One-loop renormalization of Coulomb-gauge QED

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In this article I present a physically motivated renormalization scheme for Coulomb-gauge QED. This scheme is useful in calculations involving QED bound states. I implement this scheme to one loop by calculating the electron self-energy, the electron self-mass, and the renormalization constants Z_1 and Z_2. Formulas for the dimensional regularization of some noncovariant integrals useful in one-loop Coulomb-gauge calculations are given.

I. INTRODUCTION

The gauge invariance of quantum electrodynamics allows one to choose a formulation of the theory to suit the problem at hand. For many problems the best formulation is in terms of a covariant gauge. Bound-state problems, however, are best handled in the Coulomb gauge.

Although the advantages of the Coulomb gauge for bound-state problems have been recognized for some time, the consistent use of the gauge has been frustrated by the lack of a renormalization scheme tailored to the Coulomb gauge. The typical response to ultraviolet-divergent graphs in a bound-state problem has been to gather the offending graphs (along with others as required) into a "gauge-invariant subset" of graphs, and to evaluate these graphs in a covariant gauge. However, since the constituent fermions in a QED bound state are not precisely on-shell, the gauge independence of the gauge-invariant subset is not exact, and corrections must be considered. Bound-state calculations in QED can be simplified by using the Coulomb gauge consistently throughout.

In this paper, I give a renormalization scheme that is convenient to use for bound-state problems in Coulomb-gauge QED. I define the scheme in Sec. II. In Sec. III, I present an explicit calculation of the one-loop electron self-energy function in a form that is useful for many applications. In Sec. IV, I use the self-energy function to find the one-loop electron self-mass and wave-function renormalization constant. In Sec. V, I find the vertex renormalization constant by examining the one-loop contribution to the vertex function. Finally, in the Appendix, I present formulas for the dimensional regularization of some noncovariant integrals that are encountered in one-loop calculations in the Coulomb gauge.

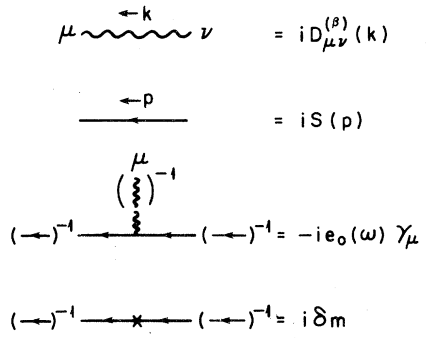

FIG. 1. The Feynman rules of QED.

II. A USEFUL RENORMALIZATION SCHEME

The (bare) QED Lagrangian in 2ω dimensions with a gauge-fixing term appropriate to the Coulomb gauge is[1]

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\beta}\{[\partial_\mu-(n\cdot\partial)n_\mu]A^\mu\}^2 +\bar{\psi}\{\gamma[i\partial-e_0(\omega)A]-m_0\}\psi\,, \tag{1}$$

where n is a timelike unit vector and β is the gauge parameter. The coupling $e_0(\omega)=e_0(\mu)^{2-\omega}$ is the product of the (dimensionless) bare coupling with a power of an arbitrary mass μ. In the following, μ will be identified with the physical electron mass m, where $m=m_0+\delta m$. The Feynman rules follow directly from the Lagrangian. They are pictured in Fig. 1. The photon propagator is

$$D^{(\beta)}_{\mu\nu}(k)=\frac{1}{k^2}\left[-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2-(nk)^2} -\frac{(nk)}{k^2-(nk)^2}(k_\mu n_\nu+n_\mu k_\nu)\right] -\beta\frac{k_\mu k_\nu}{[k^2-(nk)^2]^2}\,, \tag{2}$$

and the electron propagator is

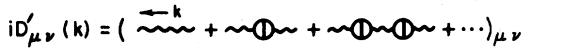

FIG. 2. Expansion of the full photon propagator in terms of the vacuum polarization tensor.

$$S(p)=\frac{1}{\gamma\cdot p-m}\ . \tag{3}$$

In the usual Coulomb gauge the gauge parameter is zero, and the photon propagator has the form

$$D_{\mu\nu}(k)=\begin{bmatrix}1/\vec{k}^2 & 0\\ 0 & \delta_{ij}^T/k^2\end{bmatrix} \tag{4}$$

in the frame where $n=(1,\vec{0})$, with $\delta_{ij}^T=\delta_{ij}-\hat{k}_i\hat{k}_j$.

