paper, I would like to propose that we can gain insight into the behavior of the gauge theories in the very-strong-coupling-constant regime by studying the extreme case, namely, the (apparently singular) limit of truly infinite coupling constant. At first glance, it would seem contradictory that a problem which is very difficult to handle for a large but finite coupling constant should become more manageable for the truly infinite case. Not suprisingly, it turns out that the infinite-coupling limit is too singular and nonrenormalizable for gauge theories, except for a small class of them, which remains well defined and renormalizable in this very singular limit. It is exactly this special class of gauge theories that we intend to study in this paper. We emphasize that this paper is not about a large-coupling approximation method. The gauge coupling is taken to be truly infinite and our results are exact and nonperturbative.

To illustrate the basic point, let us start with a gauge theory based on a semisimple Lie group G which contains fermions and scalars in given representations (not necessarily irreducible) of G (Ref. 1):

$$\mathscr{L}^{\text{gauge}}=-\frac{1}{4g^2}F^a_{\mu\nu}F_a^{\mu\nu}+\bar{\psi}(i\not{\mathscr{D}}-m)\psi+\tfrac{1}{2}(\mathscr{D}_\mu\phi)^\dagger(\mathscr{D}^\mu\phi)-V(\phi)+\text{Yukawa terms if any},$$

where $\mathscr{D}_\mu$ is the gauge-covariant differential operator, and $V(\phi)$ is the scalar self-interaction potential containing terms of order at most four. It is clear that if we go to the infinite-coupling-constant limit ($g=\infty$), the kinetic term in the Lagrangian for the gauge fields can be dropped and the gauge fields become auxiliary fields. The presence of auxiliary fields has two immediate consequences. The first one is that the system now possesses more dynamical constraints than before. The second one is that

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for the quantized case all the functional integrations over the gauge fields can be performed to eliminate them.

This apparent simplification of the Lagrangian, however, is a dangerous one for the quantized case. The Lagrangian $\mathcal{L}^{\text{gauge}}$ given above is known to be renormalizable.[2] The remaining Lagrangian, however, after we set $g=\infty$ and drop the kinetic term for the gauge fields, is, at best, suspect from the point of renormalizability. One cannot arbitrarily exclude some terms from a renormalizable Lagrangian and still maintain renormalizability unless, of course, there exists a new symmetry of the Lagrangian resulting from the exclusion of the terms dropped. In the general case, when we drop the kinetic term from the Lagrangian $\mathcal{L}^{\text{gauge}}$, we generate new constraints; unfortunately, these new constraints are of second class[3] and do not correspond to the generators of a new symmetry. We must conclude, therefore, that the limit $g=\infty$ is indeed a very singular one and that most of the gauge theories are not renormalizable without the presence of the kinetic term for the gauge fields. Although this is often the case, there are exceptions. The first example for such an exception was reported for an SU(2) gauge theory of scalars in the adjoint representation.[4] In general, for any semisimple compact Lie group G, the theory with a single multiplet of scalars in the adjoint representation retains its renormalizability in this singular limit of $g=\infty$.

To illustrate this point, let us start from

$$\mathcal{L}'=-\frac{1}{4g^2}F^a_{\mu\nu}F^{\mu\nu}_a+\tfrac{1}{2}(\mathcal{D}_\mu\phi)(\mathcal{D}^\mu\phi)_a-V(\phi)\ , \tag{1}$$

and drop the kinetic term for the gauge fields at $g=\infty$:

$$\mathcal{L}=\lim_{g\to\infty}\mathcal{L}'=\tfrac{1}{2}(\mathcal{D}_\mu\phi)_a(\mathcal{D}^\mu\phi)_a-V(\phi)\ . \tag{2}$$

Unlike the general case of a gauge theory which contains scalars in representations other than the adjoint representations, this case indeed has a larger symmetry at $g=\infty$: *The Lagrangian $\mathcal{L}$ given by Eq. (2) is not only invariant under the ordinary local gauge transformations of the group G, but also under the following new transformation*[5]*:*

$$\begin{aligned}&\phi_a(x)\to\phi_a(x)\ ,\\ &A^\mu_a(x)\to A^\mu_a(x)+\phi_a(x)\Lambda^\mu(x,\phi(x),A(x))\ ,\end{aligned} \tag{3}$$

where Λ^μ are completely arbitrary functions of space-time x, and/or arbitrary functionals of the fields $\phi_a(x)$ and $A^a_\mu(x)$.

If we failed to notice this new invariance upon the inspection of the Lagrangian $\mathcal{L}$ given by Eq. (2), we would rediscover it through the constraint analysis of this system. As expected, not all of the new constraints generated by dropping the gauge kinetic term turn out to be second class in this case[6]; indeed, some of them are first class and they generate this new invariance expressed by Eq. (3).

To describe the effect of this invariance in a more transparent way, let us make a brief digression and discuss the question of gauge fixing for classical fields. It is well known that given any potential A^a_μ, one can make a gauge transformation to the potential A'^a_μ which is in the axial gauge[7]: $\eta_\mu A'^\mu_a=0$, where η_μ is an arbitrary constant spacelike vector. This is all we can do (other than a global gauge transformation) in the case where the gauge kinetic term is present. However, for the Lagrangian $\mathcal{L}$, the invariance expressed by Eq. (3) enables us to restrict A^a_μ even further. Using that invariance, we obtain a new potential $A''^\mu_a=A'^\mu_a+\Lambda^\mu\phi_a$, where we can choose Λ^μ as we please. The choice $\Lambda_\mu=-\alpha_a A'^a_\mu/(\alpha_b\phi_b)$ for any $\alpha_a(x,\phi(x))$ is particulary interesting,[8] since we will obtain $\alpha_a A''^\mu_a=0$. But more importantly, the new potential A''^a_μ is still in the axial gauge: $\eta_\mu A''^\mu_a=0$. It is clear now that the extra invariance this system possesses allows us to restrict A^μ_a more than we could in the ordinary case. We can simultaneously satisfy $\eta_\mu A^\mu_a=0$ and $\alpha_a A^\mu_a=0$. In Ref. 4, this gauge was named *doubly axial gauge.* A particularly interesting special case is $\alpha_a=\phi_a$, where the classical field configurations are easier to deal with.[9]

Even though they are obvious, I would like to make a couple of remarks pertaining to this new extra invariance. The first remark is to emphasize that the Lagrangian $\mathcal{L}'$ is not invariant under the transformation given by Eq. (3); only $\mathcal{L}$ is. This invariance then prevents the generation of the kinetic term for the gauge fields by higher-order corrections, keeping the theory renormalizable. Secondly, this invariance we have been describing is peculiar to the adjoint representation of the scalar fields. No other representation content for any combination of scalar and/or fermion fields that possesses such an invariance is known to me presently. Stated more precisely, the theory with the scalars in the adjoint representation (and no fermions) is the only class of models known to me where the constraints generated by taking the $g=\infty$ limit are not all second class.[10]

The rest of this paper, then, is organized as follows: In the next section, we study the general problem of the quantized gauge theory given by the Lagrangian $\mathcal{L}$ of Eq. (3). In Sec. III, we will first specialize in the simple case of $G=\text{SU}(2)$, and work out the complete details of the quantized theory and show that it is equivalent to a one-component ϕ^4 real scalar theory. In this case, we will also compare the quantized result with the classical one, and also present a brief discussion of the problem for arbitrary semisimple gauge groups. Finally, in Sec. IV, we will state our conclusions and present a discussion of gauge theories in the singular limit of $g=\infty$.

