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Quantization of dynamical systems and stochastic control theory

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In the general framework of stochastic control theory we introduce a suitable form of stochastic action associated to the controlled process. A variational principle gives all the main features of Nelson's stochastic mechanics. In particular, we derive the expression for the current velocity field as the gradient of the phase action. Moreover, the stochastic corrections to the Hamilton-Jacobi equation are in agreement with the quantum-mechanical form of the Madelung fluid (equivalent to the Schrödinger equation). Therefore, stochastic control theory can provide a very simple model simulating quantum-mechanical behavior.

I. INTRODUCTION

Methods based on the theory of stochastic processes have been very useful for the investigation of the physical properties of quantum-mechanical models and quantum field theory (see, for example, Refs. 1—7 and references quoted therein). In fact, in the last few years there have been continuous efforts to enlarge the overlapping area between quantum mechanics and probability theory, not only from the point of view of methods and techniques, but also with reference to physical interpretation (see, for example, Refs. 8 and 9). Moreover, the merging of quantum mechanics and probability theory stimulated research aiming at the foundations of a new "quantum probability" theory (see, for example, Ref. 10 and previous references quoted therein).

It is very well known that there is a close relation between classical mechanics and deterministic control theory.[11] In particular, the Hamilton-Jacobi equation can be understood as a programming equation for an optimal control problem, as will also be recalled in Sec. II. Therefore, on the basis of the previous considerations, it is very natural to investigate possible connections between quantum mechanics and stochastic control theory. Here a typical difficulty arises. In fact, the programming equations of the standard versions of stochastic control theory are not time-reversal invariant in general and describe a typical parabolic behavior. On the other hand, quantum mechanics is time-reversal invariant and the resulting candidate for a programming equation is the Madelung form of the Hamilton-Jacobi equation with quantum-mechanical corrections. In this case the programming equation describes some kind of hyperbolic behavior, with no essential dissipation involved.

But it is easy to see that a proper choice of the stochastic action allows a generalization of the classical case. Then it is possible to reach the objective of connecting stochastic control theory and quantum-mechanical behavior. The purpose of this paper is to give a complete and concise account of the emerging general structure.

For related work on the connections between quantum mechanics and stochastic control theory, we refer to Refs. 12—14, where different ideas and techniques are developed. The analogy between nonlinear filtering and quantum physics is investigated in Ref. 15. The relevance of stochastic control theory for classical mechanics is stressed in Ref. 16.

The paper is organized as follows: Section II is dedicated to a brief review of the classical case, showing the connections between classical mechanics and deterministic optimal control theory. For a more detailed and precise treatment we refer to Ref. 11. Our considerations are introduced in order to familiarize the reader with the general structure of the problems in control theory and prepare the grounds for a natural generalization to the stochastic case. In particular, we find it of some interest to recall that, even in classical mechanics, the Hamilton-Jacobi principal function S can be introduced as a field of Lagrangian multipliers, connected with a constrained variational problem, where initial and final densities of the classical fluid in configuration space are kept constant.

In Sec. III we introduce the general frame of the stochastic control theory employed in the following. In particular, we recall some of the useful formulas connected with the solutions of stochastic differential equations.

Section IV contains the definition of the stochastic action associated to the general controlled pro-

cess. Various different action functionals are introduced and their connections are pointed out. We also find it convenient to give the explicit expression for the variations of the action functionals as a consequence of a change in the controlling drift field.

The main results of this paper are found in Sec. V. In fact, we establish two equivalent versions of the stochastic variational principle. In one version the variations of the controlling drift field are constrained in such a way that the density of the controlled process is kept fixed at initial and final times. In the other version only the initial density is fixed and no constraints are put on the control field, but additional terms are introduced in the action functional. The equivalence of the two principles allows the interpretation of the phase function S as a field of Lagrangian multipliers connected with the constraint of fixed density.

The stochastic variational principle then allows us to derive the whole structure of stochastic mechanics.[17,18] In particular, we show that the expression of the mean velocity field as the gradient of the phase action follows easily as a consequence of the variational principle, as in the classical case. Moreover, the stochastic corrections to the classical Hamilton-Jacobi programming equation are shown to reproduce correctly the quantum corrections, provided the strength of the disturbing noise in the control equation is properly adjusted as a function of the Planck constant. Therefore the Schrödinger equation can be interpreted as a consequence of the stochastic variational principle.

Finally, Sec. VI deals with possible further development and applications of the theory.

II. CLASSICAL MECHANICS AND DETERMINISTIC CONTROL THEORY

We only recall some basic facts. For a more complete treatment we refer to Ref. 11. Our account will be very simple and pedagogical. In fact, we consider this section as a preparation for the more complicated case of stochastic control theory developed later, where all essential features are more or less direct generalizations of the classical case.

Let us consider a dynamical system with configuration space $\mathcal{M}$. For the sake of simplicity we assume $\mathcal{M}\equiv\mathbb{R}^n$, but our considerations can be easily generalized to the case where $\mathcal{M}$ is a general Riemannian manifold, with metric given by the kinetic energy, as will be shown elsewhere.[19]

We consider trajectories of the type

$$[t_0,t_1]\ni t\to q(t)\in\mathcal{M} \tag{1}$$

evolving in time according to a feedback control law enforced by the differential equation

$$\dot{q}(t)=v(q(t),t)\ . \tag{2}$$

The control field v is some given time-dependent vector field on $\mathcal{M}$. Here

$$v(\cdot,t)\in\mathbb{R}^n\ ,\quad t_0\le t\le t_1\ .$$

Standard conditions of regularity on v (see, for example, Refs. 20 and 21) assure existence and uniqueness for the solution of (2) for some given condition at time t', $t_0\le t'\le t_1$,

$$q(t')=x'\ . \tag{3}$$

In some cases it is relevant to show explicitly in the notation the conditions (3). Then we write $q(t;x',t')$ for the solution of (2) satisfying (3).

The following properties hold:

$$q(t;x,t)=x\ ,\quad q(t;q(t';x,t),t')=x\ ,$$
$$q(t;q(t';x'',t''),t')=q(t;x'',t'')\ . \tag{4}$$

For sufficiently regular fields $F(\cdot,t)$ on $\mathbb{R}^n$, we define the substantial time derivative through

$$(DF)(x,t)=\lim_{\Delta t\to 0}(\Delta t)^{-1}[F(q(t+\Delta t;x,t),t+\Delta t)-F(x,t)]\ ; \tag{5}$$

clearly it is enough to know $q(t')$ for t' in a small interval containing t in order to evaluate (5).

