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Background-field method and the renormalization of non-Abelian gauge theories in curved space-time

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The gauge-invariant background-field approach is used to discuss a non-Abelian gauge theory containing fermions in curved space-time. Renormalization of the theory at the one-loop level is presented using dimensional regularization, heat-kernel techniques, and a curved—space-time momentum-space method.

I. INTRODUCTION

An area of recent investigation concerns interacting quantum field theory in curved space-time. It is particularly important to know if such theories are renormalizable, a fact which is not completely obvious to show since standard flat—space-time momentum-space techniques may no longer be used. So far renormalizability has only been shown for interacting scalar fields[1–4] and quantum electrodynamics[5] in fairly general space-times. (Weaker results have also been obtained in specific space-times. See Ref. 3 and below for more details.) In this paper we study the renormalizability of a non-Abelian gauge theory containing fermions on a curved background.

The method used here consists of a computation of the effective action using the gauge-invariant background-field method.[6–10] Dimensional regularization[11] along with heat-kernel techniques[12,13] and the curved—space-time momentum-space method of Bunch and Parker[1] are used to analyze the divergences. This approach offers several advantages over the standard diagrammatic analysis. First, by explicit construction it can contain only gauge-invariant combinations of the background field. Second, it cuts down the number of graphs which need to be considered, involving only one-particle-irreducible vacuum bubbles. There is no need to examine the n-point functions separately for each n since they are obtained by functional differentiation of the effective action with respect to the background fields.

These, or similar, techniques have been used for non-Abelian gauge theories in flat space-time.[7,14–16] Results for scalars or spinors in combined classical gauge plus gravitational backgrounds may be found in Refs. 6, 17, and 18. On S^4 massless QED and scalar electrodynamics have been studied by Drummond and Shore[19] and Shore.[20,21] Lüscher[4] has recently given results for Yang-Mills theory on S^4 in a multi-instanton background. In weak gravitational backgrounds, Utiyama[22] has analyzed QED, and Ichinose and Omote[23] have studied non-Abelian gauge theories. Panangaden[5] has shown renormalizability of QED at the one-loop level in an arbitrarily curved, but topologically trivial space-time.

In this paper we prove that a non-Abelian gauge theory containing fermions is one-loop renormalizable in curved space-time. In the special case where the gauge group is U(1), the theory considered here reduces to QED; thus, this paper contains an alternative to and generalizes the proof of Panangaden.[5] In our view the background-field method provides a simpler approach than the conventional diagrammatic method. (This is true even in flat space-time.) For the theory containing non-Abelian gauge fields plus fermions considered here, using the conventional approach it would be necessary to calculate 18 diagrams which contribute to the two-, three-, and four-point functions. (Details may be found in Itzykson and Zuber,[24] for instance.) In curved space-time the situation is even worse since the expressions for the propagators calculated using the momentum-space approach[1] involve more terms than in flat space-time, leading to more complicated expressions to evaluate. This is clear even in Panangaden's[5] QED calculation where only three graphs need to be evaluated.

In Sec. II we present our notation, and the relevant expression for the one-loop effective action obtained via the gauge-invariant background-field method is given. In Sec. III the divergences are computed, and it is shown that the theory is renormalizable to one-loop order. A brief discussion is contained in Sec. IV.

II. THE GAUGE-INVARIANT BACKGROUND-FIELD METHOD

Let $\{T^a\}$ be the set of generators of the Lie algebra of some group G which has dimension N. The generators will be taken to be anti-Hermitian and to satisfy

$$[T^a,T^b]=f^{abc}T^c \; . \tag{2.1}$$

Any repeated Latin indices will be summed over from 1 to N. Suppose that the generators T^a correspond to a representation G_R of G with dimension d_R (i.e., T^a may be taken to be a $d_R \times d_R$ anti-Hermitian matrix). Let $C_2(G_R)$ denote the quadratic Casimir operator for this representation:

$$C_2(G_R)=-T^aT^a \; . \tag{2.2}$$

Then,

$$\mathrm{tr}(T^aT^b)=-\frac{d_R}{N}C_2(G_R)\delta_{ab} \; . \tag{2.3}$$

The usual Yang-Mills action is

$$I_{\mathrm{YM}}[A]=\tfrac{1}{4}\int dv_x F^a_{\mu\nu}F^{a\mu\nu} \; , \tag{2.4}$$

where $dv_x=[g(x)]^{1/2}d^4x$ is the invariant volume element on the space-time manifold, and

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{bca}A^b_\mu A^c_\nu \; . \tag{2.5}$$

(We choose to work on a Riemannian space-time rather than a Lorentzian one.) It then follows that I_{YM} is invariant under the infinitesimal gauge transformation

$$A^a_\mu \to A^a_\mu + f^{bca}\theta^b A^c_\mu - \frac{1}{g}\partial_\mu\theta^a \; , \tag{2.6}$$

where $\theta^a(x)$ are the parameters of the transformation.

It proves convenient in the following to deal with the Lie-algebra or matrix-valued connection. Define

$$A_\mu = gA^a_\mu T^a \; , \tag{2.7}$$

where T^a is a matrix in the adjoint representation G_{ad} of G. It is this object which has the geometrical meaning of a connection in a principal fiber bundle. Define also

$$F_{\mu\nu}=gF^a_{\mu\nu}T^a \; . \tag{2.8}$$

Then $F_{\mu\nu}$ is the curvature given in terms of A_μ by

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu] \; . \tag{2.9}$$

Equation (2.6) is the infinitesimal form of

$$A_\mu \to U(g)A_\mu U^{-1}(g)-[\partial_\mu U(g)]U^{-1}(g) \tag{2.10}$$

for $U(g)=\exp(\theta^aT^a)$. (θ^a are the coordinates of $g\in G$ in the group manifold in a neighborhood of the identity.) From (2.9),

$$F_{\mu\nu}\to U(g)F_{\mu\nu}U^{-1}(g) \; . \tag{2.11}$$

In place of (2.4) we may take

$$I_{YM}=\frac{-1}{4g^2C_2(G_{\mathrm{ad}})}\int dv_x \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \; . \tag{2.12}$$

Let $\psi(x)$ be a multicomponent spinor field which transforms under G as

$$\psi(x)\to U(g)\psi(x) \; . \tag{2.13}$$

It is assumed that $U(g)$ provides a unitary d_F-dimensional representation G_F of G. [We are concerned here with the case where $\psi(x)$ contains d_F four-component Dirac spinors.] The adjoint spinor $\bar\psi(x)$ transforms as

$$\bar\psi(x)\to\bar\psi(x)U^{-1}(g) \; . \tag{2.14}$$

Let D_μ be the covariant derivative computed using the appropriate spin connection as well as the gauge connection. For $\psi(x)$ transforming as in (2.13),

$$D_\mu\psi\to U(g)D_\mu\psi \tag{2.15}$$

provided that

$$D_\mu=\nabla_\mu+A_\mu \tag{2.16}$$

with A_μ acting on ψ by matrix multiplication. The matrix A_μ is taken in the same representation as ψ. ∇_μ contains the spin connection and is a multiple of the group identity. If $\phi(x)$ represents a set of fields transforming covariantly in the same way as the curvature in (2.11), then

$$D_\mu\phi=\nabla_\mu\phi+[A_\mu,\phi] \tag{2.17}$$

gives its covariant derivative.