II. GENERAL FORMALISM

We start with the Lagrangian $\mathscr{L}$ given by Eq. (2) and possessing the new invariance expressed by Eq. (3). The systematic quantization of this system requires the analysis of its constraint structure first. After that the Faddeev-Popov-Senjanovic (FPS) determinant[11] can be computed and the path integration can be performed (wherever possible) in the generating functions $W[J]$.[12]

We start the constraint analysis by evaluating the canonical momenta. The momenta canonical to the gauge fields A_μ^a vanish: $E_a^\mu \equiv \partial\mathscr{L}/\partial(\partial_0 A_\mu^a)\approx 0$. These are the primary constraints.[3] The momenta canonical to the scalar fields are

$$\xi_a \equiv \partial\mathscr{L}/\partial(\partial_0\phi_a)=(\mathscr{D}^0\phi)_a$$
$$\equiv \partial^0\phi_a + f_{abc}\phi_b A_c^0 \ .$$

The canonical Hamiltonian density is then computed to be

$$\mathscr{H}_c = \tfrac{1}{2}\xi_a\xi_a - f_{abc}\xi_a\phi_b A_c^0 - \tfrac{1}{2}(\mathscr{D}_i\phi)_a(\mathscr{D}^i\phi)_a + V(\phi) \ . \tag{4}$$

The primary Hamiltonian is obtained by adjoining the primary constraints

$$\mathscr{H}_p = \mathscr{H}_c + K_a^\mu E_\mu^a \ , \tag{5}$$

where the coefficients K_μ^a are arbitrary for the first-class constraints and are determined in terms of the fields and momenta for the second-class ones.[3] The evaluation of these coefficients K_a^μ is only necessary for the classical Hamiltonian equations. Fortunately, the functional quantization method bypasses this point: there is no need to compute K_μ^a that correspond to the second-class constraints.

As usual, the requirement of consistency is stated by

$$\{E_\mu^a, H_p\} \equiv \left\{E_\mu^a(\vec{x},t),\ \int d^3y\, \mathscr{H}_p(\vec{y},t)\right\} \approx 0 \ ,$$

which generates the secondary constraints χ_a and ψ_a^i, which are easily computed to be[13]

$$\{E_a^0, H_p\} \approx -f_{abc}\xi_b\phi_c \equiv -\chi_a \approx 0 \ , \tag{6}$$

$$\{E_a^i, H_p\} \approx -f_{abc}\phi_b(\mathscr{D}^i\phi)_c \equiv -\psi_a^i \approx 0 \ . \tag{7}$$

For consistency, these secondary constraints have to be maintained.[3] By direct computation we find

$$\{\chi_a, H_p\} \approx -f_{abc}\chi_b A_c^0 - (\mathscr{D}_i\psi^i)_a \approx 0 \ , \tag{8}$$

$$\{\psi_a^i, H_p\} \approx f_{abc}\phi_b\{2(\mathscr{D}_i\xi)_c - f_{cde}\phi_d((\mathscr{D}^i A^0)_e - K_e^i)\} \approx 0 \ . \tag{9}$$

Unfortunately the algebra leading to Eqs. (8) and (9) is quite long and tedious and is not very illuminating from the point of view of physics. For this reason, the proofs of these two equations are relegated to Appendix A for the reader who wishes to trace the steps of the computation.

Equation (8) clearly states that no new constraints are generated by the requirement that χ_a have to be maintained. Equation (9) takes more care to interpret correctly. At first sight, it appears that all Eq. (9) does is to determine the coefficients K_e^i in terms of ϕ, ξ, and A^μ. However, K_e^i are only partially determined from Eq. (9). It is easy to see that all the combinations $\phi_a\{\psi_a^i, H_p\}$, $\xi_a\{\psi_a^i, H_p\}$, $(\mathscr{D}_i\phi)_a\{\psi_a^i, H_p\}$ vanish weakly regardless of what values K_e^i may take. To analyze this system of equations let us denote the order of the group G by N, and its adjoint representation matrices by θ_a $[(\theta_a)^{bc} = -\mathrm{i}f_{abc}]$. Let us also define the matrix A by $A \equiv (i\theta_a\phi_a)^2$, and the vectors $\vec{X}^i$ and $\vec{Y}^i$ in the N-dimensional group space by $X_a^i \equiv K_a^i - (\mathscr{D}^i A^0)_a$, and $Y_a^i = 2(i\theta_b\phi_b)(\mathscr{D}^i\xi)_a$, for $i=1,2,3$. Then Eq. (9) can be cast into the following matrix equation:

$$i\{\vec{\psi}^i, H_p\} \approx A\vec{X}^i + Y^i \approx 0 \quad \text{for } i=1,2,3 \ . \tag{10}$$

Equation (10) does not completely determine $\vec{Y}^i$ since A is not invertible.[14] As a matter of fact the three N-dimensional matrix equations expressed by Eq. (10) can be consistent if and only if

$$\mathrm{rank}(A) = \mathrm{rank}(A, \vec{Y}^i) \quad \text{for } i=1,2,3 \ , \tag{11}$$

where $(A, \vec{Y}^i)$ is the $N\times(N+1)$ augmented matrix. If Eq. (11) is identically satisfied for each i, then the set of equations given by Eq. (9) are consistent and no new secondary constraints are generated by the requirement $\{\psi_a^i, H_p\}\approx 0$. If, on the other hand, Eq. (11) fails to be true, additional constraints are then imposed (generated) to satisfy Eq. (9).

In this case it turns out that Eq. (11) is identically satisfied. The proof of this statement, even though straightforward, is quite unrelated to the physics of the problem, and for this reason it will be given in Appendix B in order not to disrupt the flow of the formalism we are developing.

Having determined all the constraints, we have to do a careful counting now. At first sight, it appears as if we have $8N$ constraints ($4N$ of E_a^μ, N of χ_a, and $3N$ of ψ_a^i). But the constraints χ_a and ψ_a^i are not all independent from each other. Clearly $\phi_a\chi_a=0$, $\phi_a\psi_a^i=0$. It must be emphasized that these are (strong) equalities, not the weak equalities we have been using for the constraints following Dirac's notation.[3] Therefore, the question of how many of these strongly satisfied relations exist must be answered at this point. Another way of stating the problem is that we have to find the rank of the $N\times N$ matrix $\Phi\equiv\theta_a\phi_a$. Just from the fact that $\Phi_{ab}\phi_b=0$, we conclude that rank $\Phi\leq N-1$. Furthermore, since Φ is purely imaginary and antisymmetric, we conclude that rank $\Phi\leq N-2$ if N is an even integer. So far we have only made use of the completely antisymmetric nature of the structure constants f_{abc} for compact semisimple groups. To determine the rank of Φ, however, we must make more use of the group structure. It is well known that the rank of any semisimple group G is equal to the number of maximum possible mutually commuting generators, or equivalently to the dimension of its associated Cartan subalgebra. We will denote the rank of G by r. Up to a normalization constant the sth-order Casimir invariant C_s that can be formed from the vector $\vec{\phi}$ can be written as $\mathrm{Tr}(\Phi^s)$, where $2\leq s\leq N$.[15] However, not all of these invariants are independent from each other. Some of them vanish identically. The number of independent Casimir invariants for a semisimple group G was first shown by Racah[16] to be equal to the rank of the group. Racah also gave the complete listing of these invariants for all simple compact groups.[17] We can then prove that $v_a^{(s)}\equiv\mathrm{Tr}(\theta_a\Phi^s)$ is an eigenvector of the matrix Φ with zero eigenvalue: $\Phi_{ab}v_b^{(s)}=0$. To see this we write

$$\begin{aligned}\Phi_{ab}v_b^{(s)}&=-if_{abc}\phi_c\mathrm{Tr}(\theta_b\Phi^s)\\&=+\phi_c\mathrm{Tr}([\theta_a,\theta_c]\Phi^s)\\&=\mathrm{Tr}(\theta_a\Phi^{s+1})-\mathrm{Tr}(\Phi\theta_a\Phi^s)=0\;.\end{aligned}$$

Therefore, Φ has at least as many zero eigenvectors as the number of independent nonzero eigenvectors $v^{(j)}$. This is exactly the number of Casimir invariants, or the rank of the group G. We then conclude that rank $(\Phi)\leq N-r$. For semisimple groups, then, rank $(\Phi)=N-r$.[18] Therefore, we conclude that $N-r$ of the constraints χ_a and $3(N-r)$ of the constraints ψ_a^i are independent; hence, the total number of constraints is not $8N$, but $8N-4r$.

