Time-dependent solutions of Yang-Mills equations related to the Wu-Yang magnetic monopole

H. Arodź*
The Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark

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A class of time-dependent, spherically symmetric solutions of Yang-Mills equations is found. The solutions describe a transformation of the initial color-electric and magnetic fields into the Wu-Yang magnetic monopole.

I. INTRODUCTION

Because of the nonlinearity of the Yang-Mills equations it is very difficult to investigate the fundamental problem of time evolution of classical non-Abelian gauge fields. None of the systematic methods developed in electrodynamics, such as Fourier decomposition or the Green's function method, can be applied. The only way to investigate the time development of Yang-Mills fields seems to be a rather difficult guessing of examples of solutions.

The known time-dependent solutions of Yang-Mills equations in Minkowski space-time have the form of non-Abelian plane waves.[1–5] The purpose of this paper is to present other types of time-dependent solutions. They are obtained for the time-dependent version of the equation considered by Wu and Yang.[6] The solutions describe a complicated process during which an initial configuration of color-electric and magnetic fields turns into the Wu-Yang magnetic monopole when time goes to infinity.

First, we describe singular solutions of this type in Sec. II. Next, in Sec. III we find a more regular solution. Unfortunately, the Wu-Yang ansatz which we use in this paper is too poor to allow for nontrivial solutions regular at the origin $r=0$. Therefore, even the more regular solutions are still singular at $r=0$, similarly as the Wu-Yang magnetic monopole is. Our solutions are regular for all other values of r. In other words, we have regular solutions in the space obtained from R^3 by excluding the origin. Nevertheless, we think that our solutions are interesting because they allow us to watch nontrivial time evolution of non-Abelian gauge fields.

The solution described in Sec. III is rather complicated. We give a precise proof of its existence, its asymptotic forms in the most interesting regions of values of an independent variable, and the result of a numerical estimate. This is enough in order to state that the solution exists and in order to determine the solution by numerical methods with any required accuracy.

The process described by the regular solution does not seem to allow for a description in terms of signals propagating with definite universal velocity. For example, there are local extrema of density of energy which propagate with arbitrarily small velocity. On the other hand, the singular solution is characterized by the velocity of light.

We also observe a peculiar universality of the Wu-Yang magnetic monopole. Namely, this single configuration is the limiting configuration when $t\to+\infty$ for the entire one-parameter family of gauge fields specified at the initial time t_0.

II. THE SINGULAR TRAIN-WAVE SOLUTIONS

In the following we consider sourceless Yang-Mills equations [the gauge group is SU(2)] in the region $r>0$. Thus,

$$\partial_\mu \hat{F}^{\mu\nu}+i[\hat{A}_\mu,\hat{F}^{\mu\nu}]=0\ , \tag{1}$$

where

$$\hat{F}_{\mu\nu}=\partial_\mu\hat{A}_\nu-\partial_\nu\hat{A}_\mu+i[\hat{A}_\mu,\hat{A}_\nu] \tag{2}$$

and $\hat{A}_\mu=\frac{1}{2}\sigma^a A^a_\mu$. We assume the following time-dependent, spherically symmetric ansatz:

$$A^a_0=0,\ \ A^a_i(\vec{x},t)=-\epsilon_{aij}\frac{x^j}{r^2}[1+H(r,t)]\ , \tag{3}$$

where $r=|\vec{x}|$. It gives the field strengths

$$F^a_{0i}=-\epsilon_{aij}\frac{x^j}{r^2}\dot{H}\ , \tag{4}$$

$$F^a_{ik}=-\frac{(1+H)^2}{r^2}\frac{x^a x^j}{r^2}\epsilon_{ikj}-2\frac{1+H}{r^2}\epsilon_{aki}-\left[\frac{1+H}{r^2}\right]'\frac{x^i x^j\epsilon_{akj}-x^k x^j\epsilon_{aij}}{r}\ , \tag{5}$$

where a dot (prime) denotes derivative with respect to time (radius r). The ansatz (3) reduces the Yang-Mills equation (1) to the single equation