Let $\{e_{\hat\mu}{}^{\nu}(x)\}$ denote the vierbein field which satisfies

$$\delta_{\hat\mu\hat\nu}=e_{\hat\mu}{}^{\rho}e_{\hat\nu}{}^{\sigma}g_{\rho\sigma} \; , \tag{2.18a}$$

$$g_{\mu\nu}=e^{\hat\rho}{}_{\mu}e^{\hat\sigma}{}_{\nu}\delta_{\hat\rho\hat\sigma} \; . \tag{2.18b}$$

The caret denotes an orthonormal frame index. Define Hermitian γ matrices to satisfy

$$\{\gamma_{\hat{\mu}},\gamma_{\hat{\nu}}\}=2\delta_{\hat{\mu}\hat{\nu}} \,. \tag{2.19}$$

Then $\bar{\psi}(x)\not{D}\psi(x)$ is invariant under G, as well as under general coordinate transformations, where

$$\not{D}=\gamma^{\hat{\mu}}e_{\hat{\mu}}{}^{\nu}D_{\nu} \,. \tag{2.20}$$

The notation of Coleman[25] is followed here. $\bar{\psi}$ and ψ are to be treated as independent anticommuting c numbers. Let M be the spinor mass matrix. If $\bar{\psi}(x)M\psi(x)$ is to be invariant under G, from (2.13) and (2.14) it is seen that

$$[M,U(g)]=0 \,. \tag{2.21}$$

The infinitesimal form of (2.21) is

$$[M,T^a]=0 \,. \tag{2.22}$$

The fermion part of the action is taken to be[25]

$$I_F[\bar{\psi},\psi,A]=-i\int dv_x\bar{\psi}(\not{D}-M)\psi \,, \tag{2.23}$$

where M is Hermitian and satisfies (2.22).

The gravitational part of the action is taken to be

$$I_G=-\int dv_x(-2\Lambda+\kappa R+\alpha_1 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} +\alpha_2 R^{\mu\nu}R_{\mu\nu}+\alpha_3 R^2) \,, \tag{2.24}$$

where Λ, κ, α_1, α_2, and α_3 are constants. [Λ is the cosmological constant and $\kappa=(16\pi G)^{-1}$ where G is the Newtonian gravitational constant.] The curvature conventions of Ref. 26 are used here. Note that there are no possible nonminimal terms which may be added to give a gauge-invariant renormalizable theory. This is no longer true, however, if scalars are included.

Let $\hat{A}_\mu(x)$ denote the arbitrary background gauge field, and $\hat{\psi}(x)$, $\hat{\bar{\psi}}(x)$ the background spinor fields. The partition function is taken to be

$$Z[J,\bar{\rho},\rho;\hat{A},\hat{\bar{\psi}},\hat{\psi}]=\int [d\bar{\psi}][d\psi][\mathscr{D}A]\exp\{-I_{\text{YM}}[\hat{A}+A]-I_F[\hat{\bar{\psi}}+\bar{\psi},\hat{\psi}+\psi,\hat{A}+A] -I_G+J^\mu_a A^a_\mu+\bar{\psi}\rho+\bar{\rho}\psi\} \,, \tag{2.25}$$

where J^μ_a, ρ, and $\bar{\rho}$ are source terms, with ρ and $\bar{\rho}$ independent anticommuting c numbers. Here $[\mathscr{D}A]$ is the relevant functional measure for the gauge fields which includes the well-known gauge-fixing and ghost terms,[27] and may also depend on the background field. It is required to be invariant under the action of G.

From (2.25) it is clear that the fields may be regarded as being split up into the sum of a background-field part $\hat{A}_\mu$, $\hat{\bar{\psi}}$, or $\hat{\psi}$, and a quantum part A_μ, $\bar{\psi}$, or ψ which gets integrated over in the functional integration. Only the quantum parts are coupled to the sources. $\hat{A}_\mu+A_\mu$ must transform under G as in (2.10); however, we are free to choose the background gauge field to transform as

$$\hat{A}_\mu\to U\hat{A}_\mu U^{-1}-(\partial_\mu U)U^{-1} \,, \tag{2.26}$$

with the quantum part of the gauge field transforming covariantly as

$$A_\mu\to UA_\mu U^{-1} \,. \tag{2.27}$$

The background gauge field therefore transforms in the usual way. Both the background and quantum parts of the fermion fields are taken to transform as in (2.13) and (2.14). The sources J^a_μ, ρ, and $\bar{\rho}$ are taken to transform in the same way as A^a_μ, ψ, and $\bar{\psi}$, respectively. [Note that

$$J^a_\mu A^{a\mu}=\frac{-1}{g^2C_2(G_{\text{ad}})}\text{tr}(J_\mu A^\mu) \,,$$

where $J_\mu=gJ^a_\mu T^a$ with T^a in the adjoint representation.] It then follows that the expression for Z in (2.25) will involve the background fields only in a gauge-invariant manner provided that at the same time we perform the gauge transformation (2.26) on the background field, we perform the change of variables (2.27), (2.13), and (2.14) in the functional integration, and transform the sources in a relevant fashion.