The next step in the constraint analysis is to determine the nature of these constraints. The reader can easily verify that all of the following constraints are first class[3]: E_a^0, $(\mathscr{D}_iE^i)_a-\chi_a$, $v_a^{(s)}E_a^i$, $(\mathscr{D}_i\phi)_aE_a^i$, $\xi_aE_a^i$, $(\mathscr{D}_i\phi)_aE_a^i$, and $\xi_a\psi_a^i$. Other combinations can also be written in a similar manner. It is easy to see that not all of these are independent. To begin with, the reader will recognize E_a^0 and $(\mathscr{D}_iE^i)_a-\chi_a$ as the (only) constraints of the theory with the gauge kinetic term [i.e., constraints of the Lagrangian density $\mathscr{L}'$ given by Eq. (1)]. These are the generators of the ordinary (local) gauge transformations, and it is reassuring that they are still among the list of our first-class constraints. After all, the theory given by $\mathscr{L}$ is still gauge invariant. In addition to these $2N$ constraints[19] there are $2r$ more independent combinations of the others. In conclusion this system has $2(N+r)$ first-class and $6(N-r)$ second-class constraints.

At this point one can continue with the classical analysis of the system; fix a gauge and solve the constraints.[20] Even though this may be an interesting problem to solve classically, it takes us beyond the original reason of studying such models. After all, as long as one stays within classical physics and does not worry about things like renormalization, one might as well study a more general model: a gauge theory without the gauge kinetic term, but with Higgs fields and fermions in arbitrary representations. Therefore, instead of further elaborating on the model at a classical level, we will proceed to the quantized case. To do so we must write out the generating functional for this system:

$$W\sim\int\prod_{x,i,a}dA_a^0dA_a^idE_a^0dE_a^id\phi_ad\xi_a\left[\prod_{s=1}^{6(N-r)}\delta(\zeta_s)\right]\prod_{s=1}^{2(N+r)}\delta(\mu_s)\delta(v_s)(\det M)^{1/2}\times\exp\left[i\int d^4x(\dot{A}_\mu^aE_a^\mu+\dot{\phi}_a\xi_a-\mathscr{H}_c)\right]\,,\qquad(12)$$

where $\zeta_1,\ldots,\zeta_{6(N-r)}$ are the independent second-class constraints, $\mu_1,\ldots,\mu_{2(N+r)}$ the independent first-class constraints, $v_1,\ldots,v_{2(N+r)}$ the gauge-fixing conditions, and M is the FPS determinant. Studying the general case of W is quite difficult for two reasons; the difficulty of computing the $10N-2r$ by $10N-2r$ FPS determinant[19] could be prohibitive, at least in most gauges, and the functional integrals are very difficult to evaluate unless the gauge choice simplifies them considerably. For this reason we will demonstrate the details of the calculations by performing them only for the simplest possible case, namely, the case when $G=\mathrm{SU}(2)$. This calculation was first reported in Ref. 4. Later on, we will argue in general about what the results should be for an arbitrary semisimple group.

III. QUANTIZATION

A. Gauge fixing and the FPS determinant for the case with $G=\mathrm{SU}(2)$

Using the results of the general counting of the various kinds of constraints we obtained in the previous section, we see at once that SU(2) has 20 constraints,[19] 8 of which are first class. This requires that we introduce 8 gauge-fixing constraints. The difficulty of quantizing and the particle content (such as ghost fields) of the theory is, of course, gauge dependent. To quantize the SU(2) model, we will choose the following: $A_a^0-C_a=0$, $\eta_i A_a^i=0$, and $\alpha_a A_a^i=0$, where C_a are arbitrary functionals of all fields and momenta and space-time, $\eta^\mu\equiv(0,\eta)$ is a constant spacelike vector, and α_a are arbitrary functionals of space-time and the scalar fields. C_a has no physical meaning and it disappears from all results, even though only one set of choices for C_a corresponds to classical paths in the phase space, the other choices do not. Furthermore, depending on the functional dependence of C_a on fields and momenta, the gauge-fixing constraints may or may not have vanishing Poisson brackets with each other as required by the FPS formalism. For the gauge fields, however, this condition need not be imposed at all.[21] If we just set $C_a=0$, it makes no difference in the calculations. One other point worth emphasizing is that the conditions $\eta_i A_a^i=0$ and $\alpha_a A_a^i=0$ appear to have specified six conditions for SU(2), but in fact any one of the six can be solved in terms of the other five; therefore, the true number of conditions is five, as it should be, to bring the total to eight.

Our choice of gauge appears to be somewhat arbitrary at this point, perhaps lacking physical motivation. The only argument we can offer to motivate this choice is "the end justifies the means" type of argument: At the end, when the dust settles, this gauge produces a ghost-free and explicitly renormalizable effective action. We have found other gauges where this is possible, but in general, in an arbitrary gauge, the final result will not be as transparent and as simple.

Remembering that only two of χ_a and six of ψ_a^i are independent, we choose the constraints χ_2,χ_3 and ψ_2^i, ψ_3^i to be the independent ones. (Any other choice will be just as good.) The five independent combinations of $\alpha_a A_a^i$ and $\eta_i A_a^i$ we choose are $\alpha_a A_a^i$, $\eta_i A_2^i$, and $\eta_i A_3^i$. With these choices, the FPS determinant is formally a 28×28 determinant,[19] which factors out in block-diagonal form to two pieces: a 6×6 field-independent[22] piece coming from $A_a^0-C_a$, and E_a^0, and the 22×22 piece coming from the rest of the constraints. For the purpose of functional integration, we will drop all field-independent contributions.[23] Therefore, the only piece of the determinant which contributes is the 22×22 subdeterminant.