$$\ddot{H}-H''-\frac{1}{r^2}(H-H^3)=0 . \quad (6)$$

For time-independent H this equation reduces to the equation considered by Wu and Yang.[6] This static equation possesses three obvious solutions, $H=0,\pm1$. $H=-1$ gives zero gauge potentials A_i^a and $H=+1$ gives pure gauge A_i^a, i.e., $F_{0i}^a=F_{jk}^a=0$. $H=0$ gives the Wu-Yang magnetic monopole

$$F_{0i}^a=0 \ , \ \ F_{ik}^a=\frac{x^a x^j}{r^4}\epsilon_{ikj} \ . \quad (7)$$

Equation (6) possesses the following time-dependent solution. We observe that the term $H-H^3$ in (6) vanishes when $H(r,t)$ is piecewise constant with values $0,\pm1$. Then, the rest of (6), i.e.,

$$\ddot{H}-H''=0 \quad (8)$$

governs the propagation of discontinuities of H. From (8) it follows that they move with the velocity of light. For example, let us consider for $t>0$

$$H(r,t)=\sum_{i=1}^{N}\lambda_i\theta(t-r+a_i)\theta(r-t-b_i) \ , \quad (9)$$

where $\theta(x)=0$ for $x<0$ and $\theta(x)=1$ for $x>0$, $\lambda_i=\pm1$, and the constants b_i,a_i are such that

$$a_N>b_N>a_{N-1}>b_{N-1}> \cdots a_1>b_1>0 \ .$$

The differentiations with respect to t and r will produce derivatives of δ functions which cancel themselves in (6) in a rigorous mathematical way. Thus, (9) is the correct solution of Eq. (6).

The solution (9) is analogous to the train-wave solutions known in nonlinear optics, see, e.g., Ref. 7. The nonlinearity of Eq. (6) manifests itself in the restriction of values of H to $0,\pm1$. The train (9) moves toward spatial infinity with the velocity of light. It is also easy to produce trains with some or all wagons moving in the opposite direction. Such solutions have to be completed with some rules stating what happens when two wagons collide or when a wagon reaches $r=0$—we will not discuss this point here.

The field strengths (4) and (5) contain δ functions for H given by (9) and therefore we must be careful with saying that the potentials (3) with H given by (9) solve Yang-Mills equations, because there is no satisfactory definition of the product of singular functions. Let us propose a solution to this difficulty for the case of potentials (3) when H has steplike discontinuities. The usual way to solve such a problem is to introduce a regularization of singularities. In our case a much less drastic, almost unnoticeable prescription is sufficient. The method is that first one should calculate all contractions of color and space indices. Here we exploit the fact that in (3) these indices are separated from H. Yang-Mills equations evaluated in this way do not contain the products $HH',H\dot{H},\dot{H}H'$. The above prescription implies no change in equations when H is a smooth function. Therefore, the prescription can be regarded as a generalization of Yang-Mills equations. Thus, (9) can be said to solve Yang-Mills equations generalized by the above prescription to the case of potentials (3) with H having steplike discontinuities.

Now we shall consider the field strengths (4) and (5). Let us take

$$H=\pm\theta(r-t) \ , \quad (10)$$

for $t>0$. Then we have a nonzero electric field F_{0i}^a concentrated on the future light cone, and a nonzero magnetic field F_{ik}^a. For any fixed $t_0>0$ we have $H=\pm1$ for $r\geq t_0$ and $H=0$ for $r<t_0$. Thus, outside of the ball of radius t_0 we have zero field strengths, because $H=\pm1$ corresponds to the pure gauge or zero A_i, respectively. On the sphere $r=t_0$ we have singular electric and magnetic fields. Inside the ball, the field is that of the Wu-Yang magnetic monopole, formula (7).

We see that for $t_0\rightarrow\infty$ for all finite r we shall have the field of the Wu-Yang monopole. In fact, any solution of the form (9) has this property.

The real trouble with the solution (9) is that because the solution has singular field strength for $r=t_0$ it is difficult to give a meaning to physical quantities like, e.g., energy density for $r=t_0$. However, for the limited purpose of watching the time development of gauge fields such a solution is still interesting. This is the reason for presenting this solution.

III. THE REGULAR SOLUTION

The solution we shall present here can be regarded as a regular counterpart of the solution (10). That is, the regular solution of Eq. (6) has the value H_0 for $t=0$ and the value 0 for $t\rightarrow\infty$, for any r in the interval $0<r<\infty$. H_0 can be made arbitrarily close to $+1$ or -1.

