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Transverse vertex and gauge technique in quantum electrodynamics

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It is known that the Schwinger-Dyson equation for the electron propagator $S(p)$ in quantum electrodynamics is linear if the full vertex in this equation is approximated by a special form (the longitudinal vertex) which satisfies the Ward identity and which yields exact results in the infrared regime. However, the approximate equation cannot be multiplicatively renormalized (nor is it properly gauge covariant) using only the longitudinal vertex. In the present work, we construct a transverse (i.e., identically conserved) vertex which, when added to the longitudinal vertex, yields an equation for $S(p)$ which remains linear and exact in the infrared, but which is multiplicatively renormalizable and gauge covariant. In the ultraviolet regime, the equation gives the known results of renormalization-group-improved perturbation theory. The essential difficulty which is overcome by the present analysis is that of overlapping divergences, which are mishandled if only the longitudinal vertex is kept.

I. INTRODUCTION

It is no news that the main difficulties in using Schwinger-Dyson (SD) equations are their nonlinearity and lack of an adequate scheme for truncating the infinite set of equations. Remarkably, in gauge theories there is an approximation to the fermion—gauge-boson vertex which is exact in the infrared and which removes both the above hindrances at one stroke for the fermion propagator.[1] The approximation method, called the gauge technique, expresses part of the vertex in terms of the fermion propagator in such a way that the Ward identity is exactly satisfied [see Eqs. (1)—(3) below]. This part of the vertex, called the longitudinal vertex, is exact in the infrared limit of vanishing gauge-boson momentum. The remaining part—the transverse vertex—is identically conserved and must therefore vanish at least linearly in the gauge-boson momentum when this quantity is small. When only the longitudinal vertex is used in the SD equation for the fermion propagator, this equation becomes both linear and self-contained.[2]

The main problem with the gauge technique is that, while correctly yielding the infrared behavior of the propagator, the ultraviolet behavior is incorrect, and, in fact, the gauge-technique SD equation is unrenormalizable due to the improper handling of overlapping divergences. Moreover, the propagator does not have the correct gauge-transformation properties in the ultraviolet regime,[3,4] a problem closely related to the lack of renormalizability.

In the present paper, we propose a solution to these problems for quantum electrodynamics (QED) by constructing an approximate transverse vertex to all orders of perturbation theory. A similar treatment for non-Abelian gauge theories, such as quantum chromodynamics, will be given elsewhere. We study QED in the absence of fermion loops and are concerned with only the leading logarithmic effects in the IR and UV regions. Within the context of our approximation, we shall show that the transverse vertex can be represented by a simple form which is generalized to all orders in perturbation theory, that when this transverse vertex is included in the kernel of the SD equation, the SD equation is still linear, and that this effective kernel obeys a simple power law. Renormalization is demonstrated such that all the overlapping divergences are removed, leaving a cutoff-independent equation whose ultraviolet behavior is that of renormalization-group-improved (RGI) perturbation theory. The mass term for the full electron propagator yields two solutions, both with the bare mass vanishing in the limit of infinite cutoff. The first describes chiral fermions with no mass term, and the second yields a mass term, showing chiral-symmetry breaking in the Baker-Johnson-Willey sense.[5]

We now possess a propagator which is exact in the infrared (i.e., near the mass shell) and behaves as expected for the ultraviolet regime in QED with a fixed point. This does not guarantee that the approximate propagator, which solves the linear SD equation, is close to the exact propagator in the intermediate-energy regime. Nevertheless it is reasonable to hope that a smooth joining of exact ultraviolet and infrared results—which is what the linear SD equation accomplishes—will not depart too far from the truth at middle energies. What we claim to have done here is produce a *linear* SD equation which treats correctly both momentum regimes where the characteristic QED expansion parameter $\alpha\ln(1-p^2/m^2)$ (p is the fermion momentum, m its mass) is large; where it is not large, our equation can be systematically improved by adding a few terms based on comparison of perturbation theory to the linear equation.

II. THE TRANSVERSE VERTEX AND SELF-ENERGY KERNEL

In the gauge technique, one writes down a spectral representation for the full fermion propagator and the full vertex function using the same spectral density for each, such that the vertex satisfies the Ward identity. That is, if[6]

$$S(p)=\int dw\frac{\rho(w)}{\not p-w+i\epsilon(w)\eta}\ , \tag{1}$$

then (from now on our $i\eta$ prescription is understood)

$$S(p)\Gamma_\mu^L(p,p')S(p')\equiv\int dw\,\rho(w)\frac{1}{\not p-w}\gamma_\mu\frac{1}{\not p'-w} \tag{2}$$

satisfies the QED Ward identity

$$(p-p')^\mu\Gamma_\mu^L(p,p')=S^{-1}(p)-S^{-1}(p')\ . \tag{3}$$

Unfortunately, this approach specifies only the longitudinal part of the vertex, leaving the transverse part unknown. As we have noted, the transverse vertex must vanish like $q=p-p'$ near the fermion mass shell ($\not p=\not p'=m$), thus yielding no leading contributions to the IR behavior of the propagator. This is because a conserved vertex has the kinematical structure $i\sigma_{\mu\nu}q^\nu$ or $q_\mu p\cdot q-p_\mu q^2$, etc., and there are no massless particle poles to eliminate the powers of q. However, this power of q is important in the UV, and understanding the nature of the transverse vertex is the key to resolving the problem of the overlapping divergences and the renormalization of the SD equation.

