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Inhomogeneous axial gauges

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Inhomogeneous axial gauges are investigated chiefly in the framework of free-field (Abelian) theory. According to the spacelike or timelike character of the fixed four-vector and the nature of the coupling between the Lagrangian multiplier and the gauge fields, these gauges involve 2, 3, 4, or even 5 degrees of freedom, of which only two are physical. Gauges with 4 or 5 degrees of freedom involve an indefinite-metric formalism which is built with some details in complete parallelism with relativistic gauges. It is shown that the temporal gauges are not related to the Fermi gauge by a transformation $A'_\mu = A_\mu + \partial_\mu X$. The absence of Faddeev-Popov ghosts in the non-Abelian case is also discussed as well as the nature of the $(n\cdot k)^{-\alpha}$ singularities in the propagator.

I. INTRODUCTION

Axial gauges

$$n\cdot A = 0 , \tag{1}$$

where n is a given four-vector, are frequently used in the framework of perturbative calculations in non-Abelian gauge theories because the Faddeev-Popov[1] ghosts decouple from the action. These are generalized[2] by inhomogeneous axial gauges which are characterized by a gauge-fixing term

$$\mathscr{L}_{\rm gf} = \frac{1}{2a} n\cdot A K n\cdot A , \tag{2}$$

where a is a parameter and K is the Fourier transform of a given function of k^2 and $n\cdot k$. For these inhomogeneous gauges, the ghosts are present in the action but do not play any role in perturbative calculations because any ghost loop contribution vanishes. Among these gauges, the case $K = \Box$ and $a = -1$ (planar gauge) is particularly interesting and frequently used in QCD calculations[3] because the propagator takes a very simple form. Some difficulties are, however, present.[4,5]

The aim of this paper is to discuss these gauges with respect to the number of independent variables which are involved and to build up the appropriate formalism for the free-field theory. Many little known facts are illustrated. Among them, let us mention (1) the $(n\cdot k)^{-\alpha}$ singularities in the propagator, which are free for spacelike n are fixed to the principal values for timelike n (Ref. 6) and (2) the temporal gauge is not related to the Fermi or the Coulomb gauge by a transformation

$$A'_\mu = A_\mu + \partial_\mu X . \tag{3}$$

When it is necessary, the indefinite-metric formalism is built with some details in complete parallelism with the case of relativistic gauges. The difference between the singular limit for $a = 0$ with different K's which lead to the same propagator is displayed and the absence of Faddeev-Popov ghosts in non-Abelian theories is also discussed. We show that the ghost part of the Lagrangian and the Lagrangian itself are both Becchi-Rouet-Stora (BRS) invariant, so that the ghosts do not play any role.

We organize our work as follows. In Sec. II we build up, with the help of a Lagrange multiplier, the most general Lagrangian describing inhomogeneous gauges and discuss the number of independent variables which are involved according to the different values of the parameters. In Sec. III, we discuss the Faddeev-Popov ghost problem. In Sec. IV, we briefly attack the propagator-singularity problem. In Sec. V, we build up a general formulation of the indefinite-metric formalism, which is applied to various particular cases. In Sec. VI, we discuss the relation between different inhomogeneous gauges and between them and the Fermi gauge. Finally, we conclude with some remarks in Sec. VII.

II. GENERAL INHOMOGENEOUS AXIAL GAUGES

Let us consider the axial-gauge condition (1). It can be introduced inside the Lagrangian through the help of a Lagrange multiplier S,

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+Sn\cdot A \ , \qquad (4)$$

and we may generalize the procedure by adding a term involving only the S field,

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+Sn\cdot A-\tfrac{1}{2}aS^2 \ . \qquad (5)$$

If we generalize further the procedure, the most general free Lagrangian coupling the Lagrange multiplier S to the gauge field through $n\cdot A$ is

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+fSn\cdot A+g\partial_\mu n\cdot A\,\partial^\mu S\\&+hn\cdot\partial Sn\cdot\partial n\cdot A+kSn\cdot\partial n\cdot A\\&-\tfrac{1}{2}aS^2+\tfrac{1}{2}b\partial_\mu S\partial^\mu S+\tfrac{1}{2}c(n\cdot\partial S)^2 \ . \qquad (6)\end{aligned}$$

We restrict ourselves to cases for which equal-time quantization can be carried out, i.e., we take either $n=(1,0,0,0)$ or $n=(0,\vec{n})$. In both cases, we have

$$\begin{aligned}\Pi^\mu&=\frac{\partial\mathscr{L}}{\partial(\partial_0 A_\mu)}\\&=-F^{0\mu}+n^\mu(g\partial_0 S+hn_0 n\cdot\partial S+kn_0 S) \ , \qquad (7)\end{aligned}$$

$$\begin{aligned}\Pi_S=\frac{\partial\mathscr{L}}{\partial(\partial_0 S)}=&\,g\partial_0 n\cdot A+hn_0 n\cdot\partial n\cdot A+b\partial_0 S\\&+cn_0 n\cdot\partial S \ . \qquad (8)\end{aligned}$$

From Eq. (7), it is clear that, for $n=(0,\vec{n})$, we will have $\Pi^0=0$ in contrast to the case $n=(1,0,0,0)$. These two cases must therefore be distinguished in the following.