In order to find this solution, let us assume that $H(r,t)$ depends on r and t through the variable

$$\rho=\mu(t)r ,$$

where $\mu(t)$ is a function of time. Then, Eq. (6) can be written as

$$\frac{\ddot{\mu}}{\mu^3}\rho\frac{dH}{d\rho}+\left[\frac{\dot{\mu}^2}{\mu^4}\rho^2-1\right]\frac{d^2H}{d\rho^2}=\frac{1}{\rho^2}(H-H^3) . \tag{11}$$

Thus, if

$$\frac{\dot{\mu}^2}{\mu^4}=\alpha^2,\ \ \frac{\ddot{\mu}}{\mu^3}=\beta , \tag{12}$$

where α,β are constants, then (11) becomes an equation in the single variable ρ, which is much easier to analyze than (6). The first of the conditions (12) implies that

$$\mu(t)=\pm\frac{1}{\alpha}\frac{1}{t-t_0} , \tag{13}$$

where t_0 is a constant. In the following we shall take $t_0=0$. Luckily, then the second condition in (12) happens to be satisfied too, with $\beta=2\alpha^2$. Introducing the new variable

$$\tau=(\alpha\rho)^{-1}-1=\pm\frac{t-t_0}{r}-1 , \tag{14}$$

we can write (11) in the form

$$(2+\tau)\tau\frac{d^2H}{d\tau^2}+2(\tau+1)\frac{dH}{d\tau}+H-H^3=0 . \tag{15}$$

For definiteness, we take the plus sign in (14). Then the interesting, for us, region $t>t_0$, $r>0$ corresponds to $\tau\in[-1,+\infty)$. Thus, we have to consider (15) in the half-line $\tau\in[=1,+\infty)$.

Equation (15) allows for helpful mechanical analogy for $\tau>0$. Namely, let us write (15) as

$$\frac{d}{d\tau}\left[(2+\tau)\tau\frac{dH}{d\tau}\right]=H^3-H . \tag{16}$$

This equation can be regarded as Newton's equation for a particle with time-dependent mass $m(\tau)=(2+\tau)\tau$ and momentum $p(\tau)=m(\tau)dH/d\tau$, moving in the potential $V(H)=H^2/2-H^4/4$. Equation (16) can be generated from the action

$$S=\int d\tau\left[\tfrac{1}{2}m(\tau)\left[\frac{dH}{d\tau}\right]^2-\frac{H^2}{2}+\frac{H^4}{4}\right] .$$

For $\tau\in[1,0)$ the particle becomes tachyonic. In this region the mechanical analogy can be obtained by the change of variable $\tau\to\zeta=-\tau$.

We have found that apart from the trivial $H=0,\pm1$, Eq. (15) has solutions $H(\tau)$ regular in the interval $\tau\in[-1,\infty)$ and such that

$$H(\tau=-1)=H_0,\ \ H(\tau=+\infty)=0 , \tag{17}$$

where rigorous arguments show that

$$0<|H_0|<\sqrt{2} . \tag{18}$$

However, numerical investigations strongly suggest that in fact

$$0<|H_0|<1 . \tag{19}$$

The proof of the existence of the solution is given in Appendix A. Results obtained there lead to the shape presented in Fig. 1. In Appendix B we argue that if (19) is true then Eq. (15) has no other solutions regular for $r>0$.

Now we shall present a description of gauge fields corresponding to the regular solution $H(\tau)$ of Eq. (15). The field strengths are given by (4) and (5), where

$$\frac{\partial H}{\partial t}=\frac{1}{r}\frac{dH}{d\tau},\ \ \frac{\partial H}{\partial r}=-\frac{1+\tau}{r}\frac{dH}{d\tau} . \tag{20}$$