Let $\hat{D}_\mu$ be the covariant derivative formed using the background-field gauge connection. Then because A^μ transforms as in (2.27), from (2.17) we have

$$\hat{D}_\mu A^\mu=\nabla_\mu A^\mu+[\hat{A}_\mu,A^\mu] \tag{2.28}$$

which transforms covariantly under (2.26) and (2.27). As a gauge-fixing term choose

$$I_{\text{GF}}=\frac{-1}{2\alpha g^2C_2(G_{\text{ad}})}\int dv_x\text{tr}[(\hat{D}_\mu A^\mu)^2] \tag{2.29a}$$

$$=\frac{1}{2\alpha g^2C_2(G_{\text{ad}})}\int dv_x\text{tr}[A^\mu\hat{D}_\mu\hat{D}_\nu A^\nu] \,, \tag{2.29b}$$

where the second line follows from the first upon an integration by parts. α is an arbitrary real constant. With this choice of gauge-fixing condition the ghost part of the action may be seen to be

$$I_{\rm GH}=\int dv_x \bar{\eta}(x)[-\hat{D}^2-A^\mu \hat{D}_\mu -(\hat{D}_\mu A^\mu)]\eta(x)\ , \qquad (2.30)$$

where $\bar{\eta}$ and η are the anticommuting ghost fields[27] which transform as in (2.14) and (2.15) under gauge transformations. [The result in (2.30) follows immediately from DeWitt's[9] more general result.] The result in (2.30) is manifestly gauge invariant. The functional measure indicated in (2.25) is

$$[\mathscr{D}A]=[dA][d\bar{\eta}][d\eta]\exp(-I_{\rm GF}-I_{\rm GH})\ . \qquad (2.31)$$

The effective action $\Gamma[\hat{A},\hat{\bar{\psi}},\hat{\psi}]$ may now be found, for example, by the Legendre transform method.[28] It is a functional of only the background fields and is invariant under background-field gauge transformations. Write

$$\Gamma=\Gamma_{\rm DIV}+\Gamma_{\rm REG}\ , \qquad (2.32)$$

where $\Gamma_{\rm DIV}$ contains all of the divergent terms and $\Gamma_{\rm REG}$ contains all of the regular terms. It then follows that $\Gamma_{\rm DIV}$ and $\Gamma_{\rm REG}$ must be separately gauge invariant provided that the regularization scheme respects gauge invariance. This means that if the theory is to be renormalizable $\Gamma_{\rm DIV}$ must be a linear combination of $I_{\rm YM}[\hat{A}]$ and quantities in $I_F[\hat{A},\hat{\bar{\psi}},\hat{\psi}]$ with divergent coefficients which may be absorbed by a renormalization of the background field and spinor mass. (In addition there may be a divergent term independent of the background fields.)

The effective action is formed from one-particle-irreducible vacuum bubbles only, although in the background-field formalism the propagators and the rules for vertices are changed from the usual ones. In order to obtain them the action is expanded in powers of the quantum fields. Because the effective action involves only one-particle-irreducible graphs, the terms which are linear in the quantum fields may be dropped. From (2.12), writing

$$I_{\rm YM}[\hat{A}+A]=I_{\rm YM}[\hat{A}]+\sum_{n=1}^{4} I_{\rm YM}^{(n)}[\hat{A},A]\ , \qquad (2.33)$$

where the superscript figure in parentheses represents the power of the quantum field which occurs, we have

$$I_{\rm YM}^{(2)}[\hat{A},A]=\frac{-1}{2g^2C_2(G_{\rm ad})}\int dv_x {\rm tr}[A^\mu(-g_{\mu\nu}\hat{D}^2+\hat{D}_\nu\hat{D}_\mu-2\hat{F}_{\mu\nu})A^\nu]\ , \qquad (2.34a)$$

$$I_{\rm YM}^{(3)}[\hat{A},A]=\frac{-1}{g^2C_2(G_{\rm ad})}\int dv_x {\rm tr}[(\hat{D}_\mu A_\nu-\hat{D}_\nu A_\mu)A^\mu A^\nu]\ , \qquad (2.34b)$$

$$I_{\rm YM}^{(4)}[A]=\frac{-1}{2g^2C_2(G_{\rm ad})}\int dv_x (g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma}){\rm tr}(A^\sigma A^\rho A^\mu A^\nu)\ . \qquad (2.34c)$$

In obtaining $I_{\rm YM}^{(2)}$ an integration by parts has been performed. $I_{\rm YM}^{(3)}$ and $I_{\rm YM}^{(4)}$ gives the rules for the trilinear and quartic gauge field vertices, respectively. For $I_F[\hat{A}+A,\hat{\bar{\psi}}+\bar{\psi},\hat{\psi}+\psi]$,

$$I_F[\hat{A}+A,\hat{\bar{\psi}}+\bar{\psi},\hat{\psi}+\psi]=I_F[\hat{A},\hat{\bar{\psi}},\hat{\psi}]+\sum_{n=1}^{3} I_F^{(n)}\ , \qquad (2.35)$$

where

$$I_F^{(2)}\doteq -i\int dv_x[\bar{\psi}(\hat{\not{D}}-M)\psi+\hat{\bar{\psi}}\not{A}\psi+\bar{\psi}\not{A}\hat{\psi}]\ , \qquad (2.36a)$$

$$I_F^{(3)}=-i\int dv_x \bar{\psi}\not{A}\psi\ . \qquad (2.36b)$$

$I_F^{(3)}$ gives the rule for the vertex involving two spinor and one gauge field lines. Note that the quadratic part of the action is not diagonal due to cross terms between the spinor and gauge fields. For the ghost part of the action

$$I_{\rm GH}=I_{\rm GH}^{(2)}+I_{\rm GH}^{(3)}\ , \qquad (2.37)$$

where

$$I_{\rm GH}^{(2)}=\int dv_x \bar{\eta}[-\hat{D}^2]\eta\ , \qquad (2.38a)$$

$$I_{\rm GH}^{(3)}=-\int dv_x \bar{\eta}[A^\mu\hat{D}_\mu+(\hat{D}_\mu A^\mu)]\eta\ . \qquad (2.38b)$$

$I^{(2)}_{\rm GH}$ is the inverse of the background-field ghost propagator. $I^{(3)}_{\rm GH}$ gives the rule for the vertex involving two ghost and one gauge field lines.

The tree-level contribution to the effective action is

$$\Gamma^{(0)}[\hat{A},\hat{\bar{\psi}},\hat{\psi}]=I_{\rm YM}[\hat{A}]+I_F[\hat{A},\hat{\bar{\psi}},\hat{\psi}]+I_G \tag{2.39}$$

with all fields and coupling constants bare. The one-loop contribution is

$$\Gamma^{(1)}[\hat{A},\hat{\bar{\psi}},\hat{\psi}]=-\ln\int[d\bar{\psi}][d\psi][dA][d\bar{\eta}][d\eta]\exp(-I^{(2)}_{\rm YM}-I_{\rm GF}-I^{(2)}_F-I^{(2)}_{\rm GH})\ . \tag{2.40}$$

Integration over the ghost fields using the usual rule for integrating over anticommuting fields[29] gives

$$\Gamma^{(1)}=-\ln {\rm Det}(-\hat{D}^2)-\ln\int[d\bar{\psi}][d\psi][dA]\exp(-I^{(2)}_{\rm YM}-I_{\rm GF}-I^{(2)}_F)\ . \tag{2.41}$$

Note that $\hat{A}_\mu$ is in the adjoint representation everywhere except for $I_F^{(2)}$ where it is in the fermion representation G_F.