When Eqs. (1) and (2) are used in the Schwinger-Dyson equation for the renormalized fermion propagator, and the photon propagator is given by

$$D_{\mu\nu}=\left[g_{\mu\nu}-\xi\frac{k_\mu k_\nu}{k^2}\right]\frac{1}{k^2}\ ,$$

the SD equation takes the linear form

$$(\not p-m_0)S(p)=Z_2{}^{-1}+\int dw\,\rho(w)\frac{1}{\not p-w}\Sigma_L(p,w)\ . \tag{4}$$

Here $\Sigma_L(p,w)$ is the one-loop fermion self-energy for a fermion of mass w and is independent of ρ. The subscript L reminds us that only the longitudinal vertex has been used in the SD equation. Ignoring the nonlogarithmic contributions,

$$\begin{aligned}\Sigma_L(p,w)&=\frac{\alpha}{4\pi}\int_0^1 d\beta[(4-\xi)w-2(1-\xi)(1-\beta)\not p]\ln\frac{\Lambda^2}{w^2-(1-\beta)p^2}\\&=(4-\xi)\frac{\alpha}{4\pi}w\left[\ln\frac{\Lambda^2}{w^2}+\frac{w^2-p^2}{p^2}\ln\frac{w^2-p^2}{w^2}\right]\\&\quad-(1-\xi)\frac{\alpha}{4\pi}\not p\left[\ln\frac{\Lambda^2}{w^2}+\frac{w^4-p^4}{p^4}\ln\frac{w^2-p^2}{w^2}\right]\quad(\Lambda\equiv\text{UV cutoff})\ .\end{aligned} \tag{5}$$

The w integrations in (4) are such that the IR ($\not p\simeq m$) support comes from the region $w\simeq m$. Thus in the infrared we write $\not p\simeq w$, $p^2-w^2\simeq(\not p-w)(\not p+w)\simeq 2w(\not p-w)$, and Σ_L behaves as

$$\begin{aligned}\Sigma_L(p,w)&\simeq\frac{3\alpha w}{4\pi}\ln\frac{\Lambda^2}{w^2}+\xi\frac{\alpha}{4\pi}(\not p-w)\ln\frac{\Lambda^2}{w^2}\\&\quad-(2+\xi)\frac{\alpha}{2\pi}(\not p-w)\ln\frac{w^2-p^2}{w^2}\ .\end{aligned} \tag{6}$$

When UV problems are ignored, Σ_L as given in (5) or (6) yields the correct exact infrared behavior at the propagator. To see this, remove the pole term in the spectral density by writing

$$\rho(w)=\delta(w-m)+\sigma(w)\ . \tag{7}$$

Then, using Σ_L as given by (6) in (4), carrying out a straightforward mass renormalization using the lowest-order perturbative result $m_0=m$

$-(3\alpha m/4\pi)\ln\Lambda^2/m^2$, and taking the imaginary part of the resulting SD equation, we arrive at

$$(w-m)\epsilon(w)\sigma(w)=-(2+\xi)\frac{\alpha}{2\pi}\theta(w^2-m^2) -\frac{3\alpha}{4\pi}m\epsilon(w)\sigma(w)\ln w^2/m^2 -(2+\xi)\frac{\alpha}{2\pi}\int d\omega'\sigma(w') \times\theta(w^2-w'^2)\ . \tag{8}$$

The second term on the right-hand side of (8) is an ultraviolet effect (it diverges for large w^2 and vanishes as $w^2 \rightarrow m^2$). If we omit this term, then (8) yields the solution

$$\sigma(w)=-(2+\xi)\frac{\alpha}{2\pi}\frac{\epsilon(w)}{w-m} \times\left[\frac{w^2-m^2}{w^2}\right]^{-(2+\xi)\alpha/2\pi}\theta(w^2-m^2)\ . \tag{9}$$

[Compare this to (28b) and the resultant propagator given by (31b).]