(1) $n=(1,0,0,0)$:

$$\Pi^k=F^{k0} \ , \qquad (9)$$

$$\Pi^0=(g+h)\partial_0 S+kS \ , \qquad (10)$$

$$\Pi_S=(g+h)\partial_0 A_0+(b+c)\partial_0 S \ . \qquad (11)$$

Except for particular cases, we can conclude from Eqs. (9) to (11) that 5 independent degrees of freedom are involved with inhomogeneous axial timelike gauges. The number of independent degrees of freedom can be reduced if there exist some constraints between Π^0, S, and Π_S. In particular, we will have two primary constraints[7]:

$$\Pi^0-kS=0 \ , \qquad (12)$$

$$\Pi^S=0 \qquad (13)$$

if

$$g+h=b+c=0 \ . \qquad (14)$$

If $k\neq 0$, the constraints (12) and (13) are second class and the number of independent pairs of variables reduces to 4. If $k=0$, (12) and (13) are first-class constraints. In that case, the Hamiltonian density can be written

$$\begin{aligned}\mathscr{H}=&\tfrac{1}{2}\,|\vec{\Pi}\,|^2+\tfrac{1}{2}\,|\vec{B}\,|^2-fSA_0+g\partial_k A_0\partial_k S\\&+\tfrac{1}{2}aS^2+\tfrac{1}{2}b\partial_k S\partial_k S-\Pi^k\partial^k A_0 \ . \qquad (15)\end{aligned}$$

The primary first-class constraints (12) and (13) imply the secondary constraints

$$(f+g\Delta)S-\partial^k\Pi^k=0 \ , \qquad (16)$$

$$(f+g\Delta)A_0+(b\Delta-a)S=0 \ . \qquad (17)$$

Except for the case $a=b=c=f=g=h=k=0$ for which the gauge is not fixed, Eqs. (12), (13), (16), and (17) form a set of four second-class constraints and the number of independent pairs of variables reduces to 3.

We will have only one primary constraint (12) if $g+h=0$ but $b+c\neq 0$. Then

$$\begin{aligned}\mathscr{H}=&\tfrac{1}{2}\,|\vec{\Pi}\,|^2+\tfrac{1}{2}\,|\vec{B}\,|^2-\Pi^k\partial^k A^0-fSA_0\\&+g\partial_k A_0\partial_k S+\tfrac{1}{2}aS^2+\tfrac{1}{2}b\partial_k S\partial_k S \ . \qquad (18)\end{aligned}$$

Equation (19) implies the secondary constraint

$$(f+g\Delta)S-\partial^k\Pi^k-\frac{k}{b+c}\Pi_S=0 \qquad (19)$$

which, with (12), forms a second-class set if $k\neq 0$. Again, if $k\neq 0$, there are four independent variables.

If $k=0$, Eq. (12) implies the following chain of secondary constraints:

$$\begin{aligned}\Pi^0=0\Longrightarrow(f+g\Delta)S-\partial^k\Pi^k=0\Longrightarrow(f+g\Delta)\Pi_S=0\\\Longrightarrow(f+g\Delta)[(f+g\Delta)A_0-(a-b\Delta)S]=0\end{aligned} \qquad (20)$$

forming a second-class set. Again, the number of independent degrees of freedom is equal to 3.

For timelike n, we have 3, 4, or 5 degrees of freedom. When only three independent pairs of variables are involved, the inhomogeneous axial gauges are class II (Ref. 8) generalizing the temporal gauge. This happens when no time derivatives are involved in the coupling between the Lagrange multiplier and the gauge field. If a time derivative of the potential but not of the Lagrange multiplier is involved, 4 degrees of freedom are present and the gauge is of class III and involves an indefinite-metric formalism. We shall study it in the following. If a time derivative of the Lagrange multiplier is involved, 5 degrees of freedom are necessary and this gauge does not fall into the usual classes. Here, again, an indefinite metric is necessary. This can be easily seen in the particular case of the Lagrangian (6) with $g=-1$, $a=b=c=f=h=0$. The diagonalizing transformation

$$\begin{bmatrix} A_0 \\ S \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \tag{21}$$

induces a wrong sign for the kinetic term of the V field.

Let us recall that it is precisely in that case that the self-energy is nontransverse in Yang-Mills theory.[4] This could be related to the fact that the gauge does not fall into the usual classes.

(2) $n=(0,\vec{n})$

Here, we always have a primary constraint

$$\Pi^0=0 \tag{22}$$

while

$$\Pi^k=F^{k0}+gn^k\partial_0 S\ , \tag{23}$$

$$\Pi_S=-g\partial_0 n\cdot\vec{A}+b\partial_0 S\ . \tag{24}$$

We can have a second primary constraint

$$\Pi_S-g(\vec{n}\cdot\vec{\Pi}-\vec{n}\cdot\vec{\partial}A_0)=0 \tag{25}$$

if $b=g^2$. If $g\neq 0$, Eqs. (22) and (25) are second-class constraints and 4 degrees of freedom are required.

If $g=0$, Eqs. (22) and (25) are first class and we can write the Hamiltonian as

$$\mathscr{H}=\tfrac{1}{2}\,|\vec{\Pi}\,|^2+\tfrac{1}{2}\,|\vec{B}\,|^2-\Pi^k\partial^k A^0+fS\vec{n}\cdot\vec{A} +h\vec{n}\cdot\vec{\partial}S\vec{n}\cdot\vec{\partial}\vec{n}\cdot\vec{A}-kS\vec{n}\cdot\vec{\partial}\vec{n}\cdot\vec{A} +\tfrac{1}{2}aS^2+\frac{1}{2}(\vec{n}\cdot\vec{\partial}S)^2\ . \tag{26}$$

Equations (22) and (25) imply the following chains of secondary constraints:

$$\Pi^0=0 \Longrightarrow \partial^k\Pi^k=0 \Longrightarrow [f-h(\vec{n}\cdot\vec{\partial})^2]S=0\ , \tag{27}$$

$$\Pi_S=0 \Longrightarrow [f-h(\vec{n}\cdot\vec{\partial})^2]\vec{n}\cdot\vec{A} +[a-c(\vec{n}\cdot\vec{\partial})^2]S=0 \Longrightarrow [f-h(\vec{n}\cdot\vec{\partial})^2](\vec{n}\cdot\vec{\Pi}-\vec{n}\cdot\vec{\partial}A_0)\ , \tag{28}$$

which form a set of six second-class constraints. Only two pairs of independent variables are involved and the corresponding gauges are class I.