An immediate consequence of (2.27) and (2.17) is that

$$[\hat{D}_\mu,\hat{D}_\nu]A_\lambda=R_{\mu\nu\lambda}{}^{\rho}A_\rho+[\hat{F}_{\mu\nu},A_\lambda] \tag{2.42}$$

with the curvature conventions of Ref. 26. This leads to

$$I^{(2)}_{\rm YM}+I_{\rm GF}=\tfrac{1}{2}\int dv_x A^{a\mu}[-g_{\mu\nu}\hat{D}^2+(1-1/\alpha)\hat{D}_\mu\hat{D}_\nu+R_{\mu\nu}-2\hat{F}_{\mu\nu}]^{ab}A^{b\nu}\ , \tag{2.43}$$

where $[\cdots]^{ab}$ denotes the matrix element of the quantity enclosed by the square brackets.

The remaining functional integrals in (2.41) may be evaluated by completing the square in the argument of the exponential. Make the following change of variables in the integration over the fermion fields, assuming as usual that the measure is translationally invariant:

$$\bar{\psi}(x)\to\bar{\psi}(x)+ig\int dv_{x'}\hat{\bar{\psi}}(x')A\!\!\!/(x')S(x',x)\ , \tag{2.44}$$

$$\psi(x)\to\psi(x)+ig\int dv_{x'}S(x,x')A\!\!\!/(x')\hat{\psi}(x')\ , \tag{2.45}$$

where we have defined

$$-i[\hat{D\!\!\!/}_x-M]S(x,x')=\delta(x,x')\ . \tag{2.46}$$

[Here $\delta(x,x')$ denotes the biscalar Dirac distribution which satisfies $\int dv_x\delta(x,x')f(x)=f(x')$.] The one-loop effective action is now

$$\Gamma^{(1)}=\tfrac{1}{2}\ln {\rm Det}[\Delta_{\mu\nu}^{-1ab}(x,x')+2g^2\hat{\bar{\psi}}(x)\gamma_\mu T^a S(x,x')T^b\gamma_\nu\hat{\psi}(x')]-\ln{\rm Det}(-\hat{D}^2)-\ln{\rm Det}S^{-1}(x,x')\ , \tag{2.47}$$

where $\Delta^{ab}_{\mu\nu}(x,x')$ has been defined by (matrix notation)

$$[-\delta^\mu{}_\lambda\hat{D}^2+(1-1/\alpha)\hat{D}^\mu\hat{D}_\lambda+R^\mu{}_\lambda-2\hat{F}^\mu{}_\lambda]\Delta^\lambda{}_\nu(x,x')=\delta^\mu{}_\nu\delta(x,x')\ . \tag{2.48}$$

The matrices T^a and T^b appearing in (2.47) are those appropriate to the fermion representation.

III. THE DIVERGENCES OF THE ONE-LOOP EFFECTIVE ACTION

In order to obtain the divergent part of $\Gamma^{(1)}$ we shall adopt dimensional regularization[11] and heat-kernel techniques.[6,12,13] The notation of Ref. 3 is followed here. Let I_2 be a second-order elliptic operator of the form

$$I_2=-\hat{D}^2+Q(x) \tag{3.1}$$

for some (matrix-valued) $Q(x)$. Then the pole part of $\ln {\rm Det}I_2$ is

$$\mathrm{PP}\{\ln \mathrm{Det} I_2\} = 2\epsilon^{-1}\int dv_x \,\mathrm{tr} E_2(x,I_2)\ , \tag{3.2}$$

where $\epsilon=(4\pi)^2(n-4)$ (with n the space-time dimension) and

$$E_2(x,I_2)=\tfrac{1}{360}[(12\Box R+5R^2-2R^{\mu\nu}R_{\mu\nu}+2R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})I+30W^{\mu\nu}W_{\mu\nu}+180Q^2-60RQ-60\hat{D}^2Q]\ . \tag{3.3}$$

I is the identity matrix in the appropriate representation for the type of field upon which I_2 acts, and $W_{\mu\nu}$ is the matrix-valued curvature defined by

$$[\hat{D}_\mu,\hat{D}_\nu]=W_{\mu\nu}\ . \tag{3.4}$$

The quantity $E_2(x,I_2)$ appears in the asymptotic expansion of the heat kernel. The result in (3.3) may be found in Ref. 30. (The result for $Q=0$ is in Ref. 6.)

If we now attempt to analyze the divergences of (2.47) directly using (3.2) there are immediate problems. The first is that although the last two terms in (2.47) involve operators of the form (3.1) so that (3.2) and (3.3) may be used, the first term involves a nonlocal object vitiating a straightforward application of the above result. If we had been studying a pure gauge theory this nonlocal object would not have been present. [In addition the last term in (2.47) would also not be there.] Even in the pure gauge case

$$\Delta^{-1}{}_{\mu\nu}(x,x')=[-g_{\mu\nu}\hat{D}^2+(1-1/\alpha)\hat{D}_\mu\hat{D}_\nu+R_{\mu\nu}-2\hat{F}_{\mu\nu}]\delta(x',x) \tag{3.5}$$

is of more general form than (3.1) if $\alpha\neq1$ so that (3.3) no longer holds. For this reason we shall restrict ourselves to the $\alpha=1$ gauge. There would seem to be no impediment to repeating DeWitt's[6] calculations, for example, to find E_2 for the operator (3.5) although we do not pursue this here. (Shore[31] has obtained results using heat-kernel techniques for $\alpha\neq1$ in the special case of a flat space-time with a covariantly constant background gauge field.)