By taking into account that

$$F(q(t+\Delta t;x,t),t+\Delta t)-F(x,t)=(\partial_t F)(x,t)\Delta t+(\nabla F)(x,t)\cdot[q(t+\Delta t;x,t)-x]+O((\Delta t)^2)\ ,$$
$$q(t+\Delta t;x,t)-x=v(x,t)\Delta t+O((\Delta t)^2)\ , \tag{6}$$

we immediately have

$$(DF)(x,t)=(\partial_t F)(x,t)+v(x,t)\cdot(\nabla F)(x,t)\ . \tag{7}$$

In particular, for the specification field

$$\hat{q}(x,t)\equiv q(t;x,t)=x\ , \tag{8}$$

we have

$$(D\hat{q})(x,t)=v(x,t)\ , \tag{9}$$

while for the generic trajectory $q(t';x,t)$ we have

$$(Dq)(t';x,t)=0\ , \tag{10}$$

as a consequence of the control equation (2). In fact,

$$(Dq)(t';x,t)=\lim_{\Delta t\to 0}(\Delta t)^{-1}[\,q(t';q(t+\Delta t;x,t),t+\Delta t) - q(t';x,t)]\,, \tag{11}$$

and (10) follows from the last of (4).

From the definition (5), we also have the transport equation

$$F(q(t_1),t_1)-F(q(t_0),t_0)=\int_{t_0}^{t_1}(DF)(q(t),t)dt\,, \tag{12}$$

where $q(t)$ is a general trajectory controlled by v with end points $q(t_0)$ at t_0 and $q(t_1)$ at t_1.

We introduce the canonical Lagrangian

$$\begin{aligned}&\mathscr{L}:T\mathscr{M}\to\mathsf{R}\,,\\ &(x,v)\to\mathscr{L}(x,v)=\tfrac{1}{2}mv^2-V(x)\,,\end{aligned} \tag{13}$$

where $T\mathscr{M}$ is the tangent bundle on $\mathscr{M}$, $x\in\mathscr{M}$, and $v\in T\mathscr{M}_x$ (the tangent space to $\mathscr{M}$ in x). Here we have simply $\mathscr{M}=\mathsf{R}^n$, $T\mathscr{M}=\mathsf{R}^n\times\mathsf{R}^n$, $x\in\mathsf{R}^n$, and $v\in\mathsf{R}^n$.

Then the action $-I$ spent by the controller, while moving the point from time t, starting at x, to time t_1, is defined as

$$-I(x,t;t_1;v)=\int_t^{t_1}\mathscr{L}(q(t';x,t)\,, v(q(t';x,t),t'))dt'\,, \tag{14}$$

for $t_0\le t\le t_1$. Notice that I in (14) depends only on the specification of v in the time interval $[t,t_1]$. The following additivity property holds:

$$I(x_0,t_0;t_1,v)=I(x_0,t_0;t;v)+I(q(t;x_0,t_0),t;t_1;v) \tag{15}$$

for $t_0\le t\le t_1$.

It is very well known that action functionals of this type find applications in the classical calculus of variations, where usually some additional conditions are enforced for the end points of trajectories. It is therefore also convenient to introduce action functionals with additional end-point contributions, which act as a kind of Lagrangian multipliers, as will be clear later.

For some given smooth field $S_1(\cdot)$ on $\mathscr{M}$, we define

$$J(x,t;t_1,S_1;v)=I(x,t;t_1;v)+S_1(q(t_1;x,t)) \tag{16}$$

for $t_0\le t\le t_1$. Then we have the property

$$J(x_0,t_0;t_1,S_1;v)=J(x_0,t_0;t;S_t;v)\,, \tag{17}$$

where $t_0\le t\le t_1$ and the field S_t is given by

$$S_t(x)\equiv J(x,t;t_1,S_1;v)\,. \tag{18}$$

When there is no danger of confusion we keep only the specification $(x,t;v)$ in (14) and (16) and suppress the dependence on t_1,S_1 in the notations. As a consequence of (7) and (10), we have that I and J satisfy the following transport equations with the appropriate final boundary conditions:

$$(DI)(x,t;v)=\mathscr{L}(x,v(x,t))\,,\quad I(\cdot,t_1)=0\,, \tag{19}$$

$$(DJ)(x,t;v)=\mathscr{L}(x,v(x,t))\,,\quad J(\cdot,t_1)=S_1(\cdot)\,. \tag{20}$$

The basic laws of classical mechanics, i.e., the Newton second principle of dynamics, or the equivalent Hamilton-Jacobi equation, can be easily established through variational principles by exploiting the functionals I or J.

Consider, for example, J. In addition to the control $v(\cdot,t)$, $t_0\le t\le t$ appearing in (14) and (16), we introduce also another control $v'(\cdot,t)$, $t_0\le t\le t_1$, and call D', J', and $q'(t;x_0,t_0)$ the corresponding substantial derivative, action functional, and controlled trajectory. Then we have

$$\begin{aligned}D'(J'-J)&=\mathscr{L}(x,v'(x,t))-\mathscr{L}(x,v(x,t))+(D-D')J\\ &=\tfrac{1}{2}m(v'^2-v^2)+(v-v')\cdot\nabla J\,.\end{aligned} \tag{21}$$

Let us now integrate along $q'(t;x_0,t_0)$, $t_0\le t\le t_1$; by exploiting the transport equation (12) and the common boundary condition in the second equation in (20) for J and J', we have

$$J(x_0,t_0;v')-J(x_0,t_0;v)=\tfrac{1}{2}m\int_{t_0}^{t_1}\left[(v'-v)\cdot\left[\frac{2\nabla J}{m}-v'-v\right]\right](q'(t;x_0,t_0),t)dt\,. \tag{22}$$

Now we exploit this basic formula along two lines. First of all let us define

$$\delta v(x,t)=v'(x,t)-v(x,t) \tag{23}$$

and consider in (22) only first-order variations in δv. Then we have

$$(\delta J)(x_0,t_0;v)=m\int_{t_0}^{t_1}\left[\delta v\cdot\left[\frac{\nabla J}{m}-v\right]\right](q(t;x_0,t_0),t)dt\,. \tag{24}$$

Therefore, if we assume that under any small variation of the control $v \to v + \delta v$ the variation of J is zero, then we must necessarily have

$$v(x,t) = \frac{1}{m}(\nabla J)(x,t;v) , \tag{25}$$

for any point x reached by the controlled trajectory at time t, $t_0 \le t \le t_1$. Therefore if we impose $\delta J = 0 \ \forall x_0$, then (25) must hold everywhere.

Formula (24) shows the typical structure of the classical variational problem. In fact, we can impose $\delta J(x_0,t_0;v)=0$ for $x_0 \in \mathcal{O}$, where $\mathcal{O}$ is some open set of the configuration space $\mathcal{M}$. Then the condition (25) will hold in the casual shadow of $\mathcal{O}$, i.e., for all points x which can be reached at time t by trajectories controlled by v and starting in $\mathcal{O}$ at time t_0. In the next sections it will be apparent that this typical "fibration" of the classical variational problem does not carry through to the stochastic case, where some average procedure will be necessary in order to introduce action functionals.