Calculating a 22×22 functional determinant appears to be an insurmountable task at first sight; however, in this case, due to the nature of the constraints and the gauge-fixing terms, this task can be accomplished without any serious difficulty. To simplify the computations, we may choose the constant spacelike vector η_μ to have only one nonvanishing component, for example, $\eta^\mu=(0,1,0,0)$. The computation procedure is straightforward, but tedious, and it will be given in Appendix C for the interested readers. The result is

$$(\det M)^{1/2}\sim\alpha_1\phi_1{}^4(\alpha\cdot\phi)^2(\phi\cdot\phi)^3\ , \tag{13}$$

where the asymptotic sign is used rather than the equality sign to remind the readers that some field-independent pieces are dropped. We also note that $\alpha_1\neq 0$ is necessary for consistency.[24]

B. Evaluating the functional integrals for the case with $G=\mathrm{SU}(2)$

We start with the amplitude given by Eq. (12):

$$\begin{aligned}W\sim\int\prod_{x,i,a} & dA_a^0 dA_a^i dE_a^0 dE_a^i d\phi_a d\xi_a\,\delta(E_a^0)\delta(A_a^0-C_a)\delta(E_a^i)\\ & \times\delta(\alpha\cdot A^i)\delta(\psi_2^i)\delta(\psi_3^i)\delta(\chi_2)\delta(\chi_3)\delta(\eta_j A_2^j)\delta(\eta_j A_3^j)(\det M)^{1/2}\\ & \times\exp\left[i\int d^4x(\dot{A}_a^\mu E_\mu^a+\dot{\phi}_a\xi_a-\mathcal{H}_c)\right]\ ,\end{aligned}$$

where $(\det M)^{1/2}$ is given by Eq. (13) and $\eta^\mu=(0,1,0,0)$. Furthermore, let us choose $\alpha=(\beta,0,0)$ for simplicity.[25] It is then clear that the integrations over the variables E_a^0, E_a^i, A_1^1, A_1^2, A_1^3, A_2^1, and A_3^1 can be done by means of δ functions. This gives a factor of β^{-3}. In the next step, we use the constraints ψ_2^i and ψ_3^i to do the integrations over the remaining gauge fields. For that we observe

$$\prod_{i=1}^{3}\delta(\psi_2^i)\delta(\psi_3^i)=\delta(\phi_3\partial^1\phi_1-\phi_1\partial^1\phi_3)\delta(\phi_1\partial^1\phi_2-\phi_2\partial^1\phi_1)\delta^{(4)}(-Q\mathscr{A}+\mathscr{B})\ ,$$

where

$$\mathscr{A}=\begin{bmatrix}A_2^2\\ A_3^2\\ A_2^3\\ A_3^3\end{bmatrix},\quad \mathscr{B}=\begin{bmatrix}\phi_3\partial^2\phi_1-\phi_1\partial^2\phi_3\\ \phi_1\partial^2\phi_2-\phi_2\partial^2\phi_1\\ \phi_3\partial^3\phi_1-\phi_1\partial^3\phi_3\\ \phi_1\partial^3\phi_2-\phi_2\partial^3\phi_1\end{bmatrix},\quad Q=\begin{bmatrix}\bar{Q} & 0\\ 0 & \bar{Q}\end{bmatrix},\ \text{and}\ \bar{Q}=\begin{bmatrix}\phi_1{}^2+\phi_3{}^2 & -\phi_2\phi_3\\ -\phi_2\phi_3 & \phi_1{}^2+\phi_2{}^2\end{bmatrix}. \tag{14}$$

The integration over $\mathscr{A}$ gives us a factor $(\det Q)^{-1}=\phi_1{}^{-4}(\phi\cdot\phi)^{-2}$, thus, we obtain

$$W\sim\int\prod_{x,a}d\phi_a d\xi_a\delta(\chi_2)\delta(\chi_3)\delta(\phi_3\partial^1\phi_1-\phi_1\partial^1\phi_3)\delta(\phi_1\partial^1\phi_2-\phi_2\partial^1\phi_1)$$
$$\times[\beta^3\phi_1{}^6(\phi\cdot\phi)^3]\beta^{-3}\phi_1{}^{-4}(\phi\cdot\phi)^{-2}\exp\left[i\int d^\mu x\,\mathscr{L}'_{\rm eff}\right], \tag{15}$$

where

$$\mathscr{L}'_{\rm eff}=\dot{\phi}_a\xi_a-\tfrac{1}{2}\xi_a\xi_a+\tfrac{1}{2}(\partial_i\phi_c)(\partial^i\phi_c)+\tfrac{1}{2}\mathscr{B}^TQ^{-1}\mathscr{B}-V(\phi)\ . \tag{16}$$

The direct substitution of $\mathscr{B}$ from Eq. (14) gives

$$\tfrac{1}{2}\mathscr{B}^TQ^{-1}\mathscr{B}=\frac{1}{2}\left[-(\partial_i\phi)\cdot(\partial^i\phi)+\frac{1}{\phi\cdot\phi}(\phi\cdot\partial_i\phi)(\phi\cdot\partial^i\phi)\right]. \tag{17}$$

The integrations over the canonical momenta ξ_2 and ξ_3 can be done by the δ functions of the constraints χ_a:

$$\delta(\chi_2)\delta(\chi_3)d\xi_2d\xi_3=\delta(\phi_3\xi_1-\phi_1\xi_3)\delta(\phi_1\xi_2-\phi_2\xi_1)d\xi_2d\xi_3$$
$$=\phi_1{}^{-2}\delta(\xi_3-\phi_3\xi_1/\phi_1)\delta(\xi_2-\phi_2\xi_1/\phi_1)d\xi_2d\xi_3\ .$$

The integration over the momentum ξ_1 is a Gaussian integration after a shift of variables yielding another factor of $\phi_1(\phi\cdot\phi)^{-1/2}$:

$$W\sim\int\prod_{x,a}d\phi_a\delta(\phi_3\partial^1\phi_1-\phi_1\partial^1\phi_3)\delta(\phi_1\partial^1\phi_2-\phi_2\partial^1\phi_1)$$
$$\times[\beta^3\phi_1{}^6(\phi\cdot\phi)^3]\beta^{-3}\phi_1{}^{-4}(\phi\cdot\phi)^{-2}\phi_1{}^{-2}[\phi_1(\phi\cdot\phi)^{-1/2}]$$
$$\times\exp\left[i\int d^4x\left[\frac{1}{2}\frac{(\phi\cdot\dot{\phi})^2}{\phi\cdot\phi}+\frac{1}{2}\frac{(\phi\cdot\partial_i\phi)(\phi\cdot\partial^i\phi)}{\phi\cdot\phi}-V(\phi)\right]\right]. \tag{18}$$

The factors coming from the FPS determinant and various integrations just combine to give a factor of $\phi_1(\phi\cdot\phi)^{1/2}$ in Eq. (18). The integrations over ϕ_3 and ϕ_2 can be performed by observing

$$d\phi_2d\phi_3\delta(\phi_3\partial^1\phi_1-\phi_1\partial^1\phi_3)\delta(\phi_1\partial^1\phi_2-\phi_2\partial^1\phi_1)$$
$$=du\,dv\,\delta(\partial^1u)\delta(\partial^1v)\ ,$$

where $u\equiv\ln(\phi_2/\phi_1)$ and $v\equiv\ln(\phi_3/\phi_1)$. Dropping a field-independent factor of $[\det\partial^1\delta(\vec{x}-\vec{x}\,')]^{-2}$, we obtain the final form of the generating functional

$$W\sim\int\prod_x d\phi\exp\left\{i\int d^4x[\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-V(\phi)]\right\}, \tag{19}$$

where the real scalar variable ϕ is related to ϕ_1 by $\phi=\sqrt{3}\phi_1$.

We note that ϕ is a color singlet and all the references to the non-Abelian degrees of freedom have disappeared from the problem.