The process described by our solution is the following. Consider the field strengths at a given point $\vec{x},r=|\vec{x}|>0$. For $t=0$, i.e., $\tau=-1$, we have the magnetic field of type (5) with constant $1+H=1+H_0$. This constant can be made arbitrarily small by taking $H_0\to-1$. We also have the electric field. Namely,

$$F^a_{0i}=-\epsilon_{aij}\frac{x^j}{r^3}H_1 , \tag{21}$$

$$B^a_i=\tfrac{1}{2}\epsilon_{ikl}F^a_{lk}$$
$$=-2(1+H_0)\frac{\delta_{ia}}{r^2}+\frac{2(H_0{}^2-1)}{r^2}\frac{x^ix^a}{r^2} , \tag{22}$$

where H_1 is the value of $dH/d\tau$ for $\tau=-1$. Now,

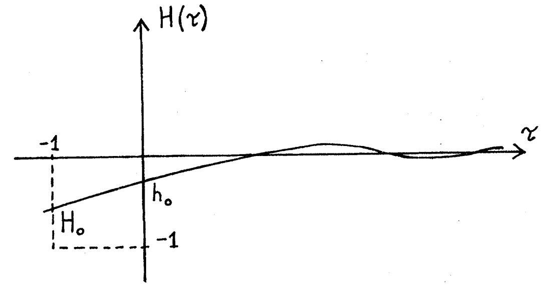

FIG. 1. The shape of the solution $H(\tau)$ of Eq. (15). For $h_0=-0.50$ we have obtained $H_0=-0.76$.

for $t>0$ we distinguish two regions, I and II, within the interval $[-1,+\infty)$ of the τ variable:

I: $r \ll t$, i.e., $\tau \gg -1$,

II: $r \gg t$, i.e., $\tau \approx -1$.

In region I, using the asymptotics (A4) for large τ we see that the field strengths [and the potentials (3)] approach the Wu-Yang magnetic monopole (7) located at $r=0$. In region II the fields have the form (21) and (22). Region II shrinks as $t \rightarrow \infty$. Thus the initial configuration (21) and (22) transforms into a Wu-Yang magnetic monopole when $t \rightarrow \infty$. Observe that this happens irrespectively of the allowed values of the constants H_0 and H_1. Thus, a Wu-Yang magnetic monopole is the universal limiting configuration for the whole family of initial configurations. This family is described by one parameter h_0 determining H_0 and H_1; see Appendix A.

The energy density of the field is

$$E(r,t)=\tfrac{1}{4}F_{ik}^{a}F_{ik}^{a}+\tfrac{1}{2}F_{0i}^{a}F_{0i}^{a}$$
$$=\frac{1}{\gamma^4}\left[[(1+\tau)^2+1]\left[\frac{dH}{d\tau}\right]^2 + \tfrac{1}{2}(H^2-1)^2\right] . \quad (23)$$

For the initial configurations (21) and (22) we have

$$r^4E_i(r,t)=\tfrac{1}{2}({H_0}^2-1)^2+{H_1}^2$$

and for the Wu-Yang monopole $r^4E_f(r,t)=\frac{1}{2}$.

Numerical investigations lead to the conclusion that

$$E_i<E_f \ . \quad (24)$$

Thus, the initial configuration has smaller density of energy than the final one. The total energy for both configurations is infinite, due to the singularity at $r=0$. For fixed r, $r^4E(r,t)$ has local extrema as the function of time t. For example, for each τ such that $H(\tau)=0$ we have $r^4E(r,t)>\frac{1}{2}$, while at each local extremum of $H(\tau)$ we have $r^4E(r,t)<\frac{1}{2}$. Let the position of such a local extremum be τ_k. The position of this extremum in space moves according to $t/r-1=\tau_k$, i.e.,

$$r=\frac{1}{1+\tau_k}t \ . \quad (25)$$

From the asymptotics (A4) we see that such extrema occur for arbitrary large τ_k. Therefore, the solution possesses local extrema of $r^4E(r,t)$ which move arbitrarily slowly. Thus, due to nonlinearity of equations Yang-Mills fields become effectively massive. A similar phenomenon, although in much simpler form, was observed for non-Abelian plane waves.[2-5] Observe also that for $-1<\tau<0$ we have a superluminal region. In this region $r^4E(r,t)$ seems to have no extrema, it grows monotonically—this is what numerical calculations for several values of h_0 show.

IV. CONCLUDING REMARKS

The above considerations obviously suggest an extension by including Higgs fields in the adjoint representation. In this way we may hope to avoid the singularity at $r=0$ by replacing the Wu-Yang monopole by the 't Hooft-Polyakov magnetic monopole.[8] Unfortunately, for this purpose the catch (11) and (12) is not sufficient. The deep reason is that the potentials (3), where $H(r,t)=H(\tau)$ is a function of the dimensionless variable τ, contain no dimensional constant (a scale). The total energy of any such field has to be zero or infinity, because of the lack of a dimensional constant. In order to have a finite energy, τ should be replaced by a dimensional variable or the ansatz (3) has to be modified.