It is clear that the gradient condition (25) implies the Hamilton-Jacobi equation. In fact, if we call S the functional J corresponding to the stationarizing control (25),

$$S(x,t) \equiv J\left[x,t;\frac{\nabla J}{m}\right] , \tag{26}$$

then we must necessarily have

$$(\partial_t S)(x,t) + \frac{1}{2m}(\nabla S)^2(x,t) + V(x) = 0 , \tag{27}$$

with the boundary condition

$$S(\cdot,t_1) = S_1(\cdot) , \tag{28}$$

following from the second equation in (20). It is important to recall that (27) is equivalent to the second principle of dynamics in the following hydrodynamical form. Define the acceleration field through

$$\begin{aligned} a(x,t) &= (Dv)(x,t) \\ &= (\partial_t v)(x,t) + (v \cdot \nabla v)(x,t) . \end{aligned} \tag{29}$$

Notice that for $v = \nabla S/m$, (29) reduces to

$$a(x,t) = \frac{1}{m}\nabla\left[\partial_t S + \frac{1}{2m}(\nabla S)^2\right] . \tag{30}$$

Therefore (27) implies

$$a(x,t) = -\frac{1}{m}(\nabla V)(x) , \tag{31}$$

and this is the Newton equation in local form. Vice versa if (30) and (31) hold, then

$$\nabla\left[\partial_t S + \frac{1}{2m}(\nabla S)^2 + V\right] = 0 . \tag{32}$$

Therefore for some $\alpha(t)$ we must have

$$\partial_t S + \frac{1}{2m}(\nabla S)^2 + V = \alpha(t) . \tag{33}$$

Putting

$$\hat{S}(x,t) = S(x,t) - \int_t^{t_1} \alpha(t')dt' , \tag{34}$$

we have that $\hat{S}$ satisfies (27) and (28), while still $v = \nabla \hat{S}/m$.

The second way of exploiting formula (22) leads to considerations of optimal control theory. In fact, assume that a solution S exists for the Hamilton-Jacobi equation (27) with the boundary condition (28). Consider (22) with v given by the Hamilton-Jacobi control $v = \nabla S/m$, so that $J = S$, and for a generic v'. Writing v in place of v' for convenience, we have

$$\begin{aligned} &J(x_0,t_0;v) - S(x_0,t_0) \\ &\quad = -m\int_{t_0}^{t_1} (v - \nabla S/m)^2(q(t;x_0,t_0),t)dt . \end{aligned} \tag{35}$$

Since the right-hand side is negative unless v satisfies $v = \nabla S/m$, we have that the Hamilton-Jacobi control is optimal in the sense that

$$J(x_0,t_0;v) \le S(x_0,t_0) , \quad \forall v, x_0 \in \mathcal{M} , \tag{36}$$

while

$$J(x_0,t_0;\nabla S/m) = S(x_0,t_0) . \tag{37}$$

It is also possible to show that the field S_1, appearing in the definition of the action functional J in (16), can be interpreted as a field of Lagrangian multipliers, corresponding to a new variational problem where the action functional now is I, defined in (14), but the variations of the control field are not free but subject to suitable constraints. We give some details about this rather remarkable fact, which holds also in the stochastic case, as will be shown in Sec. V.

First of all let us introduce a kind of hydrodynamic picture in classical mechanics, also in reference to the control problem. Instead of considering single trajectories we consider clouds of particles moving under the action of the control field. Therefore we assume that some density $\rho_0(\cdot)$ is given at the initial time t_0 on the configuration space. By acting on each single particle the control field will let the density $\rho(\cdot,t)$ evolve in time according to the continuity equation

$$(\partial_t \rho)(x,t) = -\nabla \cdot [\rho(x,t)v(x,t)] \tag{38}$$

with initial condition

$$\rho(\cdot,t_0)=\rho_0(\cdot)\ . \tag{39}$$

The connection between the fluid picture and the particle picture is easily established by introducing the transition "probability density" from time t' to time t defined by

$$p(x,t;x',t')=\delta(x-q(t;x',t'))\ . \tag{40}$$

The δ function on the right-hand side assures the deterministic motion of the particles forming the fluid. Notice that (40) is meaningful also for $t\leq t'$. The deterministic transition probability (40) satisfies the forward equation

$$(\partial_t p)(x,t;x',t')+\nabla\cdot[p(x,t;x',t')v(x,t)]=0 \tag{41}$$

with boundary condition

$$\lim_{t\to t'} p(x,t;x',t')=\delta(x-x')\ , \tag{42}$$

and the backward equation

$$(D'p)(x,t;x',t')=0\ , \tag{43}$$

where D', defined in (7), acts on the (x',t') variables. The following identity of the Kolmogorov type is also easily checked as a consequence of (4):

$$p(x,t;x'',t'')=\int p(x,t;x',t')p(x',t';x'',t'')dx' \tag{44}$$

with no restriction on the ordering of t,t',t''. Finally, the evolution of the density can be easily expressed in the form

$$\rho(x,t)=\int p(x,t;x',t')\rho(x',t')dx'\ . \tag{45}$$

Variations of p and ρ under a change of the control variable $v\to v'=v+\delta v$ can be easily found starting from (43). In fact, by using the same method as given by (21) and (22), we find

$$p'(x_1,t_1;x_0,t_0)-p(x_1,t_1;x_0,t_0)=\int_{t_0}^{t_1}\int \nabla p'(x_1,t_1;x,t)\cdot(v'-v)(x,t)p(x,t;x_0,t_0)dx\,dt\ , \tag{46}$$

where p and p' refer to the controls v and v', respectively. By exploiting (45) we have also the variation of the density in the form

$$\rho'(x_1,t_1)-\rho(x_1,t_1)=\int_{t_0}^{t_1}\int \nabla p'(x_1,t_1;x,t)\cdot(v'-v)(x,t)\rho(x,t)dx\,dt\ . \tag{47}$$

Now we are ready to establish a constrained variational principle. We consider $I(x_0,t_0;t_1;v)$ and its first-order variation

$$\begin{aligned}(\delta I)(x_0,t_0;t_1;v)&=m\int_{t_0}^{t_1}[\delta v\cdot(\nabla I/m-v)](q(t;x_0,t_0),t)dt\\&=m\int_{t_0}^{t_1}\int[\delta v\cdot(\nabla I/m-v)](x,t)p(x,t;x_0,t_0)dx\,dt\ ,\end{aligned} \tag{48}$$

which is found by following the same method leading to (24). Notice that I *does not* contain the boundary term with S_1, as J in (16). It is also convenient to introduce the average

$$A(t_0,t_1;\rho_0;v)=-\int I(x_0,t_0;t_1;v)\rho_0(x_0)dx_0\ . \tag{49}$$

Then (48) and (45) give

$$(\delta A)(t_0,t_1;\rho_0;v)=-m\int_{t_0}^{t_1}\int[\rho\delta v\cdot(\nabla I/m-v)](x,t)dx\,dt\ . \tag{50}$$

Now we explore the consequences of the variational principle

$$(\delta A)(t_0,t_1;\rho_0;v)=0 \tag{51}$$

under the following constraint for δv:

$$(\delta\rho)(x_1,t_1)=0\ ,\quad \forall x_1\ . \tag{52}$$

From (47) we have

$$\delta\rho(x_1,t_1)=\int_{t_0}^{t_1}\int\nabla p(x_1,t_1;x,t)\cdot(\rho\delta v)(x,t)dx\,dt\ . \tag{53}$$