The infrared behavior predicted in (9) is correct.[7] If one attempted to include the UV contributions in (5) or (8), the SD equation would yield the correct $O(\alpha)$ propagator, but any attempt to go beyond this in a self-consistent way would yield a cutoff-dependent propagator, due to the appearance of overlap logarithms of the form

$$\alpha^2\ln\frac{\Lambda^2}{m^2}\ln\frac{m^2-p^2}{m^2}+O(\alpha^3)\ ,$$

which are not removable by multiplicative renormalization (i.e., Z_2^{-1} would necessarily be momentum-dependent). Using only the longitudinal vertex in the SD equation leads to a mishandling of overlapping divergences.

Recognizing that the vertex given in (2) is incomplete, we proceed to improve the SD equation such that it also yields the correct RGI UV behavior. We will construct a transverse self-energy kernel $\Sigma_T(p,w)$ ("transverse" since it is due to the transverse vertex) such that, when the full self-energy kernel $\Sigma\equiv\Sigma_L+\Sigma_T$ is used in place of Σ_L in Eq. (4), the resulting SD equation is correctly renormalized while remaining linear in ρ. (Σ is shown schematically in Fig. 1, where the photon line is a *free* covariant propagator, and the fermion line is a *free* propagator of mass w.)

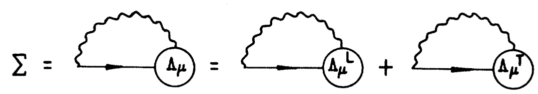

FIG. 1. Diagrammatic representation of the self-energy kernel Σ of the linear Schwinger-Dyson equation, in terms of the longitudinal and transverse vertex functions.

To show this we resort to perturbation theory in powers of α. Transverse effects first appear in the fourth-order $[O(\alpha^2)]$ self-energy, or in the second-order vertex. Define the second-order transverse vertex as

$$\Lambda_\mu^{T(2)}(p,p')=\Lambda_\mu^{(2)}(p,p')-\Lambda_\mu^{L(2)}(p,p')\ . \tag{10}$$

$\Lambda_\mu^{(2)}$ is the renormalized second-order Feynman vertex and $\Lambda_\mu^{L(2)}$ is the second-order longitudinal vertex as given by the gauge technique. The kinematical structure of Λ_μ^T is such that it must vanish at least as fast as $q=p-p'$ when $q\rightarrow 0$ (e.g., $\Lambda_\mu^T\sim\sigma_{\mu\nu}q^\nu$). All leading infrared effects are contained in the longitudinal vertex. To avoid the possibility of spurious kinematical singularities in the transverse vertex as $q\rightarrow 0$, we regulate Λ_μ^T as given by (10) with the mass-shell boundary condition

$$\Lambda_\mu^T(m,m)=0,\quad w=\not p=m\ , \tag{10a}$$

which is consistent with $Z_1=Z_2$ when the vertex and propagators are renormalized on-shell.

We have calculated $\Lambda_\mu^{T(2)}$ as defined in Eqs. (10) and (10a), and discuss details in the Appendix. We should also mention at this point that recently Ball and Chiu[8] have calculated the second-order longitudinal and transverse vertex in QED. They use a similar ansatz of solving the Ward identity to define a longitudinal vertex, though our $\Lambda_\mu^{L(2)}$ differs from theirs by a transverse piece. They present the transverse vertex in much more detail than we do (in our appendix), breaking it down to one common scalar Feynman integral and writing it in terms of manifestly conserved forms. However, our interest in the vertex is motivated by the linear SD equation and we are more concerned with the UV and IR limits with an eye towards generalization to all orders, so we give as an alternative an *effective* transverse vertex which retains two important properties of the true transverse vertex: the correct UV behavior and the infrared condition (10a). Our effective transverse vertex is given by $(q=p-p')$

$$\Lambda_\mu^{T(2)}(p,p')=\tfrac{1}{3}\{(1-\xi)(\not p'\gamma_\mu \not p-\not p\gamma_\mu \not p')+\tfrac{1}{2}(4-\xi)w[\not q,\gamma_\mu]\}K^{(2)}(p,p')\,,$$
$$K^{(2)}(p,p')=2\left[\frac{3\alpha}{4\pi}\right]\int\frac{d\alpha_1 d\alpha_2 d\alpha_3\delta(1-\sum\alpha_i)(1-\alpha_1)}{(1-\alpha_1)w^2-\alpha_1\alpha_2 p^2-\alpha_1\alpha_3 p'^2-\alpha_2\alpha_3 q^2}\,. \tag{11}$$

Far off-shell, $-p_>{}^2 \gg -p_<{}^2 \gg m^2$, $w^2 \gg m^2$, where $p_> = p$ if $|p^2| > |p'^2|$, and vice versa, we find

$$K^{(2)}(p,p')\simeq\frac{3\alpha}{4\pi}(-p_>)^{-2}\ln[(-p_>{}^2)/(-p_<{}^2)]\,. \tag{12a}$$