If $b\neq g^2$, there is only one primary constraint and

$$\mathscr{H}=\tfrac{1}{2}\,|\vec{\Pi}\,|^2+\tfrac{1}{2}\,|\vec{B}\,|^2-\Pi^k\partial^k A^0+\frac{1}{2(b-g^2)}(\Pi_S-g\vec{n}\cdot\vec{\Pi}-g\vec{n}\cdot\vec{\partial}A_0)^2+fS\vec{n}\cdot\vec{A}-g\partial_k\vec{n}\cdot\vec{A}\partial_k S +h\vec{n}\cdot\vec{\partial}S\vec{n}\cdot\vec{\partial}\vec{n}\cdot\vec{A}-kS\vec{n}\cdot\vec{\partial}\vec{n}\cdot\vec{A}+\tfrac{1}{2}aS^2+\tfrac{1}{2}b\partial_k S\partial_k S-\tfrac{1}{2}c(\vec{n}\cdot\vec{\partial}S)^2\ . \tag{29}$$

From Eq. (22) results the secondary constraint

$$\partial^k\Pi^k-\frac{g}{b-g^2}\vec{n}\cdot\vec{\partial}(\Pi_S-g\vec{n}\cdot\vec{\Pi}+g\vec{n}\cdot\vec{\partial}A_0)=0\ . \tag{30}$$

Equations (22) and (30) are second class if $g\neq 0$ while, if $g=0$, the following chain of constraints is implied by (22):

$$\Pi^0=0 \Longrightarrow \partial^k\Pi^k=0 \Longrightarrow \vec{n}\cdot\vec{\partial}[f-h(\vec{n}\cdot\vec{\partial})^2]S=0 \Longrightarrow \vec{n}\cdot\vec{\partial}[f-h(\vec{n}\cdot\vec{\partial})^2]\Pi_S=0 \Longrightarrow \vec{n}\cdot\vec{\partial}[f-h(\vec{n}\cdot\vec{\partial})^2]\vec{n}\cdot\vec{A}=0 \Longrightarrow \vec{n}\cdot\vec{\partial}[f-h(\vec{n}\cdot\vec{\partial})^2](\vec{n}\cdot\vec{\Pi}-\vec{n}\cdot\vec{\partial}A_0)=0\ . \tag{31}$$

This is a set of six second-class constraints reducing to 2 the number of degrees of freedom.

The situation is similar to but simpler than that in the timelike case. If a time derivative is involved in the coupling between the Lagrange multiplier and the potentials, 4 degrees of freedom are present. As in the timelike case, it can be noted that an indefinite metric is necessary. Moreover, since the constraint (25) or (30) implies a spatial derivative of A_0, the variable canonically conjugate to the primary constraint (22), the theory is nonlocal and such gauges are again fundamentally different from the usual gauges. If such a time derivative does not occur in the Lagrangian, we are faced with the physical degrees of freedom only and there is no essential difficulty as in the usual axial gauge.

III. THE FADDEEV-POPOV GHOSTS

Before introducing the ghosts in non-Abelian theory through BRS invariance,[9] let us recall the result of the Noether theorem in a weaker form than that usually used. Let $\doteq$ mean equality when field equations are used. The net result of the Noether theorem is the following. Let $\mathscr{L}$ be a Lagrangian depending on a set of fields u_i and their first derivatives $\partial_\mu u_i$ submitted to the transformation

$$\delta u_i = \psi_{ij}\delta\omega^j, \quad \partial_\mu \delta\omega^j = 0 . \tag{32}$$

Then,

$$\delta\mathscr{L} \doteq \partial_\mu j_i^\mu \delta\omega^i \tag{33}$$

with

$$j_i^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu u_j)}\psi_{ji} . \tag{34}$$

If the Lagrangian is invariant, the current (22) is conserved. Equation (33) is, however, more powerful than this particular result. We may indeed have a noninvariant Lagrangian satisfying

$$\delta\mathscr{L} \doteq \partial_\mu k_i^\mu \delta\omega^i . \tag{35}$$

The conserved current is then $j_i^\mu - k_i^\mu$ instead of j_i^μ. This remark allows us to add to the Lagrangian four-divergences or terms of the form $n\cdot\partial g(x)$, which could alter the variation of the Lagrangian but not the conserved current. Adding such terms, the Lagrangian (6) generalized to the non-Abelian case becomes

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}^\alpha F_\alpha^{\mu\nu} + n\cdot A_\alpha L S^\alpha - \tfrac{1}{2}S_\alpha M S^\alpha , \tag{36}$$

where

$$L = f - g\Box - h(n\cdot\partial)^2 + kn\cdot\partial , \tag{37}$$

$$M = a + b\Box + c(n\cdot\partial)^2 . \tag{38}$$

Since the field equations with respect to S_α read

$$MS_\alpha = Ln\cdot A_\alpha , \tag{39}$$

we can write, when M is nonsingular,

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}^\alpha F_\alpha^{\mu\nu} + \tfrac{1}{2}n\cdot A_\alpha K n\cdot A^\alpha , \tag{40}$$

where

$$K = LM^{-1}L \tag{41}$$

is a symmetric operator.