Then the constraint (52) obliges $\rho\delta v$ to be "orthogonal" in

$$L^2(\mathbb{R}^n\times[t_0,t_1],dx\,dt)$$

to all functions of the type

$$\nabla p(x_1,t_1;x,t)\ ,\quad \forall x_1\ . \tag{54}$$

Therefore from the condition (51) and the identity (50), we cannot conclude that $v=\nabla I/m$ must be true but only that the difference $v-\nabla I/m$ must be

in the linear hull spanned by functions of the type (54) for all values of x_1. Therefore there must be some $S_1(\cdot)$ such that

$$mv(x,t)=\nabla I(x,t;v) + \nabla \int S_1(x_1)p(x_1,t_1;x,t)dx_1 \ . \quad (55)$$

Therefore, if we *define,* for v given by (55),

$$S(x,t)=I(x,t;v)+\int S_1(x_1)p(x_1,t_1;x,t)dx_1 \ , \quad (56)$$

then the constrained variational problem (51) and (52) enforces the Hamilton-Jacobi form of the control

$$v(x,t)=(\nabla S)(x,t)/m \ , \quad (57)$$

while it is simple to check that Eq. (27) holds for S, with the boundary condition (28). In conclusion, we have seen that the constrained variational problem (51) and (52) is equivalent to the variational problem based on J. They both lead to the Hamilton-Jacobi structure. The principal function S can be interpreted as a field of Lagrangian multipliers associated to the constraints of fixed density. From this point of view S is the field conjugated to ρ, in agreement with the fluid picture of classical mechanics as expressed by Eqs. (27), (38), and (57) (the Hamilton-Jacobi fluid).

Let us remark that the constrained problem corresponds to making stationary the average action A of (49), under variations of v, while keeping fixed the boundary densities $\rho(\cdot,t_0)$ and $\rho(\cdot,t_1)$. Since it is always possible to smoothly interpolate many $\rho(\cdot,t)$ between fixed end-point densities, there will be many possible choices of v compatible with the constraints, as follows from (38) interpreted now as a condition on v in correspondence of some interpolating ρ.

Let us conclude this section with a few remarks related to obvious extensions of the methods recalled here. Clearly it is possible to consider more general Lagrangians than those of type (13). Of particular physical interest is the introduction of terms corresponding to generalized Lorentz forces,

$$\mathcal{L}\to\mathcal{L}+A(x,t)\cdot v(x,t) \ . \quad (58)$$

All considerations developed in this section can be extended to (58) with obvious modifications. For example, now the Hamilton-Jacobi condition (57) would read

$$v(x,t)=[A(x,t)+(\nabla S)(x,t)]/m \ . \quad (59)$$

Moreover, it is possible, by time-reversal invariance, to consider variational problems where, for example, J is modified through the introduction of terms related to the initial time t_0. In the following sections the whole scheme will be generalized to the stochastic case.

III. KINEMATICS OF THE CONTROLLED STOCHASTIC THEORY

In stochastic theory we promote the classical variable $q(t)$ to a diffusion Markov stochastic process, still denoted by $q(t)$, taking values on the configuration space $\mathcal{M}=\mathbb{R}^n$, with a given initial distribution $\rho_0(\cdot)$ at time t_0. The control equation (2) is replaced by the Itô stochastic differential equation[22]

$$dq(t)=v_{(+)}(q(t),t)dt+dw(t) \ , \quad (60)$$

where now $v_{(+)}(\cdot,t)$ plays the role of feedback control field and $w(t)$ is a Brownian motion on $\mathbb{R}^n$ with

$$E(dw(t)\,|\,q(t)=x)=0 \ , \quad (61)$$

$$E(dw(t)dw(t)\,|\,q(t)=x)=2\nu I\,dt \quad (dt>0) \ . \quad (62)$$

In (62), I is the unit matrix corresponding to the Euclidean metric on $\mathbb{R}^n$ and ν is some fixed diffusion coefficient. We refer to Refs. 23 and 19 for the generalization to nonflat Riemann manifolds.

In (61) and (62), $E(A/B)$ are conditional expectations denoting the average of the random variable A under the condition that the event B happens. In general, we denote by $E(A)$ the (unconditional) expectation of the random variable A. For the density $\rho(\cdot,t)$ of the process we have

$$E(F(q(t),t))=\int F(x,t)\rho(x,t)dx \ , \quad (63)$$

while (60) implies the forward diffusion equation

$$(\partial_t\rho)(x,t)=-\nabla\cdot[\rho(x,t)v_{(+)}(x,t)]+\nu(\Delta\rho)(x,t) \quad (64)$$

with initial condition

$$\rho(\cdot,t_0)=\rho_0(\cdot) \ . \quad (65)$$

For the transition probability density

$$p(x,t;x',t') \ , \quad t\geq t' \ ,$$

we have the forward Fokker-Planck equation

$$(\partial_t p)(x,t;x',t')=-\nabla\cdot(pv_{(+)})+\nu\Delta p \quad (66)$$

with initial condition

$$\lim_{t\to t'+} p(x,t;x',t')=\delta(x-x') \ . \quad (67)$$

Notice that p depends only on $v_{(+)}$ (not on ρ_0). The density is propagated in time through

$$\rho(x,t)=\int p(x,t;x',t')\rho(x',t')dx' , \quad t\geq t' . \tag{68}$$

It is also convenient to introduce

$$v_{(-)}(x,t)=v_{(+)}-2\nu\nabla\rho/\rho , \tag{69}$$

and define the osmotic velocity[17]

$$u(x,t)=\tfrac{1}{2}(v_{(+)}-v_{(-)})(x,t)=\nu\nabla\rho/\rho =2\nu\nabla R , \tag{70}$$

where

$$\rho(x,t)=\exp[2R(x,t)] . \tag{71}$$

The current velocity is defined as

$$v(x,t)=\tfrac{1}{2}(v_{(+)}+v_{(-)})(x,t) . \tag{72}$$

Then the diffusion equation (64) can be equivalently written either as a continuity equation

$$(\partial_t\rho)(x,t)=-\nabla\cdot(\rho v) \tag{73}$$

or in the backward form

$$(\partial_t\rho)(x,t)=-\nabla\cdot(\rho v_{(-)})-\nu\Delta\rho . \tag{74}$$

It is also useful to introduce the linear diffusion operators

$$\begin{aligned}&(\mathscr{D}_{(\pm)}F)(x,t)=-\nabla\cdot(Fv_{(\pm)})\pm\nu\Delta F ,\\ &(\mathscr{D}F)(x,t)=-\nabla\cdot(Fv) ,\\ &(\delta\mathscr{D}F)(x,t)=-\nabla\cdot(Fu)+\nu\Delta F ,\\ &\mathscr{D}=\tfrac{1}{2}(\mathscr{D}_{(+)}+\mathscr{D}_{(-)}) ,\\ &\delta\mathscr{D}=\tfrac{1}{2}(\mathscr{D}_{(+)}-\mathscr{D}_{(-)}) .\end{aligned} \tag{75}$$

Then (64), (74), (73), and (70) can be written as

$$\begin{aligned}&(\partial_t\rho)(x,t)=(\mathscr{D}_{(\pm)}\rho)(x,t)=(\mathscr{D}\rho)(x,t) ,\\ &(\delta\mathscr{D}\rho)(x,t)=0 .\end{aligned} \tag{76}$$

Notice that we could also assume either v or $v_{(-)}$ as control fields. In the first case, ρ would be defined through (73) with initial condition (65). Then (70) and (72) would give $v_{(+)}$ and $v_{(-)}$ in the form

$$v_{(\pm)}(x,t)=v\pm u . \tag{77}$$

In the second case, ρ would be defined through (74) and (65), and (70) and (72) would give v and $v_{(+)}$ as functions of $v_{(-)}$ and ρ. We prefer to keep $v_{(+)}$ in the central position as a control field because it appears in the basic control law (60) and in the following Itô-Girsanov formula.