Note that for the $p=p'$, $q=0$, the term in curly brackets in (11) vanishes while $K^{(2)}(p,p')$ is regular on-shell: $p^2=w^2=m^2$. There is thus no infrared logarithm in (11). [However, this effective transverse vertex yields an incorrect gauge-dependent anomalous magnetic moment. It is straightforward to modify the coefficient of $K^{(2)}(p,p')$ in (11) such that our effective transverse vertex is correct in the UV *and* reproduces the correct anomalous magnetic moment. The resultant function is modestly more unwieldy than (11), and, for our purposes, unnecessary.] Calculating $\Sigma_T^{(4)}(p,w)$, we find

$$\Sigma_T^{(4)}(p,w)\equiv ie^2\int\frac{d^4q}{(2\pi)^4}\frac{(-g^{\mu\nu}+\xi q^\mu q^\nu/q^2)}{q^2}\times\Lambda_\mu^{T(2)}(p,p-q)\frac{1}{\not p-\not q-w}\gamma_\nu$$
$$\simeq-\frac{3}{2}\left[\frac{\alpha}{4\pi}\right]^2[(4-\xi)w-(1-\xi)\not p]\times\ln^2\frac{\Lambda^2}{w^2-p^2}\,, \tag{13}$$

for p^2 or $w^2 \gg m^2$. This makes explicit the leading logarithms in the ultraviolet coming from $\Lambda_{\mu T}^{(2)}$. Near-shell, $\Sigma_T^{(4)}(p,w)$ is

$$O\left[\alpha^2\ln\frac{w^2-p^2}{w^2}\right],$$

which is nonleading. The leading ultraviolet logarithms contain an overlapping ($\sim\ln\Lambda^2\ln p^2$) divergence.

We have continued our study of Σ_T through sixth order. No straightforward definition of the fourth-order transverse vertex has been found that relates $\Lambda_\mu^{T(4)}$ to the full vertex and the gauge-technique longitudinal vertex, while preserving a linear SD equation. The straightforward generalization of (10)—that $\Lambda_\mu^{T(4)}$ is the difference between the full fourth-order vertex and $\Lambda_\mu^{L(4)}$—does not work. This is to be expected, though, since through the SD equation and the gauge technique, the fourth-order longitudinal vertex "remembers" the second-order transverse vertex, thus obscuring the definition of $\Lambda_\mu^{T(4)}$. That is, the sixth-order fermion propagator as given by the right-hand side of the SD Eq. (4) depends not only on the sixth-order self-energy kernel, but also on a convolution of the fourth-order spectral function ρ with the second order Σ_L. We have corrected $\Sigma_L^{(4)}$ by adding the transverse contribution $\Sigma_T^{(4)}$. Thus the fourth-order ρ already contains some transverse information.

These sixth-order calculations do tell us how to fix up (4) such that it does renormalize correctly. This, in turn, gives us a clue as to how to construct the transverse vertex. We find, with $K^{(2)}(p,p')$ as given in Eq. (6), that a useful fourth-order transverse vertex $\Lambda_\mu^{T(4)}$ is found by substituting $K^{(4)}$ in the right-hand side of (11), where

$$K^{(4)}(p,p')\simeq-2\left[\frac{3\alpha}{4\pi}\right]\int d\beta_1 d\beta_2 d\beta_3\times\delta(1-\Sigma\beta_i)(1-\beta_1)\times K^{(2)}(\sqrt{\beta_2}p,\sqrt{\beta_3}p')\,. \tag{14}$$

This form is valid in the UV, and as $K^{(2)}$ has no leading IR singularities, IR difficulties generated by (14) will be less than leading. In general,

$$K^{(2n+2)}(p,p')=(-1)^n 2\left[\frac{3\alpha}{4\pi}\right]\int d\beta_1 d\beta_2 d\beta_3\delta(1-\Sigma\beta_i)(1-\beta_1)\times K^{(2n)}(\sqrt{\beta_2}p,\sqrt{\beta_3}p')\,. \tag{15}$$

Of course, there is no reason to expect the kinematical form of $\Lambda_\mu^{T(2)}(p,p')$ to be retained order by order in perturbation theory, but this suggests *defining* a full transverse vertex $\Lambda_\mu^T(p,p')$ with the same kinematical structure as $\Lambda_\mu^{T(2)}(p,p')$, and with $K(p,p')$ satisfying the integral equation

$$K(p,p')=K^{(2)}(p,p')-2\left[\frac{3\alpha}{4\pi}\right]\times\int d\beta_1 d\beta_2 d\beta_3 \delta(1-\Sigma\beta_i)(1-\beta_1)\times K(\sqrt{\beta_2}p,\sqrt{\beta_3}p')\ , \quad (16)$$

where $K^{(2)}(p,p')$ is given by Eq. (12). The remainder, $\Lambda_\mu-\Lambda_\mu^L-\Lambda_\mu^T$, is less than leading in both the IR and UV. Then, far off-shell, $-p'^2 \gg -p^2 \gg w^2$, (16) can be solved by iteration beginning with the asymptotic form (12a) to yield

$$K(p,p')\simeq\frac{1}{3}\frac{1}{p'^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left[\frac{3\alpha}{4\pi}\right]^n\times\{\ln[(-p'^2)/(-p^2)]\}^n\ . \quad (17)$$