Renormalization deals with the behavior of the full propagator and vertex function. These functions are defined as follows. The full photon propagator (times i) is the sum of all connected photon-photon graphs. The full photon propagator is pictured in Fig. 2 in terms of the vacuum polarization tensor

$$\Pi_{\mu\nu}(k)=(k^2g_{\mu\nu}-k_\mu k_\nu)\Pi(k^2)\ , \tag{5}$$

where the vacuum polarization tensor (times $-i$) is the sum of all one-photon-irreducible amputated photon-photon graphs. The graphical definition of $\Pi_{\mu\nu}$ is given in Fig. 3. The transversality of $\Pi_{\mu\nu}$ follows from a Ward identity.[2] The series indicated in Fig. 2 can be summed, with the result

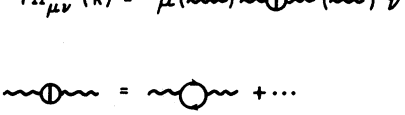

FIG. 3. Graphical definition of the vacuum polarization tensor.

$$iD'_{\mu\nu}(k)=\frac{i}{k^2[1+\Pi(k^2)]}\left[-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2-(nk)^2}-\frac{(nk)}{k^2-(nk)^2}(k_\mu n_\nu+n_\mu k_\nu)\right]\ . \tag{6}$$

The full electron propagator (times i) is the sum of all connected electron-electron graphs. The full-electron propagator is depicted graphically in Fig. 4. The graphs in Fig. 4 involve the electron self-energy, which (times $-i$) is the sum of all one-electron-irreducible amputated electron-electron graphs. The self-energy is defined graphically in Fig. 5. Summing the series for the full electron propagator, one finds

$$iS'(p)=\frac{i}{\gamma\cdot p-m-\Sigma(p)}\ . \tag{7}$$

The full vertex function is the sum of all one-photon-irreducible, one-electron-irreducible, amputated vertex graphs. These graphs are pictured in Fig. 6.

I assume that the renormalization of QED in the Coulomb gauge proceeds essentially as it does in the covariant gauges. Indeed, in writing the bare Lagrangian in the form (1), I assume that the counterterms generated in the Coulomb gauge have the usual covariant forms

$$(F_{\mu\nu}F^{\mu\nu},\bar{\psi}\gamma(i\partial)\psi,\bar{\psi}\gamma A\psi,\bar{\psi}\psi)\ .$$

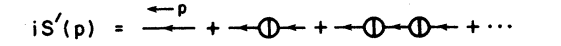

FIG. 4. Expansion of the full electron propagator in terms of the electron self-energy.

This assumption is justified by Heckathorn.[3] A set of finite renormalized propagation and vertex functions can be defined according to

$$D'_{\mu\nu}(k)=Z_3D^R_{\mu\nu}(k)\ , \tag{8a}$$

$$S'(p)=Z_2S^R(p)\ , \tag{8b}$$

$$\Gamma'_\mu(p',p)=Z_1^{-1}\Gamma^R_\mu(p',p)\ . \tag{8c}$$

The physical mass m is defined by requiring electron propagation when $p^2=m^2$. The physical coupling is defined by requiring the interaction energy of two distant stationary charges to behave as α/r, where $\alpha=e^2/4\pi\cong\frac{1}{137}$. The normalization conditions for the electron propagator and the vertex function will be chosen to correspond to the physical conditions encountered in QED bound-state problems.

The statement that m is the physical electron mass means that the electron propagates over arbitrarily large distances when $p^2=m^2$. This condition can be translated to a condition on the full electron

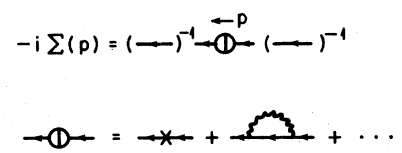

FIG. 5. Graphical definition of the electron self-energy.