At the moment we do not see any interesting application of our solutions in quantized Yang-Mills theory. However, we think that the solutions are interesting because they allow us to watch the time development of Yang-Mills fields on a rather nontrivial example.

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Appendix A: The proof of existence of the solution

In this appendix we assume that $H(\tau)$ is differentiable for all $\tau \in [-1,+\infty)$. Equation (15) has the singular point at $\tau=0$ (the other singularity at $\tau=-2$ is outside the considered region of the τ variable). Assuming that $H(\tau)$ is regular at $\tau=0$, we expand $H(\tau)$ around this point,

$$H(\tau)=h_0+h_1\tau+h_2\tau^2+h_3\tau^3+h_4\tau^4+\cdots \ . \quad (A1)$$

From (15) it follows that

$$h_1=\tfrac{1}{2}h_0(h_0{}^2-1),\ \ h_2=\tfrac{3}{16}h_0(h_0{}^2-1)^2\ ,$$
$$h_3=\tfrac{7}{96}h_0(h_0{}^2-1)^3\ , \qquad (A2)$$
$$h_4=\tfrac{1}{32}h_0(h_0{}^2-1)^3(\tfrac{87}{96}h_0{}^2-\tfrac{91}{96})\ .$$

Observe that all derivatives at $\tau=0$ are determined by the only free parameter $h_0=H(0)$.

Now we shall consider the two regions $\tau>0$ and $-1\leq\tau\leq 0$ separately. First, let us consider $\tau>0$. For this region we prove the following two lemmas.

Lemma AI.

$$\int_{\tau_n}^{\tau}(1+\tau')\dot H^2(\tau')d\tau'+\frac{(2+\tau)\tau}{2}\dot H^2(\tau)=\frac{H^4(\tau)-H^4(\tau_n)}{4}-\frac{H^2(\tau)-H^2(\tau_n)}{2}\ , \qquad (A3)$$

where $\tau_n=0$, or τ_n is a position of a local extremum of $H(\tau)$, i.e., $\dot H(\tau_n)=0$, $\tau_n>0$.

Proof. Multiply (16) by H and integrate from τ_n up to τ.

Lemma AII.

If the limit $\lim_{\tau\to\infty}H(\tau)$ exists, then it could only be 0, ± 1, or ∞.

Proof. Assume that $H_\infty=\lim_{\tau\to\infty}H(\tau)$ is finite and different from $0,\pm 1$. Then, Eq. (15) considered for $\tau\to\infty$ has the solution

$$H(\tau)=(H_\infty{}^3-H_\infty)\ln\tau+\frac{\alpha}{\tau}+\beta\ ,$$

where α and β are constants. This contradicts the assumption that H_∞ is finite. This ends the proof.

Now let us assume that $|h_0|>1$. Then (A2) implies that

$$\left.\frac{dH}{d\tau}\right|_{\tau=0}=h_1$$

has the same sign as h_0. On the other hand, Eq. (15) implies that in the region $\tau>0$, $H(\tau)$ cannot have neither local maximum such that $H(\tau)>1$ nor local minimum such that $H(\tau)<-1$ [because, e.g., $\dot H=0$, $\ddot H>0$, $H<-1$, $\tau>0$ are inconsistent with (15)]. These two facts together lead to the conclusion that $|H(\tau)|>|h_0|>1$. According to Lemma AII this implies that $H(\tau)$ has no limit or that it grows to infinity as $\tau\to\infty$. Thus, $|h_0|>0$ does not lead to an interesting solution (see also Lemma BI).

Therefore, we consider $|h_0|<1$ [the case $h_0=\pm 1$ leads to $H(\tau)=\pm 1$ for all $\tau>0$, as is seen from (A2)]. In this case, Lemma AI considered for $\tau_n=0$ implies that $|H(\tau)|<|h_0|$ for $\tau>0$, because $|H(\tau)|=|h_0|$ would contradict the strict positivity of the left-hand side of (A3). Thus, $H(\tau)$ cannot reach $\pm 1,\infty$ when $|h_0|<1$.