Standard theorems[22,11] give the sufficient regularity conditions on $v_{(+)}$ which assure existence and uniqueness for the solutions of (60). More general control fields $v_{(+)}$ can be introduced, if the solution of (60) is defined through the Itô-Girsanov formula (see, for example, Theorem V 10 2 of Ref. 11).

It is useful to introduce mean forward and backward derivatives $D_{(\pm)}$, which generalize the substantial derivative D in (7), according to the standard definitions[17,18]

$$(D_{(\pm)}F)(x,t)=\pm\lim_{\Delta t\to 0^+}(\Delta t)^{-1}E(F(q(t\pm\Delta t),t\pm\Delta t)-F(q(t),t)\,|\,q(t)=x) . \tag{78}$$

It can easily be shown that

$$(D_{(\pm)}\hat{q})(x,t)=v_{(\pm)}(x,t) , \tag{79}$$

where $\hat{q}$ is the specification field $\hat{q}(x,t)=x$, already introduced in (8). For a generic field $F(\cdot,t)$, under appropriate conditions of smoothness, we have

$$(D_{(\pm)}F)(x,t)=(\partial_tF)(x,t)+v_{(\pm)}(x,t)\cdot(\nabla F)(x,t)\pm\nu(\Delta F)(x,t) . \tag{80}$$

We can also introduce the transport operators

$$(\mathscr{A}_{(\pm)}F)(x,t)=v_{(\pm)}\cdot\nabla F\pm\nu\Delta F , \tag{81}$$

$$\begin{aligned}&(\mathscr{A}F)(x,t)=v\cdot\nabla F,\\ &(\delta\mathscr{A}F)(x,t)=u\cdot\nabla F+\nu\Delta F ,\\ &\mathscr{A}=\tfrac{1}{2}(\mathscr{A}_{(+)}+\mathscr{A}_{(-)}),\ \delta\mathscr{A}=\tfrac{1}{2}(\mathscr{A}_{(+)}-\mathscr{A}_{(-)}) ,\end{aligned}$$

so that

$$\begin{aligned}&D_{(\pm)}=\partial_t+\mathscr{A}_{(\pm)},\ D=\partial_t+\mathscr{A},\ \delta D=\delta\mathscr{A} ,\\ &D=\tfrac{1}{2}(D_{(+)}+D_{(-)}),\ \delta D=\tfrac{1}{2}(D_{(+)}-D_{(-)}) .\end{aligned} \tag{82}$$

Notice the following conjugation properties in $L^2(\mathbb{R}^n,dx)$:

$$\mathscr{D}_{(\pm)}^{\dagger}=\mathscr{A}_{(\pm)},\ \mathscr{D}^{\dagger}=\mathscr{A},\ (\delta\mathscr{D})^{\dagger}=\delta D . \tag{83}$$

While on the Hilbert space $L^2(\mathbb{R}^n\times\mathbb{R},\rho\,dx\,dt)$, we have[17]

$$D^{\dagger}_{(\pm)}=-D_{(\mp)}\ . \tag{84}$$

It is convenient to recall that conditional expectations can be easily expressed in terms of transition probability densities. In particular, we have, for $t\geq t'$,

$$E(F(q(t),t)\,|\,q(t')=x')=\int F(x,t)p(x,t;x',t')dx\ , \tag{85}$$

while, in the case of forward conditioning,

$$E(F(q,(t'),t')\,|\,q(t)=x)=\rho(x,t)^{-1}\int p(x,t;x',t')F(x',t')\rho(x',t')dx'\ . \tag{86}$$

Clearly, $\rho(x,t)$ must be different from zero if (86) has meaning. In fact, only in this case is it possible to impose the conditioning $q(t)=x$.

Now we list some useful expressions involving the forward and backward derivatives defined in (80). First of all let us recall that the backward Fokker-Planck equation for the transition probability can be written as

$$(D'_{(+)}p)(x,t;x',t')=0,\quad t\geq t'\ , \tag{87}$$

where $D'_{(+)}$ is defined in (80) and acts on the (x',t') variables. We also have[17]

$$\frac{d}{dt}E(F(q(t),t)G(q(t),t))=E((D_{(\pm)}F)(q(t),t)G(q(t),t))+E(F(q(t),t)(D_{(\mp)}G)(q(t),t))\ . \tag{88}$$

Moreover, the following conditioned transport equations hold, as a counterpart of (12), for $t_0\leq t_1$:

$$E(F(q(t_1),t_1)\,|\,q(t_0)=x_0)-F(x_0,t_o)=\int_{t_0}^{t_1}E((D_{(+)}F)(q(t),t)\,|\,q(t_0)=x_0)dt\ , \tag{89}$$

$$F(x_1,t_1)-E(F(q(t_0),t_0)\,|\,q(t_1)=x_1)=\int_{t_0}^{t_1}E((D_{(-)}F)(q(t),t)\,|\,q(t_1)=x_1)dt\ . \tag{90}$$

Equations (89) and (90) follow easily from the definition (78) and the Markov property. In some cases it is useful to compare two different processes, one, $q(t)$, evolving under the control $v_{(+)}$, and the other, $q'(t)$, evolving under $v'_{(+)}$. According to the general frame outlined before, we assume that both processes have the same distribution density ρ_0 at time t_0. Then we can easily give, under suitable regularity conditions, an explicit expression for the averages of functionals $F(q')$ of the process $q'(t)$, for $t_0\leq t\leq t_1$, in terms of the same functionals $F(q)$ for the other process $q(t)$, considered as a known reference process. In fact, the Itô-Girsanov formula[22,11] gives

$$E(F(q'))=E\left[F(q)\exp\left[\frac{1}{2\nu}\int_{t_0}^{t_1}[\delta v_{(+)}(q(t),t)\cdot dw(t)-\tfrac{1}{2}\delta v_{(+)}{}^2(q(t),t)]dt\right]\right], \tag{91}$$

where

$$\delta v_{(+)}(x,t)=(v'_{(+)}-v_{(+)})(x,t)\ .$$

Finally, we give the explicit expression of the variations of the transition probability density and the density corresponding to a change (Eq. (92)] in the control drift field, while the external Brownian noise and the initial density are kept fixed.