C. Discussion and the comparison with the classical result

When there are constraints in a theory, the number of physical variables is equal to the number of apparent variables minus the number of constraints, at least in a classical theory. The quantum theories are more difficult to deal with when we count the number of degrees of freedom due to various ghost particles one obtains. One seeks for ghost-free gauges to do this. Our result for the SU(2) example expressed by Eq. (19) is explicitly ghost free and the final number of degrees of freedom is simply 1. We shall first demonstrate that a similar result holds classically as well. One can directly solve the equations of motion classically, in the gauge we used above, and obtain the desired result. However, for classical calculations, the gauge $\alpha_a=\phi_a$ is more convenient[9] than $\alpha=(\beta,0,0)$. In this case, the final answer is expressed not in terms of $\phi\equiv\sqrt{3}\phi_1$, but in terms of $\phi\equiv(\phi\cdot\phi)^{1/2}$. First, we write down the classical equations of motion:

$$\mathscr{D}_\mu(\mathscr{D}^\mu\phi))\alpha+\frac{\partial V}{\partial\phi_a}=0\ , \tag{20}$$

$$f_{abc}\phi_b(\mathscr{D}_\mu\phi)_c=0\ . \tag{21}$$

Equation (21) is singular and one cannot solve A_μ^a from it until one fixes a gauge (at least partially). In the partially fixed gauge $\phi_a A_a^\mu=0$, the solution of Eq. (21) is

$$A_\mu^a=f_{abc}\phi_b\partial^\mu\phi_c/(\phi\cdot\phi)\ . \tag{22}$$

Substitution of Eq. (22) into Eq. (20) yields

$$\partial^2\phi+\frac{\partial V}{\partial\phi}=0,\quad \text{where } \phi\equiv(\phi\cdot\phi)^{1/2}\ ,$$

which is the classical equation of motion for a single real scalar field.

This result shows us that the classical and quantized versions of the theory are equally simple. What we consider most remarkable, however, is not this similarity, but the fact that the quantized version is a well-defined renormalizable theory. In general, it is no surprise at all that the classical and quantized field theories have the same number of physical degrees of freedom, but the final effective Lagrangian (and the one we started with as well) is a renormalizable one. One must compare this result to the result one would obtain from a model containing arbitrary representation of scalars and fermions. In these models, one will have a reduction of the number of the apparent fields and momenta, but the quantum effective Lagrangian left at the end in the path integral will not be a renormalizable one.

D. A brief discussion of quantization with an arbitrary semisimple group

We will not work out the general case in the detailed fashion we worked out the SU(2) example. The reason for this is that we do not know of any method which enables us to compute the FPS determinant for an arbitrary semisimple group. For each individual example (no matter how large the group is) the procedure is straightforward; unfortunately, we have been unable to devise a scheme in which we obtain the value of the FPS determinant without actually specifying what the group G is. We could not even derive a general expression for SU(N) for arbitrary N. For this reason, instead of producing a few more examples [such as SU(3), SU(4), SO(7),...], which teach us nothing new and interesting, we will give a general argument about what the result will be.[26]

The first part of this argument is no more sophisticated than counting the degrees of freedom for the general case: One starts with $10N$ degrees of freedom, $5N$ fields, and $5N$ canonical momentum densities. The total number of constraints is $8N-4r$ as was explained in Sec. II. Of these constraints, $2(N+r)$ are first class, and we must introduce $2(N+r)$ gauge-fixing terms[20] for these as required by the FPS quantization ansatz.[12] The total number of δ-function—type constraints is then $10N-2r$, therefore, one expects only $2r$ degrees of freedom to be left at the end. If we choose our gauge-fixing terms judiciously, it is possible to arrange that the final $2r$ variables will be divided into two groups: r of them being functions of the scalar fields and the remaining r being the corresponding canonical momenta with no ghost fields remaining. For the SU(2) example we presented, $r=1$, and at the end the result was brought to a form expressed in terms of ϕ_1 and ξ_1 in the doubly axial gauge. For SU(2), the ξ_1 integration was also performed (because it was a Gaussian integration) to reduce the number of variables to only one.

In general, however, for groups other than SU(2), the elimination of the final r canonical momenta by integrating over them is not guaranteed until we can demonstrate that these integrations can indeed be done in terms of suitably chosen variables. To illustrate this point more clearly, let us first note that the only[27] dependence on the canonical momenta ξ_a is in the constraints χ_a and in the exponential in the term

$$\int d^4x\ \tfrac{1}{2}\xi_a\xi_a\ .$$

Remembering that only $N-r$ of the constraints χ_a are independent, we have then a way of solving

$N-r$ of the canonical momenta in terms of the first r of them and the fields ϕ. First let us remember that

$$\Phi_{ab}v_b^{(s)}=0 \quad \text{for } s=2,3,\ldots,\infty \ .$$

Let us denote the orders s which correspond to one of the orders of the characteristic invariants of the group by s_i. For example, for the exceptional group F_4, there are four Casimir invariants and $s_1=2$, $s_2=6$, $s_3=8$, $s_4=12$, respectively.[17] Now also note that the constraint

$$\chi_a=f_{abc}\phi_b\xi_c=0$$

merely states that ξ_c is a zero eigenvector of the matrix Φ. But the rank of Φ was determined to be $N-r$ for semisimple groups with independent zero eigenvectors $v^{(s_i)}$, $i=1,\ldots,r$. Therefore, the constraint expresses ξ_c in terms of $v^{(s_i)}$:

$$\xi_a=\sum_{i=1}^{r} C_i v_a^{(s_i)} \ , \tag{23}$$

where the coefficients C_i are arbitrary. We may choose to integrate over C_i if we wish, but it is more convenient [as we did when we quantized the SU(2) case] to choose the first r canonical momenta as our independent variables and integrate over them. In that case, the first r equations in (23) must be identities

$$\xi_j=\sum_{i=1}^{r} C_i v_j^{(s_i)}, \quad j=1,\ldots,r \ . \tag{24}$$

Differentiating (24) with respect to ξ_k shows that C_i in fact are linear with ξ_k for $k=1,\ldots,r$:

$$C_i(\xi_1,\ldots,\xi_r;\phi_1,\ldots,\phi_N)=\sum_{k=1}^{r} e_{ik}(\phi_1,\ldots,\phi_N)\xi_k \ , \tag{25}$$

where $e(\phi)=(v(\phi))^{-1}$, and the $r\times r$ matrix v has elements $v_j^{(s_i)}$. We then have our integrations as follows:

$$d\xi_1\cdots d\xi_N \prod_{a=r+1}^{N}\delta(\chi_a)=(d\xi_1\cdots d\xi_r)[\det(\rho)]^{-1}\prod_{a=r+1}^{N} d\xi_a\delta\left[\xi_a-\sum_{i=1}^{r} C_i v^{(s_i)}\right] \ , \tag{26}$$

where the C_i are determined in terms of $\xi_1,\ldots,\xi_r$ and $\phi_1,\ldots,\phi_N$ by Eq. (25), and $\det(\rho)$ is the Jacobian for the $(N-r)\times(N-r)$ lower block of the matrix $||\partial\chi_a/\partial\xi_b||$. Equation (26) first enables us to integrate over the momenta $\xi_{r+1},\ldots,\xi_N$ and express the result over $\phi_1,\ldots,\phi_N$ and $\xi_1,\ldots,\xi_r$. But it further assures that the remaining integrations over $\xi_1,\ldots,\xi_r$ can also be performed. This is so because C_i are linear in ξ_k as expressed by Eq. (25), and the term $\frac{1}{2}\xi_a\xi_a$ in the exponential becomes a Gaussian term in $\xi_1,\ldots,\xi_r$ with a ϕ-dependent coefficient. Of course, the Gaussian integrations can be carried out right away, eliminating the canonical momenta altogether. This completes the proof that the final result will be expressed in only r variables, not $2r$. Which r variables one ends up with is, of course, gauge dependent. This, then, establishes a standard way of doing the integrations, once the FPS determinant is known for the problem.

IV. CONCLUSIONS

We would like to divide the discussion and our conclusions into two parts. The first part directly follows from Lagrangian $\mathscr{L}$ given by Eq. (2). Regardless of any phenomenological implications, the existence of a renormalizable gauge theory without the gauge kinetic term is itself very remarkable and at least of academic interest. The second part of our discussion will concern the relation between $\mathscr{L}$ and $\mathscr{L}'$ [given by Eq. (1)] and what exactly we mean by infinite coupling (base or renormalized?), and what happens if the symmetry is spontaneously broken.