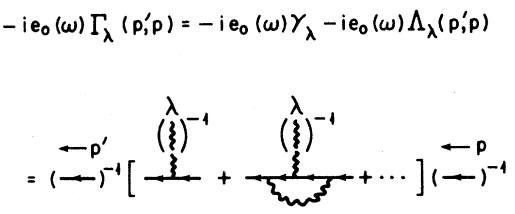

FIG. 6. Graphical definition of the vertex function.

propagator $S'(p)$. The inverse propagator has the following form:

$$[S'(p)]^{-1}=\gamma\cdot p-m-\Sigma(p)$$
$$=am+b\gamma\cdot p+c\xi\gamma\cdot n\ , \tag{9}$$

where a, b, and c (functions of p^2 and $\xi=p\cdot n$) can be calculated perturbatively from Feynman graphs. The full propagator itself is found by inverting $(S')^{-1}$,

$$S'(p)=\frac{am-b\gamma\cdot p-c\xi\gamma\cdot n}{m^2\Delta(p^2,\xi)}\ , \tag{10}$$

where

$$\Delta(p^2,\xi)=a^2(p^2,\xi)-b^2(p^2,\xi)\left[\frac{p^2}{m^2}\right]$$
$$-c^2(p^2,\xi)\left[\frac{\xi^2}{m^2}\right]$$
$$-2b(p^2,\xi)c(p^2,\xi)\left[\frac{\xi^2}{m^2}\right]\ . \tag{11}$$

Now m is the physical mass if

$$\Delta(m^2,\xi)=0\ . \tag{12}$$

The mass shift δm must be chosen so that this equation holds. It is not immediately obvious that such a δm exists, since (12) is a complicated relation involving the electron energy ξ, while δm is a constant, independent of ξ. However, δm is known to be gauge independent. In a covariant gauge the corresponding Δ has no explicit ξ dependence, and the equation

$$\Delta(p^2=m^2)=0\ \text{(covariant gauge)} \tag{13}$$

can be solved for δm. This same δm must solve (12).

Another condition is still required to fix the electron wave-function renormalization constant Z_2. This condition is given when $S^R(p)$ is normalized at some point in momentum space. Since the typical QED bound-state, positronium, has energy levels

$$E=2m+O(m\alpha^2) \tag{14}$$

and typical momenta

$$\langle\,|\,\vec{p}\,|\,\rangle=O(m\alpha)\ , \tag{15}$$

the natural point at which to define $S^R(p)$ is at $p=mn$. This momentum characterizes an electron at rest in the frame specified by the timelike vector n. I require

$$S^R(p)\underset{p\to mn}{\sim}\frac{\gamma\cdot mn+m}{(p^2-m^2)}\ , \tag{16}$$

corresponding to free propagation forward in time for an electron at rest in the given frame. This determines Z_2 as I will now show.

The form of the full propagator for p near mn is

$$S'(p)\underset{p\to mn}{\sim}\frac{a_0m-(b_0+c_0)\gamma\cdot mn}{(p^2-m^2)m^2(\partial/\partial p^2)\Delta(p^2,m)\,|_{p^2=m^2}}\ , \tag{17}$$

where $a_0=a(m^2,m)$, etc. From the vanishing of $\Delta(m^2,m)$ and the fact that

$$a=-1+O(\alpha),\quad b=1+O(\alpha),\quad c=O(\alpha)\ , \tag{18}$$

if follows that

$$a_0+b_0+c_0=0\ , \tag{19}$$

and so

$$S'(p)\underset{p\to mn}{\sim}\frac{a_0(\gamma\cdot mn+m)}{(p^2-m^2)m^2[2a_0(a'+b'+c')-b_0{}^2/m^2]}\ , \tag{20}$$

where $a'=(\partial/\partial p^2)a(p^2,m)\,|_{p^2=m^2}$, etc. In light of Eqs. (8b) and (16), the renormalization constant is in view,

$$Z_2=[2m^2(a'+b'+c')-b_0{}^2/a_0]^{-1}\ . \tag{21}$$

I define the vertex-function renormalization constant Z_1 so that the renormalized vertex function $\Gamma^R_\mu(p',p)$ is effectively γ_μ for electrons at rest:

$$(\gamma\cdot p'+m)\Gamma^R_\mu(p',p)(\gamma\cdot p+m)$$
$$\underset{p',p\to mn}{\to}(\gamma\cdot mn+m)\gamma_\mu(\gamma\cdot mn+m)\ . \tag{22}$$