Under the BRS transformation,

$$\begin{aligned}\delta A_\mu^\alpha &= \partial_\mu\theta^\alpha + g f_{\beta\gamma}^\alpha A_\mu^\beta \theta^\gamma \\ &= (D_\mu)_\gamma^\alpha \theta^\gamma = (D_\mu)_\gamma^\alpha \eta^\gamma \delta\lambda ,\end{aligned} \tag{42}$$

$$\delta\eta^\alpha = -\tfrac{1}{2} g f_{\beta\gamma}^\alpha \eta^\beta \eta^\gamma \delta\lambda , \tag{43}$$

$$\delta\xi_\alpha = -n\cdot A_\alpha \delta\lambda , \tag{44}$$

where $\delta\lambda$, η^α, ξ_α are anticommuting objects, we have, since K is symmetric,

$$\begin{aligned}\delta S &= \int d^4x\, \delta\mathscr{L} = \tfrac{1}{2}\int d^4x\, \delta(n\cdot A_\alpha K n\cdot A^\alpha) \\ &= \int d^4x\, n\cdot A_\alpha K \delta n\cdot A^\alpha\end{aligned} \tag{45}$$

and the action can be made invariant if we add the Faddeev-Popov ghosts term

$$\begin{aligned}S_{\rm FP} &= -\int d^4x\, \xi_\alpha K n\cdot D_\beta^\alpha \eta^\beta \\ &= -\int d^4x\, \xi_\alpha' n\cdot D_\beta^\alpha \eta^\beta ,\end{aligned} \tag{46}$$

where

$$\xi_\alpha' = K\xi_\alpha \tag{47}$$

and the symmetry of K has been used.

This is the usual and simplest way to obtain the ghost term. However, we want to work with a local Lagrangian containing only fields and their first derivatives. Including the ghost term, this Lagrangian reads

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{4}F_{\mu\nu}^\alpha F_\alpha^{\mu\nu} + fS_\alpha n\cdot A^\alpha + g\partial_\mu n\cdot A_\alpha \partial^\mu S^\alpha \\ &+ hn\cdot\partial S_\alpha n\cdot\partial n\cdot A^\alpha + kS_\alpha n\cdot\partial n\cdot A^\alpha \\ &- \tfrac{1}{2}aS_\alpha S^\alpha + \tfrac{1}{2}b\partial_\mu S_\alpha \partial^\mu S^\alpha \\ &+ \tfrac{1}{2}cn\cdot\partial S^\alpha n\cdot\partial S_\alpha - \xi_\alpha n\cdot D_\beta^\alpha \eta^\beta .\end{aligned} \tag{48}$$

The Euler-Lagrange equation with respect to ξ_α reads

$$n\cdot D_\beta^\alpha \eta^\beta \doteq 0 \tag{49}$$

and, with the help of Eq. (42), it implies

$$\delta n\cdot A_\alpha \doteq 0 . \tag{50}$$

The transformation law of S_α is not specified. In order to be in agreement with Eqs. (39) and (50), we take

$$\delta S_\alpha \doteq 0 \tag{51}$$

and it can be seen that the particular transformation of ξ_α does not play any role in the calculation of $\delta\mathscr{L}$ up to the validity of field equations. We could take

$$\delta\xi_\alpha \doteq 0 \tag{52}$$

if the whole set of transformations [Eqs. (42), (43), (51), and (52)] is consistent with the definition of the BRS current and canonical Poisson brackets or commutation relations.

Since $\delta\mathscr{L} \doteq 0$,

$$J^\mu \doteq \frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\nu^\alpha)}\delta A_\nu^\alpha + \frac{\partial\mathscr{L}}{\partial(\partial_\mu \eta^\alpha)}\delta\eta^\alpha \tag{53}$$

independently of the transformation on ξ_α. From the canonical Poisson brackets, we derive

$$\begin{aligned}\{\delta A_\nu^\alpha(x), A_\mu^B(y)\}_{x_0=y_0} &= \{\delta A_\nu^\alpha(x), \eta^\beta(y)\}_{x_0=y_0}\\ &= \{\delta A_\nu^\alpha(x), \xi_\beta(y)\}_{x_0=y_0}\\ &= \{\delta\eta^\alpha(x), A_\mu^\beta(y)\}_{x_0=y_0}\\ &= \{\delta\eta^\alpha(x), \eta^\beta(y)\}_{x_0=y_0}\\ &= 0\ ,\end{aligned} \tag{54}$$

so that the time component of the conserved current

$$J^0 \doteq \Pi_\alpha^\nu \delta A_\nu^\alpha + \Pi_{\eta\alpha}\delta\eta^\alpha$$

satisfies

$$\{J_0(x), A_\mu^\alpha(y)\}_{x_0=y_0} \doteq -\delta A_\mu^\alpha(x)\delta^{(3)}(\vec{x}-\vec{y})\ , \tag{55}$$

$$\{J_0(x), \eta^\beta(y)\}_{x_0=y_0} \doteq -\delta\eta^\alpha(x)\delta^{(3)}(\vec{x}-\vec{y})\ , \tag{56}$$

$$\{J_0(x), S^\alpha(y)\}_{x_0=y_0} \doteq \{J_0(x), \xi_\alpha(y)\}_{x_0=y_0} = 0 \tag{57}$$

which are consistent with the assumed BRS transformations. Let us remark that, for relativistic gauges, the situation is quite different. The consistency imposes the transformation law

$$\delta\xi_\alpha \doteq -\Pi_\alpha^0\delta\lambda\ .$$

Here, both the original and the Faddeev-Popov Lagrangians are invariant under the *modified* BRS transformations and the ghost term can *strictly* be dropped out without destroying BRS invariance.