With Λ_μ^T as defined above, we calculate $\Sigma_T(p,w)$ in the UV ($-p^2 \gg w^2$), using p' as the integration momentum:

$$\begin{aligned}\Sigma_T(p,w)&\equiv ie^2\int\frac{d^4p'}{(2\pi)^4}D^{\mu\nu}(p-p')\Lambda_\mu^T(p,p')\frac{1}{\not p'-w}\gamma_\nu\\&\simeq-\tfrac{1}{3}[\not p(1-\xi)-w(4-\xi)]\int_{-p^2}^{\Lambda^2}\frac{dp'^2}{p'^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left[\frac{3\alpha}{4\pi}\right]^{n+1}\left[\ln\frac{p'^2}{-p^2}\right]^n\\&=\tfrac{1}{3}[\not p(1-\xi)-w(4-\xi)]\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}\left[\frac{3\alpha}{4\pi}\right]^{n+1}\left[\ln\frac{\Lambda^2}{-p^2}\right]^{n+1}\ .\end{aligned} \quad (18)$$

Here $D_{\mu\nu}$ is the free gauge-covariant photon propagator and we have rotated p' to Euclidean space to do the angular integrations. Divergences have been regulated by the introduction of a momentum cutoff Λ and only leading logarithms have been kept.

When $\Sigma_T(p,w)$, as given by Eq. (18), is added to $\Sigma_L(p,w)$, as given in Eq. (5), and the resulting combination used as the self-energy kernel of the linear SD equation (4), the leading logarithmic contributions to $S(p)$ are renormalized by Z_2^{-1} and the resultant terms yield the leading UV behavior of $S(p)$ without disturbing the already correct IR behavior.

We have extended our analysis in the UV by looking for a form of Σ_T which has the UV behavior given in (18), but also *exactly* renormalizes our improved SD equation. This is possible because, with renormalizability as a constraint, the UV form of the nth order SD kernel Σ is dictated by the form of Σ and ρ at orders less than n. For example, consider the following expansion in powers of α. We write

$$\begin{aligned}&\rho=\rho^{(0)}+\rho^{(2)}+\rho^{(4)}+\rho^{(6)}+\cdots\ ,\\&\Sigma=\Sigma^{(2)}+\Sigma^{(4)}+\Sigma^{(6)}+\cdots\ ,\\&m_0=m-\delta m^{(2)}-\delta m^{(4)}-\delta m^{(6)}-\cdots\ ,\end{aligned} \quad (19)$$

where the superscript denotes the order of the expansion, e.g., $\rho^{(6)}$ is $O(e^6)$ or $O(\alpha^3)$. Of course, $\rho^{(0)}$ is just the pole term $\delta(w-m)$. Collecting fourth-order contribution to the SD equation, we find

$$\begin{aligned}(\not p-m)\int\frac{dw\rho^{(4)}(w)}{\not p-w}+\delta m^{(2)}\int\frac{dw\rho^{(2)}(w)}{\not p-w}+\frac{\delta m^{(4)}}{\not p-m}&=Z_2{}^{-1(4)}+\int dw\,\rho^{(2)}(w)\frac{1}{\not p-w}\Sigma^{(2)}(p,w)\\&\quad+\frac{1}{\not p-m}\Sigma^{(4)}(p,m)\ .\end{aligned} \quad (20)$$

We know $\Sigma^{(2)}(p,w)\equiv\Sigma_L(p,w)$ as given in Eq. (5) exactly. Σ_L contains not only the leading logarithm $\alpha\ln(\Lambda^2/-p^2)$, but also leading logarithms reduced by powers of w^2/p^2. Since we can solve for the second-order propagator exactly, we know $\rho^{(2)}$ exactly. Thus we can do all the integrals involving $\rho^{(2)}$ in (20), which will result in cutoff-dependent terms of the overlap type. The cutoff dependence must be removed by the mass and wave-function counterterms in a manner consistent with multiplicative renormalization, and hence a minimal form of $\Sigma^{(4)}$ is specified to remove overlapping divergences. We have used this procedure through sixth order in the self-energy kernel and find that this renormalization is essentially effected by the replacement of $[\ln(\Lambda^2/-p^2)]^n$ in Eq. (18) with

$$\int_0^1 d\beta\left[\ln\frac{\Lambda^2}{w^2-(1-\beta)p^2}\right]^n$$

(up to polynomials in β in the integrand). Note that, with this form, the logarithms in Σ_T have the same branch structure as those in Σ_L. Then, with the definition

$$\Sigma(p,w)=\Sigma_L(p,w)+\Sigma_T(p,w)$$
$$\equiv \not{p}A(p^2,w^2)+wB(p^2,w^2)\ , \qquad (21)$$