We follow the same method leading to (46) in the classical case. In fact, we start from (87) and exploit the trick explained in (21), but using $D_{(+)}$ in place of D. Then we use the transport equation (89) instead of (12). Also, in the stochastic case, we end up with an expression similar to (46), where now p and p' are not given by (40), as in the classical case, but are the solutions of (66) or (87) with initial condition (67). For the variation of the density in the stochastic case we still have an expression similar to (47), in fact (45) and (68) are structurally the same. The explicit expressions for the stochastic case are

$$\begin{aligned}p'(x_1,t_1;x_0,t_0)-p(x_1,t_1;x_0,t_0)&=\int_{t_0}^{t_1}\int\nabla p'(x_1,t_1;x,t)\cdot\delta v_{(+)}(x,t)p(x,t;x_0,t_0)dx\,dt\ ,\\ \rho'(x_1,t_1)-\rho(x_1,t_1)&=\int_{t_0}^{t_1}\int\nabla p'(x_1t_1;xt)\cdot\delta v_{(+)}(x,t)\rho(x,t)dx\,dt\ .\end{aligned} \tag{92}$$

This typical procedure will be also exploited in the following [see Eq. (110)].

This ends our discussion about the kinematical properties of the stochastic control theory. The reader will notice that we have written most of Sec. II having in mind the generalization to the stochas-

tic case. In fact, the classical and the stochastic cases are very similar and most of the structure is preserved, provided we take care of the fact that the Brownian disturbances produce the splitting of the velocity field v in $v_{(+)}$ and $v_{(-)}$. Some physical consequences of this splitting with reference to the quantum uncertainty principle and Bohr complementarity are analyzed in Refs. 24 and 25. In the next sections, we generalize the contrained and unconstrained forms of the variational principle to the stochastic case.

IV. THE STOCHASTIC ACTION FUNCTIONALS

The main problem for the formulation of stochastic variational principles, leading to a simulation of quantum-mechanical behavior, is to find the right action. Our proposal is the following.

In analogy with the classical case (13), we introduce the Lagrangian field defined by

$$\mathscr{L}(x,t)=\tfrac{1}{2}mv_{(+)}(x,t)\cdot v_{(-)}(x,t)-V(x)\ , \tag{93}$$

where $v_{(+)}$ is the control field and $v_{(-)}$ is defined through (69). Notice that the Lagrangian field depend not only on $v_{(+)}$ but also on the initial density ρ_0. This is a peculiar aspect of the theory. Clearly (93) is time-reversal invariant. In fact, for the time-reversal transformation we have[18]

$$\begin{aligned} &t\to t'=-t, q(t)\to q'(t')=q(t)\ ,\\ &x\to x'=x\ ,\\ &v_{(\pm)}(x,t)\to v'_{(\pm)}(x,t')=-v_{(\mp)}(x,t)\ . \end{aligned} \tag{94}$$

As $\nu\to 0$, (93) reduces to the classical case.

In analogy with (49), let us define the average stochastic action, spent by the controller in order to move the system from the time t_0 to time t_1, $t_1\geq t_0$, for some initial distribution $\rho_0(\cdot)$,

$$\begin{aligned} A(t_0,t_1;\rho_0;v_{(+)})&=\int_{t_0}^{t_1}E(\mathscr{L}(q(t),t))dt\\ &=\int_{t_0}^{t_1}\int\mathscr{L}(x,t)\rho(x,t)dx\,dt\ , \end{aligned} \tag{95}$$

with $\mathscr{L}$ given by (93). Notice the additivity property

$$\begin{aligned} A(t_0,t_1;\rho_0;v_{(+)})=&A(t_0,t;\rho_0;v_{(+)})\\ &+A(t,t_1;\rho_t;v_{(+)})\ , \end{aligned} \tag{96}$$

where ρ_t is $\rho(\cdot,t)$ and $t_0\leq t\leq t_1$.

Let us point out the following intuitive physical picture at the basis of (95). Let $s_0=t_0,s_1,\ldots,s_N=t_1$ be an equipartition of the interval $[t_0,t_1]$, with $\Delta s=(t_1-t_0)/N$. First of all let us consider a classical trajectory $q(t)$; then we can assume the limit

$$\mathscr{A}(t_0,t_1)=\tfrac{1}{2}m\lim_{\Delta s\to 0}\sum_{i=0}^{N-1}\frac{[q(s_{i+1})-q(s_i)]^2}{(\Delta s)^2}\Delta s \tag{97}$$

as an expression for the kinetic part of the action in the interval $[t_0,t_1]$ according to the definitions (14) and (49). For the stochastic case let us modify (97) in the form

$$\begin{aligned} \mathscr{A}(t_0,t_1)=\tfrac{1}{2}m\lim_{\Delta s\to 0}\sum_{i=1}^{N-1}&\frac{[q(s_{i+1})-q(s_i)]}{\Delta s}\\ &\cdot\frac{[q(s_i)-q(s_{i-1})]}{\Delta s}\Delta s\ . \end{aligned} \tag{98}$$

Clearly (97) and (98) give the same limit in the classical case, because the change is only of second order in Δs. On the other hand, the limit (98) does not exist in the stochastic case because $q(t)$ is nowhere differentiable. But we can take (conditional) averages on $\mathscr{A}$ in (98) before and then take the limit. Then it can easily be shown that we have

$$\begin{aligned} E(\mathscr{A}(t_0,t_1))=\tfrac{1}{2}m\int_{t_0}^{t_1}\int&(v_{(+)}\cdot v_{(-)})(x,t)\\ &\times\rho(x,t)dx\,dt\ , \end{aligned} \tag{99}$$

so that the kinetic part of (95) is correctly reproduced. On the other hand, the limit in (97) would give divergent results also in the case where some average is taken before. It is clear that the tendency of the Brownian motion to produce highly irregular trajectories for $q(t)$ favors (98) over (97) as far as the convergence is concerned. In fact, the contributions in (98) come from scalar products of the displacement in each infinitesimal time interval multiplied by the displacement in the next interval. It is clear that only a physical explanation of the nature of the underlying Brownian motion could also produce a deeper justification of (98). For the moment (98), and the equivalent (95), are taken as basic assumption of our theory.

Notice that (93) can be written in equivalent form,

$$\mathscr{L}(x,t)=\tfrac{1}{2}m(v^2-u^2)(x,t)-V(x)\ , \tag{93'}$$

by exploiting the current and osmotic velocities defined in (70) and (72). The minus sign before u^2 in (93') is physically relevant; it means that the osmotic part of the action plays more a role similar to a potential rather than a kinetic contribution. This is in agreement with the Schrödinger variational formulation for stationary states of quantum-mechanical

systems, where u^2 and V enter with the *same* sign in the energy functional.