A. Pure gauge theories

In order to obtain the one-loop effective action for a pure gauge theory, set $\bar{\psi}=\hat{\psi}=0$ in (2.47) and ignore the fermion loop contribution (the last term). Then,

$$\Gamma^{(1)}[\hat{A}]=\tfrac{1}{2}\ln\mathrm{Det}\Delta^{-1}{}_{\mu\nu}-\ln\mathrm{Det}(-\hat{D}^2)\ . \tag{3.6}$$

If we choose $\alpha=1$, then $\Delta^{-1}{}_{\mu\nu}$ is of the form (3.1) where

$$Q_{\mu\nu}(x)=R_{\mu\nu}-2\hat{F}_{\mu\nu}\ . \tag{3.7}$$

The matrix-valued curvature may be obtained from (2.42). Because $\Delta^{-1}{}_{\mu\nu}$ acts on the quantum part of the gauge field which transforms as in (2.27),

$$[\hat{D}_\mu,\hat{D}_\nu]A_\lambda=[W_{\mu\nu\lambda}{}^\rho,A_\rho] \tag{3.8}$$

from which [see (2.42)]

$$W_{\mu\nu\lambda}{}^\rho=R_{\mu\nu\lambda}{}^\rho+\hat{F}_{\mu\nu}\delta_\lambda{}^\rho\ . \tag{3.9}$$

Equations (3.2) and (3.3) then lead to (discarding integrals of total divergences)

$$\mathrm{PP}\{\tfrac{1}{2}\ln\mathrm{Det}\Delta^{-1}{}_{\mu\nu}\}=\epsilon^{-1}\int dv_x[-\tfrac{1}{9}NR^2+\tfrac{43}{90}NR^{\mu\nu}R_{\mu\nu}-\tfrac{11}{180}NR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-\tfrac{5}{3}\,\mathrm{tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})]\ . \tag{3.10}$$

We require next $E_2(x,-\hat{D}^2)$ when $\hat{D}^2$ acts on ghosts. Since the ghost fields transform as scalars under general coordinate transformations, and like (2.13) and (2.14) under gauge transformations $W_{\mu\nu}=\hat{F}_{\mu\nu}$. From (3.2) and (3.3),

$$\mathrm{PP}\{\ln\mathrm{Det}(-\hat{D}^2)\}=\epsilon^{-1}\int dv_x[\tfrac{1}{36}NR^2-\tfrac{1}{90}NR^{\mu\nu}R_{\mu\nu}+\tfrac{1}{90}NR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\tfrac{1}{6}\,\mathrm{tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})]\ . \tag{3.11}$$

In both (3.10) and (3.11), $\mathrm{tr}I_{\mathrm{ad}}=N$, where I_{ad} is the unit matrix in the adjoint representation, has been used.

From (3.6), (3.10), and (3.11), the divergent part of the one-loop effective action for a pure gauge theory in a general curved space-time is

$$\mathrm{PP}\{\Gamma^{(1)}[\hat{A}]\}=\epsilon^{-1}\int dv_x[-\tfrac{5}{36}NR^2+\tfrac{22}{45}NR^{\mu\nu}R_{\mu\nu}-\tfrac{13}{180}NR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-\tfrac{11}{6}\,\mathrm{tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})]\ . \tag{3.12}$$

Introduce background-field and coupling-constant renormalization factors by (see, for example, Ref. 15)

$$g_B = \mu^{2-n/2} Z_g g \,, \tag{3.13}$$

$$\hat{A}^a_{\mu B} = \mu^{n/2-2} Z_A{}^{1/2} \hat{A}^a_\mu \,. \tag{3.14}$$

(The subscript "B" denotes a bare quantity.) The 't Hooft[32] unit of mass is introduced to keep the dimensions of the renormalized coupling constant and background field the same for all n as for $n=4$. From (2.7), (3.13), and (3.14),

$$\hat{A}_{\mu B} = Z_g Z_A{}^{1/2} \hat{A}_\mu \,. \tag{3.15}$$

From (2.8),

$$\hat{F}_{\mu\nu B} = Z_g Z_A{}^{1/2} \{ \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + Z_g Z_A{}^{1/2} [\hat{A}_\mu, \hat{A}_\nu] \} \,. \tag{3.16}$$

Because we know by explicit construction that the background field can occur in the effective action only in a gauge-invariant way, we must have[15]

$$Z_g Z_A{}^{1/2} = 1 \,. \tag{3.17}$$

Thus the matrix-valued background field and field strength do not get renormalized. Write

$$Z_A = 1 + \hbar \delta Z_A^{(1)} + O(\hbar^2) \,, \tag{3.18}$$

$$Z_g = 1 + \hbar \delta Z_g^{(1)} + O(\hbar^2) \,. \tag{3.19}$$

Because of (3.17),

$$\delta Z_A^{(1)} = -2 \delta Z_g^{(1)} \,. \tag{3.20}$$

Write also,

$$\Lambda_B = \mu^{n-4} [\Lambda + \hbar \delta \Lambda^{(1)} + O(\hbar^2)] \,, \tag{3.21a}$$

$$\kappa_B = \mu^{n-4} [\kappa + \hbar \delta \kappa^{(1)} + O(\hbar^2)] \,, \tag{3.21b}$$

$$\alpha_{iB} = \mu^{n-4} [\alpha_i + \hbar \delta \alpha_i^{(1)} + O(\hbar^2)] \quad (i = 1,2,3) \,. \tag{3.21c}$$

The $O(\hbar)$ contribution to the pole part of $\Gamma^{(0)}$ is therefore

$$\mathrm{PP}\{\Gamma^{(0)}\} = \frac{\mu^{n-4}\hbar}{2g^2 C_2(G_{\mathrm{ad}})} \delta Z_g^{(1)} \int dv_x \operatorname{tr}(\hat{F}_{\mu\nu} \hat{F}^{\mu\nu}) - \mu^{n-4} \hbar \int dv_x (-2\delta\Lambda^{(1)} + \delta\kappa^{(1)} R + \delta\alpha_1^{(1)} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \delta\alpha_2^{(1)} R^{\mu\nu} R_{\mu\nu} + \delta\alpha_3^{(1)} R^2) \,. \tag{3.22}$$

The pole terms in the one-loop effective action (3.12) are seen to be of the same form as those appearing in the classical action. The theory is therefore renormalizable in a general space-time with counterterms

$$\delta Z_g^{(1)} = \tfrac{11}{3} g^2 C_2(G_{\mathrm{ad}}) \epsilon^{-1} \,, \tag{3.23a}$$

$$\delta \Lambda^{(1)} = 0 \,, \tag{3.23b}$$

$$\delta \kappa^{(1)} = 0 \,, \tag{3.23c}$$

$$\delta \alpha_1^{(1)} = \tfrac{13}{180} N \epsilon^{-1} \,, \tag{3.23d}$$

$$\delta \alpha_2^{(1)} = -\tfrac{22}{45} N \epsilon^{-1} \,, \tag{3.23e}$$

$$\delta \alpha_3^{(1)} = \tfrac{5}{36} N \epsilon^{-1} \,. \tag{3.23f}$$

In addition, from (3.20),

$$\delta Z_A^{(1)} = -\tfrac{22}{3} g^2 C_2(G_{\mathrm{ad}}) \epsilon^{-1} \tag{3.23g}$$

gives the renormalization of the background field.