This is an important difference between our and the usual formulation where a paradoxal situation occurs. Indeed, ghosts are necessary to formulate the usual BRS invariance (and consequently to deduce Slavnov-Taylor[10] identities) while any ghost loop vanishes as can be checked by a formal calculation.[5]

IV. THE PROPAGATOR SINGULARITY

We have already treated this probelm in a previous publication[6] for some particular cases (axial spacelike and timelike gauges, $\partial_0 A_0 = aS$ gauges, timelike planar gauge). Our conclusion was that the propagator singularity $(n\cdot k)^{-\alpha}$ is not fixed for spacelike n while, for timelike n, the principal value is requested. We refer to Ref. 6 for details and summarize here only the main points which are also generalized to any inhomogeneous axial gauge.

For spacelike n, $\Pi^0=0$ is always a primary constraint so that A_0 can never be an independent variable. It is given by another constraint which is either (25) or (28) or (30) or (31). All these equations involve an inhomogeneous differential equation

$$\vec{n}\cdot\vec{\partial}A_0 = X \tag{58}$$

for which the Green's function is not fixed if boundary conditions on A_0 are not given. This indetermination of the Green's function is already present in the canonical equal-time commutation relations and is obviously carried over to the propagator.

For timelike n, the situation is quite different. $n\cdot\partial$ is an evolution operator and any differential equation involving this operator sets an initial-value problem instead of a boundary-value problem in the spacelike case. In the free-field theory, commutation relations for any time can be obtained without ambiguities from the solution of the Cauchy problem and equal-time commutators. Since the propagator used in perturbation theory is defined as

$$\begin{aligned}D_{\mu\nu}^F(x) = \theta(x_0)\langle 0|A_\mu(x)A_\nu(0)|0\rangle\\ +\theta(-x_0)\langle 0|A_\nu(0)A_\mu(x)|0\rangle\ ,\end{aligned} \tag{59}$$

where A_μ is the *free* field, it is given without ambiguity. Indeed, if we set

$$D_{\mu\nu}(\xi = x-y) = \langle 0|A_\mu(x)A_\nu(y)|0\rangle\ , \tag{60}$$

we have

$$\begin{aligned}\bar{D}_{\mu\nu}(\xi) &= \langle\overline{0|A_\mu(x)A_\nu(y)|0}\rangle\\ &= \langle 0|A_\nu(y)A_\mu(x)|0\rangle\\ &= D_{\nu\mu}(-\xi)\end{aligned} \tag{61}$$

and

$$\langle 0|[A_\mu(x), A_\nu(y)]|0\rangle = 2i\,\mathrm{Im}D_{\mu\nu}(\xi)\ . \tag{62}$$

Since, in any case, as can be seen from Ref. 6 or from Sec. VI, we have in $[A_k(x), A_l(0)]$, a term of the form

$$iC\partial_k\partial_l \frac{x_0{}^m}{|\vec{x}|} ,$$

where m is an odd number and $C = C(\Delta)$ is some real constant or a real function of the Laplacian operator, a term

$$\tfrac{1}{2} iC\partial_k\partial_l \frac{x_0{}^m}{|\vec{x}|}$$

is present in $\langle 0 | A_k(x) A_l(0) | 0 \rangle$. This term induces a term

$$\tfrac{1}{2} iC\partial_k\partial_l \frac{x_0^m \epsilon(x_0)}{|\vec{x}|}$$

in the propagator. Its Fourier transform gives a principal value

$$P\left[\frac{1}{k_0{}^{m+1}}\right] .$$

The use of field equations easily implies that there is no corresponding real part in $\langle 0 | A_k(x) A_l(0) | 0 \rangle$. For timelike n, the principal-value prescription is therefore unavoidable.

Our result is in contradiction with the conclusion of a paper by Caracciolo, Curci, and Menotti.[11] These authors compute the Wilson loop in the temporal gauge and compare the results with those obtained in Fermi and Coulomb gauges. They conclude that the principal-value prescription for the propagator singularity in the temporal gauge is excluded and they add a nontranslational-invariant term to the propagator. Owing to the absence of a special gauge transformation between the temporal and the Fermi gauges (see Sec. VI) we think that such a comparison is dangerous. Therefore the conclusion of Ref. 11 could be changed into the impossibility of comparing loop calculations (which are gauge dependent) in gauges which are not related by a special transformation. Also dangerous is the decomposition of the potentials into longitudinal and transverse parts. In the temporal gauge, in contrast to the Fermi or the Coulomb gauges $\partial^i \pi^i$ does not commute with A_i^T, so that the decomposition into longitudinal and transverse parts is not a canonical transformation leading to a separation between physical and nonphysical degrees of freedom. In fact, both difficulties are related and this could explain why the results of Caracciolo, Curci, and Menotti are confirmed in Ref. 12 where the implementation of Gauss's law in the temporal gauge leads to a singular nontranslational-invariant propagator.