We have seen in Sec. II that Lagrangian $\mathscr{L}$ possesses a new translationlike local symmetry which is lost if the kinetic term is added. It was this fact which assured us that the theory is renormalizable. In general, we have shown that the quantized theory will be reduced to a theory with r variables (fields) only, where r is the rank of the gauge group. This result is verified in detail for SU(2), where $r=1$ and the manipulation of the functional integrals is quite simple. Furthermore, we observed that the classical theory [at least for SU(2)] is at least as simple as the quantum theory. For other groups, the classical result is likely to be simpler than the quantum-mechanical result.[28]

The relevance of these results to an ordinary gauge theory (one with the gauge kinetic term), however, is not very well understood. We can only offer some qualitative arguments in this case. First of all, when we established the connection between the Lagrangians $\mathscr{L}$ and $\mathscr{L}'$ by taking the infinite-gauge-coupling limit, we were deliberately vague about which coupling constant we were talking about: bare coupling or renormalized running coupling? If the theory does not suffer from infrared problems (i.e., if the running coupling constant does not depend on an artificial infrared cutoff), it turns out that it does not matter which coupling constant we are talking about. The renormalized and bare couplings are related by $g_0{}^2 = Z_3{}^{-1} g^2$, and letting the bare coupling g_0 become infinite forces g to become infinite too. What the physical relevance of the truly infinite coupling constant is, we do not know. There are examples, however, in asymptotically free[29] unbroken gauge theories where one expects confinement.[30] Consider the "hadron" made out of two scalars.[31] Beyond the confinement length scale $\Lambda_c{}^{-1}$, the gauge fields diminish very rapidly and the physics at length scales much larger than $\Lambda_c{}^{-1}$ can be described without the confining gauge fields. This example is very suggestive to us, then: In our study of the quantized theories, we found that the dropping of the kinetic gauge term always produced a new effective theory with only local color-singlet particle content. We certainly do not want to push this as a supporting argument for confinement and for the existence of only the color singlets. We merely consider it as an amusing example.

The study of these kinds of gauge theories may have classical relevance too. (In this case, we do not have to restrict ourselves to the theory with a single multiplet of adjoint scalars.) The classical solutions of this problem may not be as simple in the presence of monopoles or instantons, and these cases may be interesting to study.

Whether or not any interesting results relevant to the physical world we live in comes out of the study of the gauge theories without the gauge kinetic term, we find it an attractive and simple limit to study in our quest for understanding the structure of gauge field theories.

ACKNOWLEDGMENTS

I would like to thank Robert Ore and Chen-Ning Yang for taking a special interest in this work and for many helpful discussions. I would also like to thank Alfred S. Goldhaber, Jack Smith, and William Weisberger for useful discussions. This work was supported in part by NSF Grant No. PHY81-09110.

APPENDIX A

In this appendix we will give the derivations of Eqs. (8) and (9). We start with the consistency condition for the constraints χ_a:

$$\begin{aligned}\{\chi_a, H_p\} &\approx f_{abc}\phi_b[f_{dce}\xi_d A_e^0 + (\mathscr{D}_i\phi)_d(-\overleftarrow{\partial}^i\delta^{cd} + f_{dce}A_e^i) - \partial V/\partial\phi_c]\\ &= -f_{abc}\phi_b[(\mathscr{D}^i(\mathscr{D}_i\phi))_c + \partial V/\partial\phi_c] + \phi_b\xi_e A_d^0(f_{abc}f_{ecd} + f_{eac}f_{bcd})\\ &= -f_{acd}A_d^0\chi_c - \partial^i\psi_i^a + f_{abc}\phi_b[(\overleftarrow{\partial}^i\delta^{cd} - f_{cde}A_e^i)(\mathscr{D}_i\phi)_d - \partial V/\partial\phi_c]\\ &= -f_{abc}\chi_b A_c^0 - (\mathscr{D}_i\psi^i)_a - f_{abc}\phi_b\partial V/\partial\phi_c\ .\end{aligned}$$

This proves Eq. (8), provided that we can show that $f_{abc}\phi_b\partial V/\partial\phi_c$ vanishes. This, on the other hand, follows from the fact that $v_a^{(s)}\Phi_{ab}=0$, where $v_a^{(s)}$ and Φ_{ab} were defined in Sec. II.

Next we compute the consistency conditions for ψ_a^i:

$$\begin{aligned}\{\psi_a^i, H_p\} &\approx f_{abc}[(\xi_b - f_{bed}A_d^0\phi_e)(\mathscr{D}^i\phi)_c + f_{cde}\phi_b\phi_d K_e^i + \phi_b(\mathscr{D}^i)^{cd}(\xi_d - f_{deg}A_g^0\phi_e)]\\ &= f_{abc}\{-f_{bde}\phi_e[A_d^0(\mathscr{D}^i\phi)_c + \xi_c A_d^i] - \phi_c\partial^i\xi_b + f_{cde}\phi_b\phi_d K_e^i + \phi_b(\mathscr{D}^i)^{cd}(\xi_d - f_{deg}A_g^0\phi_e)\} - \partial^i\chi_a\\ &= -\partial^i\chi_a + f_{abc}\{-A_c^0\psi_b^i - A_c^i\chi_b - f_{bed}\phi_e[A_d^0(\mathscr{D}^i\phi)_e + \xi_e A_d^i]\\ &\qquad - \phi_c\partial^i\xi_b + f_{cde}\phi_b\phi_d K_e^i + \phi_b(\mathscr{D}^i)^{cd}(\xi_d - f_{deg}A_g^0\phi_e)\}\\ &= -f_{abc}\phi_c[2(\mathscr{D}^i\xi)_b + f_{bed}A_d^0(\mathscr{D}^i\phi)_e - f_{deg}(\mathscr{D}^i)_{bd}A_g^0\phi_e - f_{bde}\phi_d K_e^i]\\ &= -f_{abc}\phi_c[2(\mathscr{D}^i\xi)_b + (f_{deh}f_{bdg} + f_{bdh}f_{edg})\phi_e A_h^i A_g^0 - f_{bed}\phi_e\partial^i A_d^0 + f_{bde}\phi_d K_e^i]\\ &= -f_{abc}\phi_c\{2(\mathscr{D}^i\xi)_b - f_{bed}\phi_e[(\mathscr{D}^i A^0)_d - K_d^i]\}\ ,\end{aligned}$$

which reduces to Eq. (9) after renaming the dummy indices.

APPENDIX B

In this appendix we will prove that Eq. (11) is identically satisfied without any additional constraints. To start with let us call $m=\text{rank}(i\Phi)$. (We know that $m=N-r$ for semisimple groups but this fact is not important here.) We then have

$$P(\lambda)\equiv\det(i\Phi-\lambda I)=\lambda^{N-m}Q(\lambda)\ ,$$

where $Q(0)\neq 0$, and $Q(\lambda)$ is a polynomial of degree m:

$$\begin{aligned}R(\lambda)&\equiv\det(A-\lambda I)=\det[(i\Phi)^2-\lambda I)\\&=\det(i\Phi-\sqrt{\lambda}I)\det(i\Phi+\sqrt{\lambda}I)\\&=P(\sqrt{\lambda})P(-\sqrt{\lambda})\\&=(-\lambda)^{N-m}Q(\sqrt{\lambda})Q(-\sqrt{\lambda})\ .\end{aligned}\tag{B1}$$

On the other hand, since Φ is antisymmetric $Q(\lambda)$ contains only the even powers of λ; in fact $Q(\lambda)$ is a polynomial in λ^2. Using $Q(0)\neq 0$, we conclude that $\text{rank}(A)=m$.