B. The inclusion of fermions

Return now to the full expression (2.47) for the one-loop effective action. Consider first of all the pole part of the last term arising from the fermion loop:

$$\ln \operatorname{Det} S^{-1}(x,x') = \operatorname{Tr} \ln[-i(\hat{\not{D}} - M)] = \operatorname{Tr} \ln[-i(\hat{\not{D}} + M)] \,,$$

where we have used the fact that the Dirac trace of an odd number of γ matrices vanishes. (The second line follows by expanding $\ln[-i(\hat{\not{D}}-M)]$ in powers of M. Alternatively, introduce $(\gamma_5)^2=1$ where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$ on the right-hand side of $\text{Det}[-i(\hat{\not{D}}-M)]$ and commute one γ_5 through the $\hat{\not{D}}$ using $\{\gamma_5,\hat{\not{D}}\}=0$.) Therefore,

$$\ln \text{Det}S^{-1}(x,x')=\tfrac{1}{2}\text{Tr}\ln(-\hat{\not{D}}^2+M^2) . \quad (3.24)$$

[Because of (2.22), $[M,\hat{\not{D}}]=0$.]

Write

$$\hat{D}_\mu=\partial_\mu+\Gamma_\mu+\hat{A}_\mu , \quad (3.25)$$

where Γ_μ is the spin connection. Since Γ_μ is a multiple of the group identity, and $\hat{A}_\mu$ is a multiple of the unit spinor matrix, $[\Gamma_\mu,\hat{A}_\nu]=0$. Thus, we may write

$$W_{\mu\nu}=\mathscr{R}_{\mu\nu}+\hat{F}_{\mu\nu} , \quad (3.26)$$

where

$$\mathscr{R}_{\mu\nu}=\partial_\mu\Gamma_\nu-\partial_\nu\Gamma_\mu+[\Gamma_\mu,\Gamma_\nu] . \quad (3.27)$$

For Dirac spinors (see for example, DeWitt[6] with a change of sign due to curvature conventions[26])

$$\mathscr{R}_{\mu\nu}=\tfrac{1}{4}R_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma . \quad (3.28)$$

It then follows by contracting (3.4) with $\gamma^\mu\gamma^\nu$ that

$$\hat{\not{D}}^2=\hat{D}^2-\tfrac{1}{4}R+\tfrac{1}{2}\gamma^\mu\gamma^\nu\hat{F}_{\mu\nu} . \quad (3.29)$$

From (3.24)

$$\ln \text{Det}S^{-1}(x,x') =\tfrac{1}{2}\text{Tr}\ln(-\hat{D}^2+\tfrac{1}{4}R+M^2-\tfrac{1}{2}\gamma^\mu\gamma^\nu\hat{F}_{\mu\nu}) . \quad (3.30)$$

This is now in a form where (3.2) and (3.3) may be used, with

$$Q(x)=M^2+\tfrac{1}{4}R-\tfrac{1}{2}\gamma^\mu\gamma^\nu\hat{F}_{\mu\nu} , \quad (3.31)$$

$$W_{\mu\nu}=\tfrac{1}{4}R_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma+\hat{F}_{\mu\nu} . \quad (3.32)$$

The trace occurring in (3.2) is over both spinor and group indices; thus $\text{tr}I=4d_F$ where the factor of 4 comes from the Dirac trace. It is straightforward to show that

$$\text{PP}\{\ln \text{Det}S^{-1}(x,x')\}=\epsilon^{-1}\int dv_x[\tfrac{1}{72}d_FR^2-\tfrac{1}{45}d_FR^{\mu\nu}R_{\mu\nu}-\tfrac{7}{360}d_FR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+2\,\text{tr}(M^2)^2 +\tfrac{1}{3}R\,\text{tr}M^2-\tfrac{2}{3}\text{tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})] . \quad (3.33)$$

Note that the matrices appearing in the last term are those appropriate to the fermion representation, so that by (2.3),

$$\text{tr}(\hat{F}_{\mu\nu}\hat{F}^{\mu\nu})=-\frac{d_F}{N}C_2(G_F)g^2\hat{F}^a_{\mu\nu}\hat{F}^{a\mu\nu} . \quad (3.34)$$

Turn next to the first term in (2.47). Although, as we have already remarked, heat-kernel techniques are not directly applicable, we may instead proceed as follows. Use the identity

$$\ln \text{Det}(A+B)=\text{Tr}\ln(A+B)=\text{Tr}\ln A+\text{Tr}\ln(1+A^{-1}B)$$

and then expand the second term in powers of $A^{-1}B$ to give

$$\tfrac{1}{2}\ln \text{Det}(A+B)=\tfrac{1}{2}\text{Tr}\ln A-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\text{Tr}[(A^{-1}B)^n] . \quad (3.35)$$

We are interested here in

$$A=\Delta^{-1}{}^{ab}_{\mu\nu}(x,x') , \quad (3.36)$$

$$B=2g^2\hat{\bar{\psi}}(x)\gamma_\mu T^aS(x,x')T^b\gamma_\nu\hat{\psi}(x') . \quad (3.37)$$

The pole part of the first term in (3.35) is just that for the pure gauge theory given in the preceding section [see Eq. (3.10)].

The $n=1$ term in the summation in (3.35) involves

$$\text{Tr}(A^{-1}B)=2g^2\int dv_xdv_{x'}\Delta^{ab}_{\mu\nu}(x,x')\hat{\bar{\psi}}(x')\gamma^\nu T^bS(x',x)T^a\gamma^\mu\hat{\psi}(x) . \quad (3.38)$$

In order to evaluate this use Bunch and Parker's[1] curved—space-time momentum-space method which involves the introduction of a Riemannian normal coordinate system. Here, because we are considering a gauge theory, we must be more general and use a synchronous coordinate frame.[33] The result is, that if $G(x,x')$ satisfies

$$[-\hat{D}_x{}^2+Q(x)]G(x,x')=\delta(x,x') , \tag{3.39}$$

then

$$G(x,x')=\int \frac{d^nk}{(2\pi)^n} e^{ik\cdot y}\left[\frac{I}{k^2}+O(k^{-4})\right] , \tag{3.40}$$

where I is the identity matrix, and all indices have been suppressed. Here $x^\mu = x'^\mu + y^\mu$ with x'^μ the origin of the coordinate system. Higher-order contributions to (3.40) may be evaluated and involve curvature invariants and the background field.[34] These terms would be required at the two-loop level. Because $\hat{A}_\mu$ vanishes at x' in a synchronous frame, we may work with a vanishing background gauge field and put it back in at the end by invoking gauge invariance.