Therefore, $H(\tau)$ reaches 0 or it does not have any limit when $\tau\to\infty$, according to Lemma AII. Now let us eliminate the second possibility. The lack of a limit means that $H(\tau)$ oscillates with constant or growing amplitude as $\tau\to\infty$. This, however, is impossible according to Lemma AI considered for $\tau_n>0$. Therefore, $H(\tau)$ has to approach 0 for $\tau\to\infty$.

Finally, we observe that Eq. (15) has the following solution for large τ when $H(\tau)$ is small:

$$H(\tau)=\frac{A}{(\tau+\tau_0)^{1/2}}\cos\left[\frac{\sqrt{3}}{2}\ln(\tau+\tau_0)\right]\ , \qquad (A4)$$

where A and τ_0 are constants. This ends the proof of existence of the solution for $\tau>0$. When we start from any $0<|h_0|<1$, Eq. (15) will drive $H(\tau)$ toward small values and then the equation possesses solution (A4), which becomes more and more accurate as $\tau\to\infty$.

Before plunging into the other part of the proof, the proof of existence of a solution for $\tau\in[-1,0)$, let us make two remarks. First, the above behavior of $H(\tau)$ as $\tau\to\infty$ can also be seen from the mechanical analogy mentioned in Sec. III. Second, for large-τ numerical estimates it is more convenient to use the variable $\mu=\ln\tau$, in which Eq. (15) takes the form

$$[1+2\exp(-\mu)]\frac{d^2H}{d\mu^2}+\frac{dH}{d\mu}+H-H^3=0\ . \qquad (A5)$$

Now we shall consider the region $-1\leq\tau<0$. This region is easier to consider because the interval is finite. Let us change variable τ on $\zeta=-\tau$. Then, Eq. (16) becomes

$$\frac{d}{d\zeta}\left[(2-\zeta)\zeta\frac{dH}{d\zeta}\right]=H-H^3\ . \qquad (A6)$$

By steps analogous to those in the proof of Lemma AI we obtain from (A6) that

$$\tfrac{1}{2}(2-\zeta)\zeta\dot H^2(\zeta)+2\int_0^{\zeta}(1-\zeta')\dot H^2 d\zeta'=\tfrac{1}{2}H^2(\zeta)-\tfrac{1}{4}H^4(\zeta)+\tfrac{1}{4}h_0{}^4-\tfrac{1}{2}h_0{}^2\ . \qquad (A7)$$

The left-hand side of (A7) is strictly positive for $0<\zeta\leq 1$. Therefore, $H(\zeta)\neq h_0$ for $0<\zeta\leq 1$. Thus, if $0<h_0<1$, then $H(\zeta)$ has to grow for $\zeta\in(0,1)$, because for the initial $\zeta=0$

$$\left.\frac{dH}{d\zeta}\right|_{\zeta=0}=-h_1>0\ .$$

Similarly, if $-1<h_0<0$, then $H(\zeta)$ has to decrease for $\zeta\in(0,1]$. From (A7) it also follows that

$$|H^2(\zeta)-1|\leq 1-h_0{}^2 \tag{A8}$$

for $\zeta\in[+1,0)$. This implies that for $\zeta=1$, $H(\zeta)$ reaches a certain finite value H_0. Also we see that $H_0\to -1$ when $h_0\to -1$. This ends the proof of existence of the solution.

In fact we have obtained a stronger result. We have shown that the type of solutions found, parametrized by $-1<h_0<1$, is the only type of regular solutions finite when $\tau\to+\infty$.

The inequality (A8) formally implies that

$$-\sqrt{2}<H_0<\sqrt{2} \tag{A9}$$

because $|h_0|<1$. For $H_0=0$ we have of course the trivial solution $H(\tau)=0$. However, all numerical solutions of Eq. (15) in the interval $\tau\in[-1,0)$ which we have found for several values of $-1<h_0<0$ do not reach the value $H(\tau)=-1$ for any $\tau\in[-1,0)$. This strongly suggests that, in fact, the bound

$$0<|H(\tau)|<1,\quad \tau\in[-1,0] \tag{A10}$$

is obeyed for nontrivial solutions.