V. INDEFINITE-METRIC FORMALISM

Let us write

$$A_\mu(x) = (2\pi)^{-3/2} \int d^4k [a_\mu(k) e^{-ik\cdot x} + a_\mu^\dagger(k) e^{ik\cdot x}] , \quad (63)$$

where $A_\mu(x)$ is assumed to be self-adjoint with respect to a sesquilinear form to be defined later. We note here that we cannot, in general, write

$$A_\mu(x) = (2\pi)^{-3/2} \int d^4k \, \theta(k_0) \delta(k^2) \sum_{i=0}^{3} \epsilon_\mu^{(i)}(k) [a_i(k) e^{-ik\cdot x} + a_i^\dagger(k) e^{ik\cdot x}] . \quad (64)$$

Equation (64) is a plane-wave development specific to the Fermi gauge where $A_\mu(x)$ satisfies

$$\Box A_\mu(x) = 0 . \quad (65)$$

We assume that $a_\mu(k)$ and $a_\mu^\dagger(k)$ are, respectively, annihilation and creation operators satisfying the following commutation relations:

$$[a_\mu(k), a_\nu(k')] = [a_\mu^\dagger(k), a_\nu^\dagger(k')] = 0 , \quad (66)$$

$$[a_\mu(k), a_\nu^\dagger(k')] = \kappa_{\mu\nu}(k) \delta^{(4)}(k - k') . \quad (67)$$

These commutation relations can be obtained in the following way. First, we solve the Cauchy problem and use canonical commutation relations to obtain $[A_\mu(x), A_\nu(0)]$ for any time x_0. Second, we derive $\langle 0 | A_\mu(x) A_\nu(0) | 0 \rangle$ according to the rules mentioned in the previous section and take the Fourier transform in order to get Eqs. (66) and (67).

For relativistic gauges,

$$\kappa_{\mu\nu}^{\rm rel}(k) = -g_{\mu\nu} \theta(k_0) \delta(k^2) - (1-a) \theta(k_0) \delta'(k^2) k_\mu k_\nu . \quad (68)$$

For the gauge

$$n\cdot\partial n\cdot A = a\cdot S \tag{69}$$

with timelike n,

$$\kappa_{\mu\nu}^{n\cdot\partial n\cdot A} = -\left[g_{\mu\nu} + \frac{k_\mu k_\nu - n\cdot k\,(k_\mu n_\nu + n_\mu k_\nu)}{(n\cdot k)^2 - k^2}\right]\theta(k\cdot n)\delta(k^2)$$
$$-\tfrac{1}{2}[k_\mu k_\nu - n\cdot k\,(k_\mu n_\nu + n_\mu k_\nu)]\frac{\delta'(n\cdot k)}{k^2} - \tfrac{1}{2}a k_\mu k_\nu \delta^{(4)}(n\cdot k) \tag{70}$$

while for the planar-type gauge with timelike n [Lagrangian (6) with $g=-1, f=h=a=c=0$],

$$\kappa_{\mu\nu}^{\mathrm{pl}} = -\left[g_{\mu\nu} + \frac{k_\mu k_\nu(1+b) - n\cdot k\,(n_\mu k_\nu + n_\nu k_\mu)}{(n\cdot k)^2 - k^2}\right]\theta(k\cdot n)\delta(k^2)$$
$$-\tfrac{1}{2}[k_\mu k_\nu(1+b) - n\cdot k\,(k_\mu n_\nu + n_\mu k_\nu)]\frac{\delta'(n\cdot k)}{k^2}\ . \tag{71}$$

The value of $\kappa_{\mu\nu}(k)$ for a general inhomogeneous axial gauge could also be obtained but the particular cases (70) and (71) are sufficiently illustrative to avoid this straightforward but tedious calculation.

A one-particle state of polarization μ and four-momentum k is obtained from the vacuum by

$$|k,\mu\rangle = a_\mu^\dagger(k)\,|0\rangle \tag{72}$$

and

$$\langle k,\mu\,|\,k',\nu\rangle = \langle 0\,|\,a_\mu(k)a_\nu^\dagger(k')\,|\,0\rangle$$
$$= \kappa_{\mu\nu}(k)\delta^{(4)}(k-k')\ . \tag{73}$$

Such states, for any k and μ, form a basis of the one-particle space and, for any state $|f\rangle$ of this space, we can write

$$|f\rangle = \int d^4k\, f^\mu(k)\,|\,k,\mu\rangle\ . \tag{74}$$

Then

$$\langle f\,|\,g\rangle = \int d^4k\, d^4k' \langle k,\mu\,|\,\bar f^\mu(k) g^\nu(k')\,|\,k',\nu\rangle$$
$$= \int d^4k\, \bar f^\mu(k)\kappa_{\mu\nu}(k) g^\nu(k)\ . \tag{75}$$

Equation (75) defines a sesquilinear form in the one-particle state. This is the form with respect to which A_μ is assumed to be a self-adjoint four-vector. It is clear, from Eq. (68) where $a=1$, that (75) reduces to the usual Gupta-Bleuler[13] sesquilinear form when the Fermi gauge is used. It is also clear that it is only in that gauge that we can speak about unphysical states as longitudinal and scalar photons, i.e., particles of zero mass. In all other gauges, such an interpretation in terms of particles is possible only for the physical states.