Writing $S(x',x)=-i(\hat{\not{D}}_{x'}+M)G(x',x)$, it follows that $G(x',x)$ satisfies (3.39) with $Q(x)=M^2$. Therefore,

$$S(x',x)=\int \frac{d^nk}{(2\pi)^n} e^{-ik\cdot y}\left[\frac{-i(i\not{k}+M)}{k^2}+O(k^{-3})\right] \tag{3.41}$$

in a synchronous coordinate frame. Also,

$$\Delta^{ab}_{\mu\nu}(x,x')=\int \frac{d^nk}{(2\pi)^n} e^{ik\cdot y}\left[\frac{\delta_{ab}\delta_{\mu\nu}}{k^2}+O(k^{-4})\right] . \tag{3.42}$$

It follows immediately from (3.41) and (3.42) that the divergent part of $\Delta^{ab}_{\mu\nu}(x,x')S(x',x)$ in a synchronous coordinate frame is identical to that in flat space-time and is given by

$$\mathrm{PP}\{\Delta^{ab}_{\mu\nu}(x,x')S(x',x)\}=-iI\delta_{ab}\delta_{\mu\nu}\epsilon^{-1}(\not{\partial}_y-2M)\delta(y) . \tag{3.43}$$

It may also be seen from (3.41) and (3.42) that all terms with $n\geq 2$ in the summation (3.35) do not contain any ultraviolet divergences.

The result in (3.43) may be written in the generally covariant form

$$\mathrm{PP}\{\Delta^{ab}_{\mu\nu}(x,x')S(x',x)\}=+iI\delta_{ab}g_{\mu\nu}(x')\epsilon^{-1}(\not{\nabla}'+2M)\delta(x,x') . \tag{3.44}$$

which reduces to (3.43) in normal coordinates. It then follows from (3.38) that

$$\mathrm{PP}\{\mathrm{Tr}(A^{-1}B)\}=4ig^2C_2(G_F)\epsilon^{-1}\int dv_x\hat{\bar{\psi}}(x)(\not{\nabla}-4M)\hat{\psi}(x) . \tag{3.45}$$

At this stage we now invoke the requirement that Γ_{DIV} contain only gauge-invariant quantities to put $\hat{A}$ back in:

$$\mathrm{PP}\{\mathrm{Tr}(A^{-1}B)\}=4ig^2C_2(G_F)\epsilon^{-1}\int dv_x\hat{\bar{\psi}}(x)(\hat{\not{D}}-4M)\hat{\psi}(x) . \tag{3.46}$$

From (3.35), (3.10), and (3.46) we therefore have

$$\begin{aligned}&\mathrm{PP}\{\ln\mathrm{Det}[\Delta^{-1ab}_{\mu\nu}(x,x')+2g^2\hat{\bar{\psi}}(x)\gamma_\mu T^a S(x,x')T^b\gamma_\nu\hat{\psi}(x')]\}\\&=\epsilon^{-1}\int dv_x[-\tfrac{1}{9}NR^2+\tfrac{43}{90}NR^{\mu\nu}R_{\mu\nu}-\tfrac{11}{180}NR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\tfrac{5}{3}C_2(G_{\mathrm{ad}})g^2\hat{F}^a_{\mu\nu}\hat{F}^{a\mu\nu}+2ig^2C_2(G_F)\hat{\bar{\psi}}(\hat{\not{D}}-4M)\hat{\psi}] .\end{aligned} \tag{3.47}$$

The complete pole part of the one-loop effective action, from (3.47), (3.11), and (3.33) is

$$\begin{aligned}\mathrm{PP}\{\Gamma^{(1)}\}=\epsilon^{-1}\int dv_x\{&-\tfrac{1}{72}(d_F+10N)R^2+\tfrac{1}{45}(d_F+22N)R^{\mu\nu}R_{\mu\nu}+\tfrac{1}{360}(7d_F-26N)R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\\&-2\,\mathrm{tr}(M^2)^2-\tfrac{1}{3}R\,\mathrm{tr}M^2+\tfrac{1}{6}g^2[11C_2(G_{\mathrm{ad}})-4(d_F/N)C_2(G_F)]\hat{F}^a_{\mu\nu}\hat{F}^{a\mu\nu}\\&+2ig^2C_2(G_F)\hat{\bar{\psi}}(x)(\hat{\not{D}}-4M)\hat{\psi}(x)\} .\end{aligned} \tag{3.48}$$

In addition to the renormalizations occurring in (3.13), (3.14) and (3.21), let

$$\hat{\bar{\psi}}_B(x)=\mu^{n/2-2}Z_F{}^{1/2}\hat{\bar{\psi}}(x)\ , \tag{3.49a}$$

$$\hat{\psi}_B(x)=\mu^{n/2-2}Z_F{}^{1/2}\hat{\psi}(x)\ , \tag{3.49b}$$

$$M_B=Z_M M\ , \tag{3.50}$$

where

$$Z_F=1+\hbar\delta Z_F^{(1)}+O(\hbar^2)\ , \tag{3.51}$$

$$Z_M=1+\hbar\delta Z_M^{(1)}+O(\hbar^2)\ . \tag{3.52}$$

Then the $O(\hbar)$ contribution to the pole part of $\Gamma^{(0)}$ is

$$\begin{aligned}\mathrm{PP}\{\Gamma^{(0)}\}=&-\tfrac{1}{2}\hbar\mu^{n-4}\delta Z_g^{(1)}\int dv_x \hat{F}^a_{\mu\nu}\hat{F}^{a\mu\nu}\\&-\hbar\mu^{n-4}\int dv_x[-2\delta\Lambda^{(1)}+\delta\kappa^{(1)}R+\delta\alpha_1^{(1)}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\delta\alpha_2^{(1)}R^{\mu\nu}R_{\mu\nu}+\delta\alpha_3^{(1)}R^2]\\&-i\hbar\mu^{n-4}\int dv_x\hat{\bar{\psi}}(x)[\delta Z_F^{(1)}\hat{\not{D}}-(\delta Z_F^{(1)}+\delta Z_M^{(1)})M]\hat{\psi}(x)\ .\end{aligned} \tag{3.53}$$