The corresponding relation for Z_1 is

$$\tfrac{1}{2}(\gamma\cdot n+1)n^\mu\Gamma'_\mu(mn,mn)\tfrac{1}{2}(\gamma\cdot n+1)$$
$$=\tfrac{1}{2}(\gamma\cdot n+1)Z_1{}^{-1}\ . \tag{23}$$

With this definition, Ward's identity can be used in the usual way to show that $Z_1=Z_2$.

Finally, the photon propagation function is normalized at $k^2=0$:

$$D^R_{\mu\nu}(k) \underset{k^2\to 0}{\sim} D_{\mu\nu}(k) . \tag{24}$$

The reason for this choice of normalization is seen when one considers the interaction of two distant ($|\vec{R}| \gg 1/m$) stationary charges. The interaction energy of two such charges is related to the $(2\omega-1)$-dimensional Fourier transform of $D'_{00}(k^0=0,\vec{k})$; it is

$$E=\frac{{e_0}^2 Z_3}{4\pi|\vec{R}|} . \tag{25}$$

I normalize e so that this interaction energy is $e^2/(4\pi|\vec{R}|)$, so the bare and physical couplings are related by

$$e^2=Z_3 {e_0}^2 . \tag{26}$$

Gauge invariance can be used to show that the vacuum polarization tensor $\Pi_{\mu\nu}$ is gauge independent, the same in the Coulomb gauge as in any of the covariant gauges. Since

$$D'_{\mu\nu}(k)=[1+\Pi(k^2)]^{-1}D_{\mu\nu}(k) , \tag{27}$$

it follows that $Z_3=[1+\Pi(0)]^{-1}$ is gauge independent as well.

III. THE ELECTRON SELF-ENERGY

To one loop the electron self-energy has two contributions (pictured in Fig. 5),

$$-i\Sigma(p)=i\delta m+\int (dk)'_\omega[-ie_0(\omega)\gamma_\mu]\frac{i}{\gamma\cdot(p-k)-m}[-ie_0(\omega)\gamma_\nu]iD^{\mu\nu}(k) . \tag{28}$$

The Coulomb-gauge photon propagator is given in Eq. (4). The momentum integral[4] in (28) can be evaluated using the formulas for the dimensional regularization of noncovariant integrals presented in the Appendix. The result is[5]

$$\begin{aligned}\Sigma(p)=&\frac{\alpha m}{4\pi}(3D+4)-\delta m-\frac{\alpha}{4\pi}D(\gamma\cdot p-m)\\&+\frac{\alpha}{4\pi}\left[\tfrac{19}{6}\vec{\gamma}\cdot\vec{p}-\tfrac{1}{2}\gamma^0 p^0\right.\\&\quad-\int_0^1\frac{dx}{\sqrt{x}}\ln X[(1-x)\vec{\gamma}\cdot\vec{p}+m]+2\int_0^1 dx\,\ln Y[(1-x)\gamma\cdot p-m]\\&\quad\left.+2\vec{\gamma}\cdot\vec{p}\int_0^1\frac{dx}{\sqrt{x}}x\int_0^1 du\,\ln Z\right]\end{aligned} \tag{29}$$

in the frame where $n=(1,\vec{0})$ with

$$D=\frac{1}{2-\omega}-\gamma_E+\ln(4\pi) \tag{30}$$

and

$$X=1+(\vec{p}^{\,2}/m^2)(1-x) , \tag{31a}$$

$$Y=1-(p^2/m^2)(1-x)-i\epsilon , \tag{31b}$$

$$Z=1-({p_0}^2/m^2)(1-u)+(\vec{p}^{\,2}/m^2)(1-xu)-i\epsilon . \tag{31c}$$

Because a real electron-photon pair can be created from a virtual electron only when $p^2>m^2$, $\Sigma(p)$ is real when $p^2<m^2$ and has an imaginary part when $p^2>m^2$. To verify this, consider the expressions

$$Y=x+\left[\frac{m^2-p^2}{m^2}\right](1-x)-i\epsilon , \tag{32a}$$

$$Z=uX+\left[\frac{m^2-p^2}{m^2}\right](1-u)-i\epsilon . \tag{32b}$$

Neither Y nor Z is negative when $p^2<m^2$, but when $p^2>m^2$ both Y znd Z are negative for some values of

their parameters. The required reality properties of the self-energy are thus obtained.