It is also useful to introduce conditioned expectations for (98). It is very simple to prove, exploiting the Markov property and the expression (85), that

$$E(\mathscr{A}(t_0,t_1)\,|\,q(t_0)=x_0) = \tfrac{1}{2}m\int_{t_0}^{t_1}\int (v_{(+)}{}^2+2\nu\nabla\cdot v_{(+)})(x,t) \times p(x,t;x_0,t_0)dx\,dt\ . \tag{100}$$

Analogous expressions hold for a forward conditioning at time t_1. Then we are motivated to introduce the following forward and backward Lagrangian fields:

$$\mathscr{L}_{(\pm)}(x,t)=\tfrac{1}{2}mv_{(\pm)}{}^2(x,t) \pm m\nu(\nabla\cdot v_{(\pm)})(x,t)-V(x)\ , \tag{101}$$

for which one can easily prove

$$E(\mathscr{L}(q(t),t))=E(\mathscr{L}_{(\pm)}(q(t),t))\ . \tag{102}$$

We also define, the $t_0\le t\le t_1$,

$$I(x,t;t_1;v_{(+)}) = -\int_t^{t_1} E(\mathscr{L}_{(+)}(q(t'),t')\,|\,q(t)=x)dt' = -\int_t^{t_1}\int \mathscr{L}_{(+)}(x',t')p(x',t';x,t)dx'dt'\ . \tag{103}$$

Notice that I satisfies the transport equation

$$(D_{(+)}I)(x,t)=\mathscr{L}_{(+)}(x,t)\ , \tag{104}$$

with the boundary condition

$$I(\cdot,t_1)=0\ . \tag{105}$$

Moreover,

$$A(t_0,t_1;\rho_0;v_{(+)}) = -\int I(x_0,t_0;t_1;v_{(+)})\rho_0(x_0)dx_0\ . \tag{106}$$

Equations (103), (104), (105), and (106) are the counterparts of (14), $(19)_1$, $(19)_2$, and (49), respectively.

In analogy with the classical case it is also useful to introduce action functionals with additional endpoint contributions. For some given smooth field $S_1(\cdot)$ on $\mathscr{M}$ we then define

$$J(x,t;t_1,S_1;v_{(+)})=I(x,t;t_1;v_{(+)}) +E(S_1(q(t_1))\,|\,q(t)=x)\ , \tag{107}$$

$$B(t_0,t_1;\rho_0,S_1;v_{(+)}) =A(t_0,t_1;\rho_0;v_{(+)})-E(S_1(q(t_1))) =-\int J(x_0,t_0;t_1,S_1;v_{(+)})\rho_0(x_0)dx_0\ . \tag{108}$$

One easily checks the following:

$$(D_{(+)}J)(x,t)=\mathscr{L}_{(+)}(x,t)\ , \quad J(\cdot,t_1)=S_1(\cdot)\ . \tag{109}$$

In order to introduce and prove variational principles it is necessary to obtain explicit expressions for the variations of the stochastic action functionals, introduced before, when the control field is changed from $v_{(+)}$ to some other $v'_{(+)}=v_{(+)}+\delta v_{(+)}$, while the initial distribution is kept fixed. We follow the same route as in (21). With obvious short-hand notations, we have

$$D'_{(+)}(J'-J)=\mathscr{L}'_{(+)}-\mathscr{L}_{(+)}+(D_{(+)}-D'_{(+)})J =[\tfrac{1}{2}m(v'_{(+)}+v_{(+)})+m\nu\nabla -\nabla J]\cdot\delta v_{(+)}\ . \tag{110}$$

Then the transport equation (89) and the boundary condition in (109) give

$$(\delta J)(x_0,t_0)=-\int_{t_0}^{t_1}\int \{[\tfrac{1}{2}m(v'_{(+)}+v_{(+)})+m\nu\nabla-\nabla J]\cdot\delta v_{(+)}\}(x,t)p'(x,t;x_0t_0)dx\,dt\ . \tag{111}$$

If we multiply by $\rho_0(x_0)$, we integrate on dx_0, and get rid of $\nabla\cdot\delta v_{(+)}$ through a simple integration by parts on dx, then we obtain

$$(\delta B)(t_0,t_1)=\int_{t_0}^{t_1}\int [\tfrac{1}{2}m(v_{(+)}+v'_{(-)})-\nabla J]\cdot\delta v_{(+)}(x,t)\rho'(x,t)dx\,dt\ . \tag{112}$$

By following the same method, but exploiting I in (110) in place of J, we also get

$$(\delta A)(t_0,t_1)=\int_{t_0}^{t_1}\int [\tfrac{1}{2}m(v_{(+)}+v'_{(-)})-\nabla I]\cdot\delta v_{(+)}(x,t)\rho'(x,t)dx\,dt\ , \tag{113}$$

which is the counterpart of (50). To the first order in $\delta v_{(+)}$ we have

$$(\delta B)(t_0,t_1)=\int_{t_0}^{t_1}\int (mv-\nabla J)\cdot\delta v_{(+)}(x,t)\rho(x,t)dx\,dt\ , \tag{114}$$

$$(\delta A)(t_0,t_1)=\int_{t_0}^{t_1}\int (mv-\nabla I)\cdot\delta v_{(+)}(x,t)\rho(x,t)dx\,dt\ . \tag{115}$$

Let us also recall that the first-order variation of the density at t_1 is

$$(\delta\rho)(x_1,t_1)=\int_{t_0}^{t_1}\int \nabla p(x_1,t_1;x,t)\cdot\delta v_{(+)}(x,t)\rho(x,t)dx\,dt\ , \tag{116}$$

as follows from the second equation in (92). Then we can state the equivalence between the variational principle based on $\delta B=0$ and the constrained variational principle based on $\delta A=0$ with $\delta\rho(x_1,t_1)=0$, $\forall x_1$. In fact, as in the classical case, the condition $\delta B=0$ for any $\delta v_{(+)}$ implies, as a consequence of (114), the Hamilton-Jacobi condition

$$v(x,t)=(\nabla S)(x,t)/m\ , \tag{117}$$

where we called S the value of J for the stationarizing field (physical control field). On the other hand, if $\delta A=0$ for any $\delta v_{(+)}$ satisfying $\delta\rho(\cdot,t_1)$, we cannot conclude that $mv-\nabla I=0$, but only

$$mv-\nabla I=\nabla\int S_1(x_1)p(x_1,t_1;x,t)dx_1\ , \tag{118}$$

for some suitable function S_1. But (118) is equivalent to (117) in view of (107), provided the boundary term S_1 in (107) is chosen in agreement with (118). Therefore, also in the stochastic case there is the interpretation of the S function as a field of Lagrangian multipliers associated to the density constraint at the final time in the variational principle.