A and B are given by

$$A(p^2,w^2) = -\tfrac{2}{3}(1-\xi) \times \int_0^1 d\beta(1-\beta) \times \left[1-\left[\frac{\Lambda^2}{w^2-(1-\beta)p^2}\right]^{-3\alpha/4\pi}\right], \qquad (21a)$$

$$B(p^2,w^2) = \tfrac{1}{3}(4-\xi) \times \int_0^1 d\beta\left[1-\left[\frac{\Lambda^2}{w^2-(1-\beta)p^2}\right]^{-3\alpha/4\pi}\right],$$

plus nonleading logarithms as below (13), for $-p^2$ or $w^2 \gg m^2$. Notice that the leading term in Σ is exactly Σ_L as given in Eq. (5). For $p^2 \simeq w^2 \simeq m^2$, these reduce to

$$A(p^2,w^2)=-2(1-\xi)\frac{\alpha}{4\pi} \times \int_0^1 d\beta(1-\beta)\ln\frac{\Lambda^2}{w^2-(1-\beta)p^2}\ , \qquad (21b)$$

$$B(p^2,w^2)=(4-\xi)\frac{\alpha}{4\pi}\int_0^1 d\beta \ln\frac{\Lambda^2}{w^2-(1-\beta)p^2}\ .$$

That is, near-shell, $\Sigma(p,w)\to\Sigma_L(p,w)$ [see Eq. (5)]. Note, incidentally, that the familiar power $3\alpha/4\pi$ appearing in these equations is gauge- (that is, ξ-) independent.

Equations (21) are the main result of this paper. When used in the linear SD equation, (21a) yields an improved UV behavior over our previous leading logarithm result given in Eq. (18). In the following two sections we use $\Sigma(p,w)$ as defined in Eqs. (21) in the SD equation, demonstrate renormalization, and then give the resulting solutions for the spectral function $\rho(w)$ and the propagator $S(p)$.

III. RENORMALIZATION

These forms (21a) and (21b) have been explicitly verified using perturbation theory through sixth order. However, a simple calculation shows renormalization to all orders.

When Σ as given in (11) is used in place of Σ_L in (4), it turns out that the SD equation is manifestly cutoff independent. To show this, separate the SD equation into Dirac even and odd parts:

$$\int dw \frac{\rho(w)}{p^2-w^2}(p^2-m_0 w) = Z_2^{-1} + \int dw \frac{\rho(w)}{p^2-w^2}[p^2 A(p^2,w^2) + w^2 B(p^2,w^2)]\ , \qquad (22a)$$

$$\int dw \frac{\rho(w)}{p^2-w^2}(w-m_0) = \int dw \frac{\rho(w)}{p^2-w^2} w[A(p^2,w^2)+B(p^2,w^2)]\ . \qquad (22b)$$

Consider (22b) first. When (21a) is used in (22b), we have

$$m_0 \int dw \frac{\rho(w)}{p^2-w^2} = \tfrac{1}{3}\int_0^1 d\beta[4-\xi-2(1-\xi)(1-\beta)] \int dw \frac{\rho(w)}{p^2-w^2} w \left[\frac{\Lambda^2}{w^2-(1-\beta)p^2}\right]^{-3\alpha/4\pi}\ . \qquad (23)$$

The spectral density ρ has the dimensions of inverse mass and must vanish like $1/w$ (up to powers of logarithms) in the UV. *Since all the w integrations are UV finite,* renormalization simply requires $(\Lambda^2)^{3\alpha/4\pi} m_0$ be finite as $\Lambda^2\to\infty$. (Note that the power $-3\alpha/4\pi$ is gauge independent.) This admits two solutions.

(1) If, as $\Lambda^2\to\infty$, m_0 vanishes faster than $(\Lambda^2)^{-3\alpha/4\pi}$, then we see the simple chiral solution

$$m_0=0,\ \ \rho(w)=\rho(-w)\ . \qquad (24)$$

From the form of the spectral function (1), we see that no mass part of the propagator exists, $m_0=0$, and the fermion remains massless.