Next we observe that

$$\text{rank}(A,\vec{Y}^i)\geq\text{rank}(A)=m\ .\tag{B2}$$

On the other hand, the augmented matrix $(A,\vec{Y}^i)$ can be written as a product:

$$(A,\vec{Y}^i)=i\Phi(i\Phi,2(\vec{\mathscr{D}}^i\xi))\ .$$

This allows us to conclude

$$\begin{aligned}\text{rank}(A,\vec{Y}^i)&\leq\min[\text{rank}(i\Phi),\text{rank}(i\Phi,2(\vec{\mathscr{D}}^i\xi))]\\&=m\ .\end{aligned}\tag{B3}$$

Equations (B2) and (B3) together prove Eq. (11).

APPENDIX C

In this appendix we will present a derivation for Eq. (13) for the FPS determinant.

This determinant is in the block form

$$M=\begin{bmatrix}A & -C^T\\ C & 0\end{bmatrix},\tag{C1}$$

where A is an 18×18 matrix and C is a 4×18 matrix. The order of various constraints used in writing M as above is as follows: E^1_1,E^1_2,E^1_3, $E^2_1,E^2_2,E^2_3,E^3_1,E^3_2,E^3_3$; $\alpha_a A^1_a$, ψ^1_2,ψ^1_3, $\alpha_a A^2_a,\psi^2_2$, $\psi^2_3,\alpha_a A^3_a,\psi^3_2,\psi^3_3$; χ_2,χ_3; $\eta_i A^i_2$; $\eta_i A^i_3$. Of course, this order does not effect the value of the determinant, but in this order things are easy to compute because

$$A=\begin{bmatrix} & -\mathscr{R}^T\\ \mathscr{R} & \end{bmatrix},\quad \mathscr{R}=\begin{bmatrix}R & & \\ & R & \\ & & R\end{bmatrix},$$

$$C=\begin{bmatrix}0 & Y\\ Z & W\end{bmatrix},$$

where $Z=(Z^1,Z^2,Z^3)$, $W=(W^1,W^2,W^3)$, and $Y=(Y^1,Y^2,Y^3)$, where R is 3×3, and Z^i, W^i, and Y^i are all 2×3 for $i=1,2,3$.

The matrix R can be computed explicitly through the relevant Poisson brackets:

$$R=\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \phi_1\phi_2 & -\phi_1{}^2-\phi_3{}^2 & \phi_2\phi_3\\ \phi_1\phi_3 & \phi_2\phi_3 & -\phi_1{}^2-\phi_2{}^2\end{bmatrix}\delta(\vec{x}-\vec{x}')\ ,$$

where $\det R=\phi_1(\alpha\cdot\phi)(\phi\cdot\phi)$, and R^{-1} exists, provided that $\alpha\cdot\phi$ is not chosen to be zero. (Since α are arbitrary and completely at our disposal to choose, we shall assume $\alpha\cdot\phi\neq 0$.) Since R^{-1} exists, A^{-1} exists and we can write from (C1)

$$\det M=(\det A)(\det CA^{-1}C^T)\ .\tag{C2}$$

The matrix $CA^{-1}C^T$ is easily computed to be

$$CA^{-1}C^T=\begin{bmatrix}0 & -Y(\mathscr{R}^{-1})^T Z^T\\ Z\mathscr{R}^{-1}Y^T & -W(\mathscr{R}^{-1})^T Z^T+Z\mathscr{R}^{-1}W^T\end{bmatrix}.$$

We see immediately that the matrix W is irrelevant for the determinant we calculate:

$$\det(CA^{-1}C^T)=\det\left[\sum_{i=1}^{3}Z^iR^{-1}(Y^i)^T\right]^2 .$$

Also, explicit calculation shows that

$$Z^i=\begin{bmatrix}0 & \eta^i & 0\\ 0 & 0 & \eta^i\end{bmatrix},$$

thus

$$\det(CA^{-1}C^T)=(\det E)^2\det\left[\sum_{i=1}^{3}\eta^i(T^i)^T\right]^2,\tag{C3}$$

where

$$\det E=(\det R)^{-2}\alpha_1\phi_1(\phi\cdot\phi)(\alpha\cdot\phi)\tag{C4}$$

and

$$\det(\eta_i(T^i)^T)=\int d\Omega^* d\Omega\, d\omega^* d\omega \exp\left[i\int d^4x\,\mathscr{L}_g\right],$$

where Ω^*, Ω, ω^*, and ω are anticommuting (Grassmann) variables and the ghost Lagrangian $\mathscr{L}_g$ is computed from the Poisson brackets $\{\chi_a,\psi^i_b\}$ directly for $a,b=2,3$. In this computation the condition $\eta_i A^i_a=0$ can be used to simplify the answer and put $\mathscr{L}_g$ in a form independent from the gauge fields:

$$\mathscr{L}_g = -i\Omega^*\phi_2\eta_i\partial^i(\phi_3\omega) - i\omega^*\phi_3\eta_i\partial^i(\phi_2\Omega) + i\Omega^*\phi_1\eta_i\partial^i(\phi_1\Omega) + i\omega^*\phi_1\eta_i\partial^i(\phi_1\Omega) + i\Omega^*\phi_3\eta_i\partial^i(\phi_3\Omega) + i\omega^*\phi_2\eta_i\partial^i(\phi_2\Omega) .$$

Using the choice $\eta^\mu = (0,1,0,0)$ made in Sec. III and the constraints ψ_2^i and ψ_3^i we can further simplify $\det(\eta_i(T^i)^T)$ by making a change of variables

$$\Omega' = \partial^1(\phi_1\Omega), \quad \Omega'^* = \Omega^*\phi_1 ,$$
$$\omega' = \partial^1(\phi_1\omega), \quad \omega'^* = \omega^*\phi_1$$

to obtain

$$\det(\eta_i(T^i)^T) \sim \int d\Omega'^* d\Omega' d\omega'^* d\omega' \exp\left[i \int d^4x \, \mathscr{L}'_g \right] ,$$

after dropping the field-independent piece $\{\det[\partial^1\delta(\vec{x}-\vec{x}\,')]\}^2$, where

$$\mathscr{L}'_g = i\phi_1^{-2}(\Omega'^*, \omega'^*) \begin{bmatrix} \phi_1^2+\phi_3^2 & -\phi_2\phi_3 \\ -\phi_2\phi_3 & \phi_1^2+\phi_2^2 \end{bmatrix} \begin{bmatrix} \Omega' \\ \omega' \end{bmatrix} .$$

This allows us to compute the determinant in a nice compact form:

$$\det(\eta_i(T^i)^T) \sim \phi_1^2(\phi\cdot\phi) . \tag{C5}$$

Combining Eqs. (C2), (C3), (C4), and (C5) we then obtain the equation for the FPS determinant:

$$\det M \sim \alpha_1^2\phi_1^8(\alpha\cdot\phi)^4(\phi\cdot\phi)^6 . \tag{C6}$$

Taking the square root gives us Eq. (13) in Sec. III.

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[1]In this paper the greek letters $\mu,\nu,\rho,\ldots$ label the four Lorentz indices, whereas the latin letters $i,j,k,\ldots$ refer to the three spatial indices. The letters $a,b,c,\ldots$ label the generators of the group G. All repeated indices imply implicit summation over the appropriate range unless otherwise stated.

[2]G. 't Hooft, Nucl. Phys. B35, 167 (1971); B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); 5, 3137 (1972).