We define physical states by restricting the admissible g by

$$k\cdot g = 0\ . \tag{76}$$

For such states and all the above $\kappa_{\mu\nu}(k)$,

$$\langle g\,|\,g\rangle = -\int d^4k\, \theta(k_0)\delta(k^2)\bar g^\mu(k) g_\mu(k)$$
$$= -\int d^4k \frac{\theta(k_0)\delta(k^2)}{k_0{}^2} \times[(\vec{k}\cdot\vec{g})^2 - |\vec{k}\,|^2\,|\vec{g}\,|^2]$$
$$\geq 0\ . \tag{77}$$

On the subspace $\mathscr{H}'$ defined by Eq. (76), the sesquilinear form is non-negative. If we define $\mathscr{H}''$ as the subspace of $\mathscr{H}'$ for which

$$\vec{g} = \vec{k}h(k),\quad g_0 = \frac{|\vec{k}\,|^2 h(k)}{k_0}\ , \tag{78}$$

the physical one-particle Hilbert space can be obtained as $\mathscr{H}'/\mathscr{H}''$ completed with respect to the norm induced by the sesquilinear form. Such a construction is well known, as is the construction of the Fock space, and can be found in any textbook[14] for the Fermi gauge. It is sufficient to make the replacement

$$-g_{\mu\nu}\theta(k_0)\delta(k^2) \to \kappa_{\mu\nu}(k) \tag{79}$$

to generalize the results to any indefinite-metric gauge.

Less known is the relation between Eq. (76) and the Maxwell equations. Let us first work with relativistic gauges for which the field equations are[15]

$$\partial^\mu F_{\mu\nu}+\partial_\nu S=0\ , \tag{80}$$

$$\partial^\mu A_\mu=aS\ . \tag{81}$$

Since classical Maxwell equations must be valid between physical states, we impose

$$\langle \psi_{\rm phys}\,|\,S\,|\,\psi'_{\rm phys}\rangle=0 \tag{82}$$

which, in a gauge different from the Landau gauge ($a\neq 0$), is equivalent to

$$\langle \psi_{\rm phys}\,|\,\partial^\mu A_\mu\,|\,\psi'_{\rm phys}\rangle=0\ . \tag{83}$$

In particular,

$$\langle 0\,|\,\partial^\mu A_\mu\,|\,g\rangle=0 \Longrightarrow k\cdot g=0$$

and so is the relation between Maxwell equations and Eq. (76) established for $a\neq 0$ gauges. For the Landau gauge, following Lautrup,[15] we use the existence of a gauge transformation

$$A'_\mu=A_\mu+\partial_\mu X,\ \ S'=S\ , \tag{84}$$

$$X=\tfrac{1}{2}(a'-a)\Delta^{-1}(x_0\partial_0 S-\tfrac{1}{2}S) \tag{85}$$

between two relativistic gauges characterized by the parameters a and a'.

Because S is invariant and satisfies

$$[S(x),S(0)]=0\ , \tag{86}$$

it is clear that $\langle 0\,|\,S\,|\,g\rangle=0$ does not depend on the particular sesquilinear form which is used to define it. Therefore

$$\langle 0\,|\,S\,|\,g\rangle=0 \Longrightarrow k\cdot g=0\ , \tag{87}$$

which is true for $a\neq 0$, is also true for $a=0$.

The same reasoning can be repeated *mutatis mutandis* for inhomogeneous axial gauges. From the Lagrangian (6) with $f=g=b=c=0$, $h=1$, for instance, the field equations are

$$\partial^\mu F_{\mu\nu}-n_\nu n\cdot\partial S=0\ , \tag{88}$$

$$n\cdot\partial n\cdot A=aS\ . \tag{89}$$

Here (timelike n),

$$\langle 0\,|\,n\cdot A\,|\,g\rangle=(2\pi)^{-3/2}\int d^4k\, g^\nu(k)n^\mu \times\kappa_{\mu\nu}^{n\cdot\partial n\cdot A}(k)e^{ik\cdot x} \tag{90}$$

and, for $a\neq 0$,

$$\langle 0\,|\,n\cdot\partial S\,|\,g\rangle=0 \Longrightarrow k\cdot g=0\ . \tag{91}$$

Moreover, the transformation (84) with

$$X(x)=\tfrac{1}{2}(a'-a)\int d^4y\,|\,x_0-y_0\,|\,\delta^{(3)}(\vec{x}-\vec{y})S(y) \tag{92}$$

changes the gauge a into the gauge a' as can be checked at the level of field equations and commutation relations for any time,

$$[A_\mu(x),S(0)]=i\partial_\mu[x_0\delta^{(3)}(\vec{x})]\ , \tag{93}$$

$$[S(x),S(0)]=0\ , \tag{94}$$

$$[A_\mu(x),A_0(0)]=-ia\partial_\mu\partial_0\left[\frac{x_0{}^3}{3!}\delta^{(3)}(\vec{x})\right]\ , \tag{95}$$

$$[A_m(x),A_n(0)]=-i\left[g_{mn}+\frac{\partial_m\partial_n}{\Delta}\right]D(x) +\frac{i}{4\pi}\partial_m\partial_n\frac{x_0}{|\,\vec{x}\,|} -ia\partial_m\partial_n\left[\frac{x_0{}^3}{3!}\delta^{(3)}(\vec{x})\right]\ . \tag{96}$$

The invariance of S and Eq. (94) will assure Eq. (91) also for $a=0$.

VI. RELATIONS BETWEEN DIFFERENT GAUGES

In the previous section, we already discussed the relation between gauges characterized by different values of the parameters a. Here, we will discuss how gauges of different types are related. In particular, we will try to relate the three gauges characterized by $a=b=c=0$ and, respectively, $f=g=0$, $h=1$, $f=h=0$, $g=-1$, $g=h=0$, and $f=1$, by a gauge transformation

$$A'_\mu=A_\mu+\partial_\mu X\ . \tag{97}$$

These three gauges give rise to the same propagator.