APPENDIX B: NO OTHER REGULAR SOLUTION OF EQ. (15) EXISTS

Lemma BI. If $|h_0|>1$, the solution $H(\tau)$ of Eq. (15) develops a singularity for finite $\tau>0$.

Proof. Because of the symmetry $H(\tau)\to -H(\tau)$ let us consider $h_0>1$. In Appendix A we have already obtained that $H(\tau)\to\infty$ when $\tau\to\infty$ if $H(\tau)$ is differentiable for all finite τ. Therefore, for large $\mu=\ln\tau$, Eq. (A5) can be written as

$$\frac{d^2H}{d\mu^2}+\frac{dH}{d\mu}-H^3=0\ . \tag{B1}$$

From this equation it follows that

$$\frac{d}{d\mu}\left[\frac{1}{2}\left[\frac{dH}{d\mu}\right]^2-\tfrac{1}{4}H^4\right]\leq 0\ .$$

This inequality implies that $dH/d\mu$ increases more slowly than H^2. Therefore, in (B1) we also can neglect the $dH/d\mu$ term. Thus, for large μ

$$\frac{d^2H}{d\mu^2}-H^3=0\ . \tag{B2}$$

However, this equation can be integrated yielding

$$H(\mu)=\frac{\sqrt{2}}{\pm\mu+\mu_0},\ \text{i.e.,}\ H(\tau)=\pm\frac{\sqrt{2}}{\ln(\tau/\tau_0)}\ , \tag{B3}$$

where μ_0 and τ_0 are constants. This solution contradicts the fact that $H(\tau)$ has to increase to infinity when $\tau\to\infty$. Therefore, the assumption that $\dot{H}$ exists for all $\tau\in(0,\infty)$ is wrong.

This lemma implies that the solution described in Appendix A is the only solution regular in the interesting interval $\tau\in[-1,\infty)$.

Lemma BII. If the numerically confirmed hypothesis (A10) is valid, then Eq. (15) has no solutions regular for all $\tau\in(-\infty,+\infty)$, except for the trivial solutions $H(\tau)=0,\pm1$.

Proof. Equation (15) is symmetrical with respect to the point $\tau=-1$, i.e., $\tau+1\to-\tau-1$ transforms the equation into itself. It follows that the solution $H(\tau)$ regular for all τ has to have zero derivative at $\tau=-1$, or that it should vanish for $\tau=-1$. However, the last possibility is excluded by (A8) because $0<|h_0|<1$ for regular nontrivial solutions. The former possibility is easily seen to be impossible when one considers the mechanical analogy for Eq. (A6) together with (A10). Namely, it would mean that the particle has zero velocity at the time $\zeta=1$ in spite of the fact that at $\zeta=0$ it has started to move with velocity pointing in the same direction as the external force $H-H^3$. For example, when $-1<h_0<0$ the initial velocity and force are negative provided that $0>H(\zeta)>-1$ for all $\zeta\in[0,1]$. The fact that $0>H(\zeta)>-1$ for $\zeta\in[0,1]$ follows from (A10) because $h_0<0$.

From Lemma BII it follows that our solution $H(\tau)$ regular for $\tau\in[-1,\infty)$ has to develop a singularity for certain $\tau<-1$. Most likely, it will be the point $\tau=-2$ which is a singular point of Eq. (15).

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*Present address: Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Cracow, Poland.

[1]S. Coleman, Phys. Lett. 70B, 59 (1977).
[2]A. Actor, Lett. Math. Phys. 2, 275 (1978).
[3]R. Casalbuoni, G. Domokos, and S. Kövesi-Domokos, Phys. Lett. 81B, 45 (1979).
[4]C. H. Oh and R. Teh, Phys. Lett. 87B, 83 (1979).
[5]G. Z. Baseyan, S. G. Matinyan, and G. K. Savvidi,

Pis'ma Zh. Eksp. Teor. Fiz. 29, 641 (1979) [JETP Lett. 29, 587 (1979)].
[6]T. T. Wu and C. N. Yang, in *Properties of Matter Under Unusual Conditions,* edited by M. Mark and S. Fernbach (Interscience, New York, 1969), p. 349.
[7]G. Rosen, Phys. Rev. 139, A539 (1965).
[8]G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974) [JETP Lett. 20, 194 (1974)].