(2) If $(\Lambda^2)^{3\alpha/4\pi} m_0$ is finite as $\Lambda^2\to\infty$, we see the realization of Baker-Johnson-Willey QED.[5] To avoid IR problems, we choose a Euclidean renormalization point $p^2=-\mu^2$ such that

$$m_0\left[\frac{\Lambda^2}{\mu^2}\right]^{3\alpha/4\pi} = m \equiv \text{physical mass}$$

as $\Lambda^2\to\infty$. Then (23) becomes

$$m\int dw\frac{\rho(w)}{p^2-w^2} =\frac{1}{3}\int_0^1 d\beta[4-\xi-2(1-\xi)(1-\beta)] \times\int dw\frac{\rho(w)}{p^2-w^2}w\left[\frac{w^2-(1-\beta)p^2}{\mu^2}\right]^{3\alpha/4\pi} . \tag{25}$$

Thus, although m_0 vanishes as $\Lambda^2\to\infty$, (23) admits a solution such that the renormalized fermion propagator has a mass term. This simple renormalization of the mass term happens because this term is really a scalar vertex, and vertices have no overlapping-divergent skeleton graphs.

Renormalization of (22a) is not as obvious due to the presence of overlapping divergences. The w integral on the right-hand side of (22a) also diverges for large w. However, using (22b) in (22a) and subtracting at $p^2=-\mu^2$ subject to the renormalization condition that $S^{-1}\simeq\not p-m$ for $p^2=-\mu^2$, namely

$$(p^2-m^2)\int dw\frac{\rho(w)}{p^2-w^2}\bigg|_{p^2=-\mu^2}=1 , \tag{26}$$

yields

$$m_0\int dw\frac{\rho(w)}{p^2-w^2}w-\frac{m_0m}{\mu^2+m^2} =-\frac{1}{3}\int_0^1 d\beta[4-\xi-2(1-\xi)(1-\beta)] \times\int dw\rho(w)\left[\frac{p^2}{p^2-w^2}\left[\frac{\Lambda^2}{w^2-(1-\beta)p^2}\right]^{-3\alpha/4\pi}-\frac{\mu^2}{\mu^2+w^2}\left[\frac{\Lambda^2}{w^2+(1-\beta)\mu^2}\right]^{-3\alpha/4\pi}\right] +\frac{1}{3}(4-\xi)\int_0^1 d\beta\int dw\,\rho(w)\left[\left[\frac{\Lambda^2}{w^2-(1-\beta)p^2}\right]^{-3\alpha/4\pi}-\left[\frac{\Lambda^2}{w^2+(1-\beta)\mu^2}\right]^{-3\alpha/4\pi}\right] . \tag{27}$$

This is now a homogeneous equation and the cutoff may be removed by our definition of m_0. Since $\rho(w)$ must vanish like $1/w$ for large w, all the integrations are UV finite, thus demonstrating the removal of overlapping divergences, and hence, the renormalization of the SD equation.

IV. SOLUTIONS

Equations (25) and (27) can be solved for the spectral density ρ to leading logarithmic accuracy, for $w^2\gg m^2$ and $w\simeq m$. [Recall the discussion following Eq. (21) concerning the UV and IR limits of the self-energy kernel.] Perhaps the simplest method of solution is to separate the pole term from ρ as before, defining $\rho(w)\equiv\delta(w-m)+\sigma(w)$, and then to recognize that σ can be further separated into two parts, σ_e and σ_o, respectively even and odd in w. Then, after taking the imaginary parts of (25) and (27) and expanding everything in a perturbation series in α, (25) and (27) become essentially a pair of simultaneous algebraic equations for σ_e and σ_o, which can be solved iteratively. Alternatively, instead of making a perturbative expansion, one can leave the resummed structure intact, in which case (25) and (27) can be converted into a pair of coupled integrodifferential equations.

We find that the part of σ which contributes to the UV propagator is given by

$$\sigma(w)\simeq(1-\xi)\frac{\alpha}{4\pi}\frac{1}{|w|}\left[\frac{w^2}{m^2}\right]^{(1-\xi)\alpha/4\pi}\theta(w^2-m^2) -(2+\xi)\frac{\alpha}{4\pi}m\frac{\epsilon(w)}{w^2}\left[\frac{w^2}{m^2}\right]^{-(2+\xi)\alpha/4\pi} \times\theta(w^2-m^2),\quad w^2\gg m^2 . \tag{28a}$$

That part of σ which contributes to the IR propagator is slightly more complicated, as we need to introduce an IR regulator mass to the w integrations,[9] or to define the w integrations by subtraction. With the latter approach we find

$$\sigma(w)\simeq-(2+\xi)\frac{\alpha}{2\pi}\frac{\epsilon(w)}{w-m} \times\left[\frac{w^2-m^2}{m^2}\right]^{-(2+\xi)\alpha/2\pi}\theta(w^2-m^2),\quad w\simeq m \tag{28b}$$

up to subtraction terms.

Exact solutions for intermediate energies are not available.