Before discussing this point, we will show that the Fermi gauge and the $\partial_0 A_0=0$ gauge cannot be related by a transformation (97). Let primed quantities be those of the $\partial_0 A_0$ gauge while unprimed objects are those of the Fermi gauge. According to Eqs. (79) and (88),

$$-n_\nu(n\cdot\partial)S'=\partial_\nu S\ . \tag{98}$$

This implies

$$(n\cdot\partial)^2 S' = -\Box S \tag{99}$$

which are both equal to zero but also

$$-n\cdot\partial S' = n\cdot\partial S \tag{100}$$

and

$$(n\cdot\partial)^2 S = 0\ . \tag{101}$$

Equation (101) is clearly incompatible with a nonvanishing Poincaré-invariant S.

The same result can also be obtained from proposition 2.2 of Strocchi and Wightman.[16] Equation (70) does not take the particular expression needed for a relation by Eq. (97) between this gauge and the Fermi gauge. Let us also remark here that the temporal gauge differs from the Evans-Fulton[17] gauge although the propagators are the same.

Let us now relate the gauge $\partial_0 A_0 = 0$ to the temporal gauge $A_0 = 0$. In this last gauge, field equations are

$$\partial^\mu F^0_{\mu\nu} + n_\nu S^0 = 0\ , \tag{102}$$

$$n\cdot A^0 = 0 \tag{103}$$

while commutation relations for any time are

$$[A^0_\mu(x), A^0_0(0)] = [S^0(x), S^0(0)] = 0\ , \tag{104}$$

$$[A^0_k(x), A^0_l(0)] = -i\left[g_{kl} + \frac{\partial_k\partial_l}{\Delta}\right] D(x) + \frac{i}{4\pi}\partial_k\partial_l \frac{x_0}{|\vec{x}|}\ , \tag{105}$$

$$[S^0(x), A^0_\mu(0)] = -i\partial_\mu \delta^{(3)}(\vec{x})\ . \tag{106}$$

Corresponding quantities in the $\partial_0 A_0 = 0$ gauge are given by taking $a = 0$ in Eqs. (88), (89), and (93) to (96).

The transformation

$$A^0_\mu = A_\mu + \partial_\mu X\ , \tag{107}$$

$$S^0 = -n\cdot\partial S' \tag{108}$$

with

$$X(x) = -\tfrac{1}{2}\int d^4y\, \epsilon(x_0 - y_0)\delta^{(3)}(\vec{x} - \vec{y})\, n\cdot A(y) \tag{109}$$

allows us to go from the $\partial_0 A_0 = 0$ to the temporal gauge. Such a transformation is however not unique since we can add to X defined by Eq. (109) a term $\alpha n\cdot\partial S$ and get the correct field equations and commutation relations. The inverse transformation is also not unique.

In the same way, the field equations and commutation relations from the Lagrangian (6) with $a = b = c = f = h = 0$, $g = -1$, are

$$\partial_\mu F'_{\mu\nu} - n_\nu \Box S' = 0\ , \tag{110}$$

$$\Box n\cdot A' = 0\ , \tag{111}$$

$$[A'_\mu(x), A'_0(0)] = [S'(x), S'(0)] = 0\ , \tag{112}$$

$$[A'_\mu(0), S'(0)] = i\partial_\mu \int_0^{x_0} D(\vec{x}, t')dt'\ , \tag{113}$$

$$[A'_k(x), A'_l(0)] = -i\left[g_{kl} + \frac{\partial_k\partial_l}{\Delta}\right] D(x) + \frac{i}{4\pi}\partial_k\partial_l \frac{x_0}{|\vec{x}|}\ . \tag{114}$$

We can go from this gauge to the $\partial_0 A_0 = 0$ gauge by a transformation

$$A_\mu = A'_\mu + \partial_\mu X\ , \tag{115}$$

$$n\cdot\partial S = \Box S' \tag{116}$$

with

$$(n\cdot\partial)^2 X = -n\cdot\partial n\cdot A'\ . \tag{117}$$

Equation (117) again does not possess a unique solution since, with

$$X(x) = -\tfrac{1}{2}\int d^4y\, \epsilon(x_0 - y_0)\delta^{(3)}(\vec{x} - \vec{y})\, n\cdot A'(y) + \alpha \Box S'(y)\ , \tag{118}$$

we can recover the correct field equations and commutation relations of the $\partial_0 A_0 = 0$ gauge for any value of α.

Nonuniqueness of the transformation between these different gauges is related to the fact that different numbers of degrees of freedom are involved. Moreover, none of these gauge conditions fixes univocally a representative for the system.

VII. CONCLUSIONS

We have discussed, from the point of view of free-field theory, many properties of inhomogeneous axial gauges which are free of Faddeev-Popov ghosts in non-Abelian Yang-Mills theory. These gauges offer a large choice of different kinds of theories, from a theory with only 2 degrees of freedom to a theory with 5 independent degrees of freedom. For spacelike n, 2 or 4 degrees of freedom are requested depending on the absence or the presence of a time derivative in the coupling of the gauge condition with the Lagrange multiplier. The case with 4 degrees of freedom is a nonlocal field theory affected with an indefinite metric and has not been studied further. This singularity may be at the origin of some difficulties with these gauges. The timelike case is always a local theory with 3, 4, or 5 degrees of freedom. Theories with 4 or 5 degrees of freedom are indefinite-metric theories for which we have developed a general formulation whose Gupta-Bleuler formalism is a particular case adapted to the Fermi gauge. There is no ambiguity in fixing the propagator singularities in these timelike gauges. We have also shown that a sometimes used transformation between the Fermi and the timelike general axial gauge does not exist while the relation between the different types of inhomogeneous axial gauges is not univocally fixed. These conclusions are not affected by the presence of an interaction.

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