IV. DETERMINATION OF δm AND Z_2

The one-loop values of δm and Z_2 can be found by using the general formulas of Sec. II with the explicit form of the self-energy found in Sec. III. The quantities a, b, and c are

$$a(p^2,\xi)=-1-\frac{\alpha}{4\pi}(3D+4)+\frac{\delta m}{m}-\frac{\alpha}{4\pi}D+\frac{\alpha}{4\pi}\left[\int_0^1\frac{dx}{\sqrt{x}}\ln X+2\int_0^1 dx\,\ln Y\right], \tag{33a}$$

$$b(p^2,\xi)=1+\frac{\alpha}{4\pi}D+\frac{\alpha}{4\pi}\left[\frac{19}{6}-\int_0^1\frac{dx}{\sqrt{x}}\ln X(1-x)-2\int_0^1 dx\,\ln Y(1-x)+2\int_0^1\frac{dx}{\sqrt{x}}x\int_0^1 du\,\ln Z\right], \tag{33b}$$

and

$$c(p^2,\xi)=\frac{\alpha}{4\pi}\left[-\frac{8}{3}+\int_0^1\frac{dx}{\sqrt{x}}\ln X(1-x)-2\int_0^1\frac{dx}{\sqrt{x}}x\int_0^1 du\,\ln Z\right]. \tag{33c}$$

To $O(\alpha)$ the mass-shell condition (12) takes the form

$$a(m^2,\xi)+b(m^2,\xi)+c(m^2,\xi)\left[\frac{\xi^2}{m^2}\right]=0\,. \tag{34}$$

A short calculation shows that this condition is satisfied when

$$\delta m=\frac{\alpha m}{4\pi}(3D+4)\,. \tag{35}$$

This is the same δm as that found in the covariant gauges.

The one-loop Coulomb-gauge wave-function renormalization constant Z_2 is found using (21). It is

$$Z_2=1-\frac{\alpha}{4\pi}D\,. \tag{36}$$

An important property of the Coulomb gauge is the fact that no infrared problems occur in the evaluation of Z_2.

V. DETERMINATION OF Z_1

As stated before, $Z_1=Z_2$ in this renormalization scheme. This statement can be verified to one loop by looking at the one-loop vertex correction graph (Fig. 6) in the Coulomb gauge,

$$-ie_0(\omega)\Lambda_\lambda(p',p)=\int(dk)'_\omega[-ie_0(\omega)\gamma_\mu]\frac{i}{\gamma\cdot(p'-k)-m}[-ie_0(\omega)\gamma_\lambda]\frac{i}{\gamma\cdot(p-k)-m}[-ie_0(\omega)\gamma_\nu]iD^{\mu\nu}(k)\,. \tag{37}$$

Now Z_1 is obtained from the vertex function by

$$\begin{aligned}\tfrac{1}{2}(1+\gamma^0)Z_1{}^{-1}&=\tfrac{1}{2}(1+\gamma^0)\Gamma'_0(mn,mn)\tfrac{1}{2}(1+\gamma^0)\\&=\tfrac{1}{2}(1+\gamma^0)[\gamma_0+\Lambda_0(mn,mn)]\tfrac{1}{2}(1+\gamma^0)\,,\end{aligned} \tag{38}$$

where $n=(1,\vec{0})$, so

$$\begin{aligned}\tfrac{1}{2}(1+\gamma^0)(Z_1{}^{-1}-1)&=\tfrac{1}{2}(1+\gamma^0)\Lambda_0(mn,mn)\tfrac{1}{2}(1+\gamma^0)\\&=\tfrac{1}{2}(1+\gamma^0)\left[\frac{-\alpha}{4\pi}\right]\int\frac{(dk)''_\omega}{(4\pi m^2)^{\omega-2}}\frac{\gamma_\mu[\gamma\cdot(mn-k)+m]\gamma_0[\gamma\cdot(mn-k)+m]\gamma_\nu}{(k^2-2k\cdot mn)^2}\\&\quad\times D^{\mu\nu}(k)\tfrac{1}{2}(1+\gamma^0)\,.\end{aligned} \tag{39}$$