Notice that, while in the classical case the variational principles are based on conditions of the type $\delta I(x_0,t_0)=0$ or $\delta J(x_0,t_0)=0$, here in the stochastic case they are based on the *averaged* conditions

$$\delta A\equiv E(\delta I(q(t_0),t_0))=0\ ,$$

$$\delta B\equiv E(\delta J(q(t_0),t_0))=0\ .$$

In this way the density enters in an essential way in the dynamical evolution, as the next considerations show. In fact, let us now investigate the consequences of (117). Since S coincides with J for the stationarizing field we have

$$(D_{(+)}S)(x,t)=\tfrac{1}{2}mv_{(+)}^2+mv\nabla\cdot v_{(+)}\ , \tag{119}$$

$$S(\cdot,t_1)=S_1(\cdot)\ . \tag{120}$$

But for $v_{(+)}$ we have

$$v_{(+)}=v+u=\nabla S/m+2\nu\nabla R \tag{121}$$

as a consequence of (117) and (70). If we substitute (121) in (119), then we find the following Hamilton-Jacobi equation with stochastic corrections:

$$(\partial_t S)(x,t)+\frac{1}{2m}(\nabla S)^2(x,t)+V(x)$$
$$-2m\nu^2[(\nabla R)^2(x,t)+(\Delta R)(x,t)]=0\ . \tag{122}$$

The connection with quantum mechanics is established in the standard way.[17,18] First of all we fix the diffusion constant so that

$$2\nu=\hbar/m\ . \tag{123}$$

Then the ansatz

$$\psi(x,t)=[\rho(x,t)]^{1/2}\exp\left[\frac{i}{\hbar}S(x,t)\right] \tag{124}$$

reduces the two real equations (73) and (122) to the Schrödinger equation

$$i\hbar(\partial_t\psi)(x,t)=-\frac{\hbar^2}{2m}(\Delta\psi)(x,t)+V(x)\psi(x,t)\ . \tag{125}$$

If we do not impose (123), then for the wave function (124) we find some nonlinear equation[26,27] instead of (125).

Clearly all our considerations extend to the case where the action functional J is modified through boundary terms at the initial time t_0 and in the case where linear terms in v are introduced in the Lagrangian field (93).

Finally, notice, on the basis of Eqs. (124), (107), (103), and the connection between S and J, that the expression of the phase function S is given in terms of the physical control field $v_{(+)}=(\nabla S+\hbar\nabla R)/m$ as follows:

$$S(x,t)=E(S_1(q(t_1))\,|\,q(t)=x)-\int_t^{t_1}E(\mathscr{L}_{(+)}(q(t'),t')\,|\,q(t)=x)dt',\quad t_0\le t\le t_1\ , \tag{126}$$

where $q(t)$ is the process subject to the physical control field. If the initial value S_0 of S is known, then we can write (126) in the equivalent form

$$S(x,t)=E(S_0(q(t_0))\,|\,q(t)=x)+\int_{t_0}^{t}E(\mathscr{L}_{(-)}(q(t'),t')\,|\,q(t)=x)dt',\quad t_0\le t\le t_1\ , \tag{127}$$

as can be easily checked on the basis of the transport equations (89) and (90) and the conditioning identities (85) and (86).

As a consistency check of our theory it is also possible to connect the stochastic Lagrangian field (93) with the standard Lagrangian

$$L(\psi(\cdot,t),\psi^*(\cdot,t),\dot{\psi}(\cdot,t),\dot{\psi}^*(\cdot,t))=\int\left[\frac{i\hbar}{2}(\psi^*\dot{\psi}-\psi\dot{\psi}^*)-\frac{\hbar^2}{2m}\nabla\psi\cdot\nabla\psi^*-V\psi\psi^*\right](x,t)dx\ , \tag{128}$$

giving rise to (124) through Euler-Lagrange independent variations of ψ and ψ^*. In fact (128), with the ansatz (124), reduces to

$$L=-\int\left[\partial_t S+\frac{\hbar^2}{2m}(\nabla R)^2+\frac{1}{2m}(\nabla S)^2+V\right](x,t)\rho(x,t)dx\ . \tag{129}$$

A simple calculation shows that (129) gives Eqs. (73) and (122) of the Madelung fluid through Euler-Lagrange independent variations of S and ρ. On the other hand, by exploiting the continuity equations (73) and (117) one easily finds

$$L=-\frac{d}{dt}\int S(x,t)\rho(x,t)dx+E(\mathscr{L}(q(t),t))\ , \tag{130}$$

where L is defined in (128) and (129), $\mathscr{L}$ is given in (93), and $q(t)$ is the associated process. Since the total time derivative is irrelevant we have the equivalence between our stochastic variational principle and the standard variational principle leading to Schrödinger equation from (128).

VI. CONCLUSIONS AND OUTLOOK

We have seen that it is possible to choose the Lagrangian field in a stochastic variational problem so that the resulting programming equation corresponds to the Hamilton-Jacobi equation with the right quantum corrections. From this point of view stochastic quantization[17,28,29,18] can be based either on the smoothed form of the second principle or dynamics or on a stochastic variational problem. The advantage of this second method relies on the fact that the gradient form of the velocity fields follows from the variational principle and is not an independent assumption.

We have given two equivalent forms of the variational principle. For one version boundary terms are inserted in the action functional and unconstrained control drift variations are allowed. For the other version the action functional does not contain boundary terms, but the control variations are restricted so that the density is kept pointwise fixed at the end points of the generic time interval of variations. This equivalence between the two principles allows the interpretation of the phase function S as a field of Lagrangian multipliers associated to the density constraints. Therefore S can be considered as a variable conjugated to ρ, according to the usual hydrodynamical picture.

While in classical mechanics and in standard stochastic control theory the variational principles are usually expressed in optimal form, where the functionals are not only made stationary with respect to control variations, but acquire optimum values (maxima or minima), we have not considered here this problem. We plan to deal with it in the future.

The extension of our considerations to the case where the configuration space is a general Riemannian manifold is straightforward and is considered in Ref. 19. In this framework it is interesting to see the interplay between geometric features and the ability of the Brownian motion to feel second-order effects, as shown, for example, in Ref. 23.

Of particular interest is the case where the underlying configuration space is not simply connected. Then the variational principles, even in the classical case, can have a more refined formulation, where boundary terms or density constraints may depend on the connectivity properties of the trajectories. This peculiar feature may lead to simulation of the half-integer-spin effects in the case of rotating bodies[30–32] or to spin-statistics effects, according to the development of general ideas already pointed out, for example, in Ref. 33.

Variational principles, very similar to those described in this paper, were considered for the ground-state process of Euclidean field theory,[34] in the frame of a general program aiming at the connection between classical statistical mechanics and quantum field theory. It would be interesting to see whether similar interpretations hold also in our case. In particular, the stochastic action functional could be connected to entropy properties of the trajectories appearing in the Itô-Girsanov formula.

Finally, we would like to remark that the kinetic metric enters in our theory in two different places. Firstly, it gives the kinetic action (as in classical case), but it also determines the underlying Brownian disturbance (see also Ref. 23). A natural exten-

sion of interesting remarks by Davidson[26] seems to suggest that the Brownian disturbance could be introduced without any relation with the kinetic metric on the configuration space.

Therefore it would be very interesting to include also in the controller domain the metric associated to the Brownian motion, providing a wide generalization of the stochastic variational principles introduced in this paper.

But it is clear that major progress in the whole program of stochastic quantization would come from a better understanding of the physical origin of the underlying Brownian motion, so that, in particular, the form of stochastic action, assumed in this paper, could be explained in more fundamental terms.

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