The divergent parts of the one-loop effective action (3.48) are of the same form as terms appearing in the original action. The theory is therefore renormalizable in curved space-time with the choice of counterterms,

$$\delta Z_g^{(1)}=\tfrac{1}{3}g^2\left[11C_2(G_{\mathrm{ad}})-4\frac{d_F}{N}C_2(G_F)\right]\epsilon^{-1}\ , \tag{3.54a}$$

$$\delta Z_A^{(1)}=-\tfrac{2}{3}g^2\left[11C_2(G_{\mathrm{ad}})-4\frac{d_F}{N}C_2(G_F)\right]\epsilon^{-1}\ , \tag{3.54b}$$

$$\delta\Lambda^{(1)}=\mathrm{tr}(M^2)^2\epsilon^{-1}\ , \tag{3.54c}$$

$$\delta\kappa^{(1)}=-\tfrac{1}{3}\,\mathrm{tr}M^2\epsilon^{-1}\ , \tag{3.54d}$$

$$\delta\alpha_1^{(1)}=\tfrac{1}{360}(7d_F-26N)\epsilon^{-1}\ , \tag{3.54e}$$

$$\delta\alpha_2^{(1)}=\tfrac{1}{45}(d_F+22N)\epsilon^{-1}\ , \tag{3.54f}$$

$$\delta\alpha_3^{(1)}=-\tfrac{1}{72}(d_F+10N)\epsilon^{-1}\ , \tag{3.54g}$$

$$\delta Z_F^{(1)}=2g^2C_2(G_F)\epsilon^{-1}\ , \tag{3.54h}$$

$$\delta Z_M^{(1)}=6g^2C_2(G_F)\epsilon^{-1}\ . \tag{5.54i}$$

The result (3.54a) for the coupling-constant renormalization agrees with that of the standard references.[35]

IV. DISCUSSION

In the preceding sections the renormalization of a non-Abelian gauge theory containing fermions has been presented at the one-loop level in a curved space-time. The result in Sec. III used the momentum-space method of Ref. 1 and therefore required attention to be restricted to trivial topologies. This means that the proof does not immediately extend to the case of twisted spinor fields which may exist.[36] It is likely that this restriction may be removed in nonsimply connected space-times whose covering space is topologically R^4 by writing the heat kernel as an image sum of heat kernels in the covering space.[37] As with the free-field case, it would be expected that only the direct contributions to the image sum (just the results used in Sec. III) would lead to divergences in the one-loop effective action. This conclusion is supported by the result of Ford[38] who calculated the vacuum polarization for both twisted and untwisted QED in $S^1\times R^3$. At the higher-loop level, the result of Banach[39] is relevant.

There is nothing in principle to prevent the inclusion of scalar fields into the analysis of this paper although the details of the calculation would become involved. With scalar self-interactions present it would be found necessary in general to include nonminimal terms involving the curvature provided that they did not violate gauge invariance. This may be seen from the RQ term in Eq. (3.3) which enters into the divergent part of the effective action, since Q [see (3.1)] will then involve the background scalar field. As a gauge-fixing term, the generalized R_ξ gauge of Shore[31] would be a convenient choice.

It is also possible to use the momentum-space technique of Ref. 1 even in flat space-time for the background-field propagators. This allows the exploitation of the use of the background-field propagators as fully as possible, although since an expansion in powers of the background field results, the details would be similar to those in Abbott's[15] paper.

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[1]T. S. Bunch and L. Parker, Phys. Rev. D 20, 2499 (1980).
[2]T. S. Bunch, Ann. Phys. (N.Y.) 131, 118 (1981).
[3]D. J. Toms, Phys. Rev. D 26, 2713 (1982).
[4]M. Lüscher, Ann. Phys. (N.Y.) (to be published).
[5]P. Panangaden, Phys. Rev. D 23, 1735 (1981).
[6]B. S. DeWitt, *The Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[7]J. Honerkamp, Nucl. Phys. B48, 269 (1972).
[8]G. 't Hooft, in Acta Univ. Wratislav. 38 (1975).
[9]B. S. DeWitt, in *Quantum Gravity II,* edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford University Press, London, 1981).
[10]D. G. Boulware, Phys. Rev. D 23, 389 (1981).
[11]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972); C. G. Bollini and J. J. Giambiagi, Nuovo Cimento 12B, 20 (1972); J. F. Ashmore, Lett. Nuovo Cimento 4, 289 (1972).
[12]J. S. Dowker and R. Critchley, Phys. Rev. D 13, 3224 (1976); 16, 3390 (1977).
[13]S. W. Hawking, Commun. Math. Phys. 55, 133 (1977).
[14]G. 't Hooft, Nucl. Phys. B62, 444 (1973).
[15]L. Abbott, Nucl. Phys. B185, 189 (1981).
[16]I. Jack and H. Osborn, DAMTP Report No. 82/10, 1982 (unpublished).
[17]G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincaré A 20, 69 (1974).
[18]D. G. Boulware, in *Recent Developments in Gravitation,* edited by M. Levy and S. Deser (Plenum, New York, 1979).
[19]I. T. Drummond and G. M. Shore, Ann. Phys. (N.Y.) 117, 89 (1979).
[20]G. M. Shore, Phys. Rev. D 21, 2226 (1980).
[21]G. M. Shore, Ann. Phys. (N.Y.) 128, 376 (1980).
[22]R. Utiyama, Phys. Rev. 125, 7127 (1962).
[23]S. Ichinose and M. Omote, Tsukuba Report No. UTHEP-84, 1981 (unpublished).
[24]C. Itzykson and J. Zuber, *Quantum Field Theory* (McGraw Hill, New York, 1980).
[25]S. Coleman, in *The Whys of Subnuclear Physics,* proceedings of the International School of Subnuclear Physics, Erice, 1977, edited by A. Zichichi (Plenum, New York, 1979), p. 805.
[26]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[27]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967).
[28]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964); C. De Dominicis and P. C. Martin, J. Math. Phys. 5, 14 (1964).
[29]F. Berezin, *The Method of Second Quantization* (Academic, New York, 1966).
[30]P. B. Gilkey, J. Diff. Geom. 10, 601 (1975); Compos. Math. 38, 201 (1979).
[31]G. Shore, Ann. Phys. (N.Y.) 137, 262 (1981).
[32]G. 't Hooft, Nucl. Phys. B61, 455 (1973); J. C. Collins and A. J. Macfarlane, Phys. Rev. D 10, 1201 (1974).
[33]M. Atiyah, R. Bott, and V. Patodi, Invent. Math. 19, 279 (1973).
[34]D. J. Toms (unpublished).
[35]D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, *ibid.* 30, 1346 (1973).
[36]C. J. Isham, Proc. R. Soc. London A364, 591 (1978).
[37]R. Banach and J. S. Dowker, J. Phys. A 12, 2527 (1979).
[38]L. H. Ford, Phys. Rev. D 21, 933 (1980).
[39]R. Banach, J. Phys. A 13, L365 (1980).