These solutions are predicted on the following forms of the wave-function renormalization coefficient $Z_2{}^{-1}$ and the bare mass m_0:

$$Z_2^{-1}=\left[\frac{\Lambda^2}{m^2}\right]^{(1-\xi)\alpha/4\pi}, \tag{29a}$$

$$m_0=m\left[\frac{\Lambda^2}{m^2}\right]^{-3\alpha/4\pi}. \tag{29b}$$

The inclusion of the transverse vertex has yielded the gauge-covariant form of Z_2^{-1} as given above. Since the fermion propagator is multiplicatively renormalized by Z_2^{-1}, the renormalized propagator will have the correct gauge-transformation properties[3] (at least to leading logarithmic accuracy). That is, if $S(x,\xi')$ is the renormalized fermion propagator in coordinate space calculated in the gauge ξ', then the coordinate-space propagator in another gauge ξ is given by

$$S(x,\xi)=(\Lambda^2x^2)^{-(\xi-\xi')\alpha/4\pi}S(x,\xi'), \tag{30}$$

where Λ^2 is the UV cutoff. This requires a redefinition of Z_2 to remove the cutoff dependence of the gauge-transformed propagator, which is consistent with (29a).

Finally, using $\rho(w)$ from Eqs. (28) to calculate the fermion propagator gives the standard[10] results

$$S(p)\simeq\frac{(-p^2/m^2)^{(1-\xi)\alpha/4\pi}}{p^2-m^2}\times\left[\not p+m\left[-\frac{p^2}{m^2}\right]^{-3\alpha/4\pi}\right], \quad -p^2\gg m^2, \tag{31a}$$

$$S(p)\simeq\frac{1}{\not p-m}\left|1-\frac{p^2}{m^2}\right|^{(2+\xi)\alpha/2\pi}, \quad p^2\simeq m^2. \tag{31b}$$

The ultraviolet results are those of RGI perturbation theory for QED with a fixed point.

V. CONCLUSION

We have shown that, in spite of the very complicated nature of QED over the full spectrum of fermion energies, the Schwinger-Dyson equation can be cast in the form of a linear, inhomogeneous integral equation which contains the correct leading physical effects for IR and UV energies. The equation is such that all cutoff dependence can be removed prior to solution without resorting to an order-by-order expansion in α, and the resulting equations are finite and can, in principle, be solved nonperturbatively.

This work was motivated by the need to understand the SD equation and the transverse vertex in quantum chromodynamics. Results from the use of the gauge technique in QCD have already been presented,[11] and incorporation of the transverse effects will be given in a later work.

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APPENDIX

Working out the Dirac algebra in (1) and (2) for the longitudinal vertex yields

$$\Gamma_\mu^L(p,p')=\gamma_\mu-\frac{1}{p^2-p'^2}[\Sigma(p)(\not p\gamma_\mu+\gamma_\mu\not p')-(\not p\gamma_\mu+\gamma_\mu\not p')\Sigma(p')], \tag{A1}$$

where $S^{-1}(p)=\not p-m-\Sigma(p,m)$. With the usual calculation for the second-order Feynman vertex we find that the transverse vertex in (5) becomes, in the Feynman gauge ($\xi=0$),

$$\begin{aligned}\Lambda_\mu^{T(2)}(p,p')=-2\int d\alpha_1 d\alpha_2 d\alpha_3\delta(1-\Sigma\alpha_i)\Bigg[\gamma_\mu\left[\ln\frac{L''}{L}-\frac{\alpha_1}{1-\alpha_1}\ln\frac{L'}{L}\right]&\\ -\{[(1-\alpha_3)\not p'-\alpha_2\not p]\gamma_\mu[(1-\alpha_2)\not p-\alpha_3\not p']&\\ -2m[(p+p')_\mu-2(\alpha_2p+\alpha_3p')_\mu]+m^2\gamma_\mu\}\frac{1}{L''}&\\ +[\alpha_1^2m^2\gamma_\mu+\alpha_1^2\not p\gamma_\mu\not p'-2m\alpha_1(\not p\gamma_\mu+\gamma_\mu\not p')]\frac{1}{L'}&\\ +(1-\alpha_1^2)m^2\gamma_\mu\frac{1}{L}\Bigg],&\end{aligned} \tag{A2}$$

where

$$L'' = (1-\alpha_1)m^2 - \alpha_1\alpha_2 p^2 - \alpha_1\alpha_3 p^2 - \alpha_2\alpha_3 q^2 ,$$

$$L' = (1-\alpha_1)m^2 - \alpha_1\alpha_2 p^2 - \alpha_1\alpha_3 p'^2 ,$$

$$L = (1-\alpha_1)^2 m^2 .$$

We have used (10a) to define our subtractions. Straightforward calculations show $\Lambda_\mu^{(2)}(p,p')$ has the same UV behavior, given by Eqs. (11) and (12a), as the effective transverse vertex defined in (11). This transverse vertex (A2) yields, of course, the correct anomalous magnetic moment and other standard IR results. Our effective vertex (11), which differs from (A2), is not correct in the IR, but these effects are beyond our approximation in any case. The important thing to note is how the transverse vertex yields a leading logarithm in the UV but not in the IR.

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