The integral is readily evaluated, and

$$Z_1 = 1 - \frac{\alpha}{4\pi} D \ . \tag{40}$$

VI. CONCLUSION

In this paper, I have given a complete account of the one-loop renormalization of Coulomb-gauge QED. No problems, ambiguities, or surprises were found. The renormalization scheme is tailored to the physical situations found in QED bound states, and should prove useful in calculations involving such systems.

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APPENDIX

Feynman graphs in the Coulomb gauge give rise to noncovariant integrals. The dimensional regularization of such integrals was considered by Heckathorn,[3] and I follow his basic method. Let $d = 2\omega$ be the dimension of spacetime. I assume that spacetime contains one time dimension and $(2\omega - 1)$ space dimensions (as opposed to the $d = \frac{1}{4}d + \frac{3}{4}d$ decomposition of Heckathorn).

Formulas for the dimensional regularization of several useful one-loop integrals are as follows[4]:

$$\int (dk)''_\omega \frac{B}{(k^2 - 2k\cdot p - M^2)^\alpha} = \frac{(-1)^\xi}{\Gamma(\xi)} C \ , \tag{A1a}$$

$$\int (dk)''_\omega \frac{B}{(k^2 - 2k\cdot p - M^2)^\alpha (-\vec{k}^2 - \lambda^2)^\beta} = \frac{(-1)^\xi}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{dx}{\sqrt{x}} x^{\alpha-1}(1-x)^{\beta-1} C \ , \tag{A1b}$$

$$\int (dk)''_\omega \frac{B}{(k^2 - 2k\cdot p - M^2)^\alpha (k^2 - 2k\cdot p' - M'^2)^\beta} = \frac{(-1)^\xi}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} C \ , \tag{A1c}$$

$$\int (dk)''_\omega \frac{B}{(k^2 - 2k\cdot p - M^2)^\alpha (k^2 - 2k\cdot p' - M'^2)^\beta (-\vec{k}^2 - \lambda^2)^\gamma} = \frac{(-1)^\xi}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 \frac{dx}{\sqrt{x}} \int_0^x dy\, y^{\alpha-1}(x-y)^{\beta-1}(1-x)^{\gamma-1} C \ , \tag{A1d}$$

$$\int (dk)''_\omega \frac{B}{(k^2 - 2k\cdot p - M^2)^\alpha (k^2 - 2k\cdot p' - M'^2)^\beta (k^2 - 2k\cdot p'' - M''^2)^\gamma} = \frac{(-1)^\xi}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx \int_0^x dy\, y^{\alpha-1}(x-y)^{\beta-1}(1-x)^{\gamma-1} C \ , \tag{A1e}$$

$$\int (dk)''_\omega \frac{B}{(k^2 - 2k\cdot p - M^2)^\alpha (k^2 - 2k\cdot p' - M'^2)^\beta (k^2 - 2k\cdot p'' - M''^2)^\gamma (-\vec{k}^2 - \lambda^2)^\delta} = \frac{(-1)^\xi}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)} \int_0^1 \frac{dx}{\sqrt{x}} \int_0^x dy \int_0^y dz\, z^{\alpha-1}(y-z)^{\beta-1}(x-y)^{\gamma-1}(1-x)^{\delta-1} C \ , \tag{A1f}$$

where C is related to B by

$$B = 1, \quad C = \frac{\Gamma(\xi - \omega)}{\Delta^{\xi-\omega}} \ , \tag{A2a}$$

$$B = k_\mu, \quad C = (AQ)_\mu \frac{\Gamma(\xi - \omega)}{\Delta^{\xi-\omega}} \ , \tag{A2b}$$