[3]For the notation and the terminology relating to the constrained dynamical systems see P. A. M. Dirac, Can. J. Math. 2, 129 (1950); J. Anderson and P. Bergmann, Phys. Rev. 83, 1081 (1951); P. Bergmann, Rev. Mod. Phys. 33, 510 (1961); P. A. M. Dirac, *Lectures on Quantum Mechanics,* Yeshiva University—Belfer Graduate School of Science, New York (Academic, New York, 1964).

[4]Sinan Kaptanoglu, Phys. Rev. D 26, 3754 (1982).

[5]Since we assumed that G is semisimple and compact, we can choose the structure constants f_{abc} to be completely antisymmetric in all three indices. This fact is extremely important to the validity of the invariance expressed by Eq. (3).

[6]Clearly every continuous local dynamical symmetry is generated by a corresponding first-class constraint, but the converse is not as firmly established. In the last of the articles given in Ref. 3, Dirac conjectured that the converse of this statement is also true. However, in the last couple of years, this point has been debated in the literature. Whether or not Dirac's conjecture is true has no relevance to the content and the conclusions of this article. For details the reader is referred to R. Cawley, Phys. Rev. Lett. 42, 413 (1979); A. Frankel, Phys. Rev. D 21, 2986 (1980); R. Cawley, *ibid.* 21, 2988 (1981); R. G. Di Stefano, this issue, Phys. Rev. D 27, 1752 (1983).

[7]R. Arnowitt and S. Fickler, Phys. Rev. 127, 1821 (1962); R. N. Mohapatra, Phys. Rev. D 4, 2215 (1971).

[8]The reader will notice that we are very casual about the upper and lower group indices. Since the gauge group G is semisimple and compact, raising and lowering group indices is completely academic and there is no need to keep track of it properly.

[9]I thank Rob Ore for pointing this out.

[10]This statement is not exactly correct. One can always add singlet fermions and/or singlet scalars to the model with one multiplet of adjoint scalars. This, however, is not a new interesting model since the singlet fermions and scalars have no gauge interactions. We caution the reader, however, that the physics of such a model, however uninteresting it may seem at this moment, is not in general the same as the physics without them. The singlets do interact: singlet fermions and singlet scalars have Yukawa interactions and the singlet scalars and the adjoint scalars interact quartically.

[11]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967); Kiev Report No. ITP-67-36, 1967 (unpublished); L. Faddeev, Teor. Mat. Fiz. 1, 3 (1969) [Theor. Math. Phys. 1, 1 (1970)]; V. N. Popov, CERN Report No. Th. 2424, 1977 (unpublished); P. Senjanovic, Ann. Phys. (N.Y.) 100, 227 (1976).

[12]For the definitions and the formalism of the functional formulation of the quantized field theories, we refer the reader to the review article by E. Abers and E. Lee, which also contains an extensive list of original references: E Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[13]The secondary constraints $\chi_a \approx 0$ and $\psi_a^i \approx 0$ correspond

to the classical Euler-Lagrange equations $f_{abc}\phi_b(\mathscr{D}_\mu\phi)_c=0$.

[14]Clearly $A_{ab}\ \phi_b=0$, therefore $\det A=0$.

[15]This inequality can be stated more strongly for semisimple groups by $2\le s\le s_{\max}\le d\equiv$ dimension of the smallest (nontrivial) irreducible representation. (For simple Lie groups $d<N$, except for E_8, where $d=N$.)

[16]G. Racah, Lincei Rend, Sci. Fis. Mat. Nat. 8, 108 (1950).

[17]For SU(n), $n\ge 2$, the invariants are $C_2,C_3,\ldots,C_n$. For SO($2n+1$) and Sp($2n$), $n\ge 1$, the invariants are $C_2,C_4,\ldots,C_{2n}$. For SO($2n$) $n\ge 3$, the invariants are $C_2,C_4,\ldots,C_{2n-2}$, and C_n. The invariants for the exceptional groups are as follows: C_2 and C_6 for G_2; C_2, C_6, C_8, and C_{12} for F_4; C_2, C_5, C_6, C_8, C_9, and C_{12}, for E_6; C_2, C_6, C_8, C_{10}, C_{12}, C_{14}, and C_{18} for E_7; C_2, C_8, C_{12}, C_{14}, C_{18}, C_{20}, C_{24}, and C_{30} for E_8.

[18]In this paper we do not consider nonsemisimple groups, for which the inequality $\text{rank}(\Phi)\le N-r$ has to be studied individually.

[19]When we talk about the numbers of various kinds of constraints we only refer to the spatial and group indices. For fields theories, of course the number is uncountably infinite due to the space-time structure. The same thing, of course, is true for the matrices of the Poisson brackets. In general they are bilocal matrices of infinite size.

[20]The name gauge fixing is loosely used here. It refers not only to the ordinary gauge fixing but also to the fixing of the extra invariance expressed by Eq. (3). A quite uncommon but a more accurate name might be "invariance fixing."

[21]Sinan Kaptanoglu, Phys. Lett. 98B, 77 (1981).

[22]The field independence of the 6×6 block is assured if C_a do not depend on A_a^0. Even if this is not the case, the result does not change. See Ref. 21.

[23]This then is a zero-temperature formalism. At finite temperature the field-independent contributions cannot be dropped arbitrarily.

[24]Requiring $\det M\neq 0$ automatically implies $\alpha_1\neq 0$. The reader may wonder why the first group index is singled out in this fashion. This is due to the choice we made for our independent constraints and the gauge-fixing terms. For example, we chose χ_2,χ_3 as our independent constraints. We could have chosen χ_1 and χ_2, or any two linear combinations of χ_a. For the choice χ_2, χ_3, ψ_2^i, ψ_3^i, $\eta_i A_2^i$, and $\eta_i A_3^i$, the requirement that the gauge be fixed completely dictates that $\alpha_1\neq 0$. If we chose instead $\chi_1,\chi_2,\psi_1^i,\psi_2^i,\eta_i A_1^i,\eta_i A_2^i$, we would have to demand $\alpha_3\neq 0$.

[25]Since $\alpha_a(x,\phi(x))$ are arbitrary subject to the requirements that $\alpha_1\neq 0$ and $\alpha\cdot\phi\neq 0$, we can make this choice.

[26]Even though the quantized case requires the computation of the FPS determinant, the classical case does not. One may attempt to solve the classical equations of motion directly. The treatment of the general classical problem is still under investigation (S. Kaptanoglu and Robert Ore, in preparation).

[27]We have implicitly assumed that the gauge-fixing conditions are chosen to be independent from ξ_a. This also assures that the FPS determinant is independent from ξ_a. These kinds of gauge-fixing terms are therefore very convenient. Choosing gauge-fixing terms which depend on ξ_a is in principle allowed, but it introduces unnecessary complications.

[28]For the groups SU(2), SU(3), SU(4), and SO(5) the classical result is the same as the quantized result, because these groups do not have Casimir invariants of order larger than 4. For other groups, however, the classical and quantum-mechanical results need not be the same: The classical potential will contain all the Casimir invariants, whereas the quantized potential will be restricted to those terms of order less than or equal to 4. This fact will affect the final form of the effective Lagrangian and its residual global (since all the gauge fields are gone) invariances.

[29]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, *ibid.* 30, 1346 (1973); D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973).

[30]All gauge theories based on a semisimple group with one multiplet of scalars in the adjoint representation and no fermions are asymptotically free. Without any spontaneous symmetry breaking (which is possible for a range of parameters in the scalar potential) these theories are then expected to be confining.

[31]Unfortunately there are no fermions (quarks) in Lagrangian $\mathscr{L}'$.