$$B = k_\mu k_\nu, \quad C = (AQ)_\mu (AQ)_\nu \frac{\Gamma(\xi - \omega)}{\Delta^{\xi-\omega}} - \tfrac{1}{2} A_{\mu\nu} \frac{\Gamma(\xi - 1 - \omega)}{\Delta^{\xi-1-\omega}} \ , \tag{A2c}$$

$$B = k_\mu k_\nu k_\lambda, \quad C = (AQ)_\mu (AQ)_\nu (AQ)_\lambda \frac{\Gamma(\xi - \omega)}{\Delta^{\xi-\omega}} - \tfrac{1}{2}[A_{\mu\nu}(AQ)_\lambda + A_{\nu\lambda}(AQ)_\mu + A_{\lambda\mu}(AQ)_\nu] \frac{\Gamma(\xi - 1 - \omega)}{\Delta^{\xi-1-\omega}} \ . \tag{A2d}$$

In the covariant integrals (A1a), (A1c), and (A1e) the matrix A is given by

$$A_{\mu\nu}=g_{\mu\nu}\ . \tag{A3}$$

In the noncovariant integrals (A1b), (A1d), and (A1f) the matrix A is

$$A_{\mu\nu}=g_{\mu\nu}+\left[\frac{1-x}{x}\right]n_\mu n_\nu=\begin{bmatrix}1/x & 0\\ 0 & -\delta_{ij}\end{bmatrix}\ . \tag{A4}$$

The quantities ξ, Q, and Δ are given for the various cases by

(a) $\xi=\alpha,\ Q=p\ ,$ (A5a)

$$\Delta=Q^2+M^2\ ;$$

(b) $\xi=\alpha+\beta,\ Q=px\ ,$ (A5b)

$$\Delta=QAQ+M^2x+\lambda^2(1-x)\ ;$$

(c) $\xi=\alpha+\beta,\ Q=px+p'(1-x)\ ,$ (A5c)

$$\Delta=Q^2+M^2x+M'^2(1-x)\ ;$$

(d) $\xi=\alpha+\beta+\gamma,\ Q=py+p'(x-y)\ ,$ (A5d)

$$\Delta=QAQ+M^2y+M'^2(x-y)+\lambda^2(1-x)\ ;$$

(e) $\xi=\alpha+\beta+\gamma,\ Q=py+p'(x-y)+p''(1-x)\ ,$ (A5e)

$$\Delta=Q^2+M^2y+M'^2(x-y)+M''^2(1-x)\ ;$$

(f) $\xi=\alpha+\beta+\gamma+\delta,\ Q=pz+p'(y-z)+p''(x-y)\ ,$ (A5f)

$$\Delta=QAQ+M^2z+M'^2(y-z)+M''^2(x-y)+\lambda^2(1-x)\ .$$

Consistency conditions such as

$$\int(dk)''_\omega\frac{k^ik^i}{(k^2-2k\cdot p-M^2)^\alpha\vec{k}^{\,2}}=\int(dk)''_\omega\frac{1}{(k^2-2k\cdot p-M^2)^\alpha} \tag{A6}$$

are satisfied by these formulas.

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[1]The notation and conventions of J. D. Bjorken and S. D. Drell [*Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964)] are used throughout this paper.

[2]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 302.

[3]D. Heckathorn, Nucl. Phys. B156, 328 (1979).

[4]The notation $(dk)_\omega=d^{2\omega}k$, $(dk)'_\omega=(dk)_\omega/(2\pi)^{2\omega}$, and $(dk)''_\omega=(dk)_\omega/i\pi^\omega$ is used to simplify the expressions.

[5]The electron propagator in the Coulomb gauge has been considered by K. Johnson [Ann. Phys. (N.Y.) 10, 536 (1960)] who presents his result in a spectral form. A spectral form for the electron self-energy is given by C. R. Hagen [Phys. Rev. 130, 813 (1963)]. See also K. Haller and L. F. Landovitz, Phys. Rev. D 2, 1498 (1970). I find the form of the self-energy given in the present paper to be more convenient for actual calculations.