# Some evaluations of Bell's inequality for particles of arbitrary spin

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Bell's inequality is applied to a gedanken experiment, where a spin singlet decays into two spin-$s$ particles. This is done by defining dichotomic observables depending on the spin components. The range of settings, for which the inequality is violated by quantum mechanics, is calculated for certain values of $s$. The classical limit is taken and the results are compared to those obtained, by an alternative method, by Mermin.

## I. INTRODUCTION

In 1964 Bell was able to show that a large class of local realistic theories must yield predictions that disagree with those of quantum mechanics,[1] thus demonstrating that the belief of Einstein, Podolsky, and Rosen[2] (EPR) was erroneous. Bell considered the following situation introduced by Bohm.[3] A pair of spin-$\frac{1}{2}$ particles are formed in a singlet spin state, and with opposite directions of motion. If a measurement of $\vec{s}\cdot\hat{n}$, where $\hat{n}$ is a unit vector, on one particle yields the value $+\frac{1}{2}$, then a measurement of $\vec{s}\cdot\hat{n}$ on the second particle must yield the value $-\frac{1}{2}$, and vice versa, because the total spin is zero. In this way, it is possible to predict in advance the result of measuring any component of $\vec{s}$ on one of the particles, by first measuring the same component of $\vec{s}$ on the other. According to the arguments of EPR, concerning locality and realism, it follows that the particle has inherent properties, which determine the value obtained when measuring any component of its spin. This implies that the quantum-mechanical description of reality is incomplete, and that a more refined description might be possible. However, using the conditions of locality and realism, Bell derived an inequality which was found to be incompatible with the predictions of quantum mechanics. Since 1964 the notion of locality and realism has been analyzed further, Bell's theorem has been proved for virtually every conceivable local realistic theory, and Bell's inequality has been generalized in several ways.[4–6] For example, it has been adapted to experimental tests. Most of the results support quantum mechanics, but so far, it has been necessary to rely on additional assumptions in order to arrive at the conclusions.

Inequalities have been derived, involving the spin components in $N$ directions. Although for every $N$ the necessary and sufficient conditions for the existence of a local realistic theory can be expressed as a set of linear inequalities of Bell's type, it is not known whether these can be generated in a simple manner. It is not even known if the inequalities for $N=3$ are sufficient for all $N$, but it is believed that they are not.

It is possible to derive inequalities for other systems.[7] Mermin has done this for a spin-$s$ singlet pair,[8] and he has shown that it is violated by quantum mechanics for a range of settings that vanishes as $1/s$ when $s\rightarrow\infty$. In this paper, an alternative to Mermin's approach is presented. Following a suggestion by Bell,[9] the spin-$s$ case is handled, not by deriving new inequalities, but by defining new dichotomic variables. Applied to quantum mechanics, this method, despite its simplicity, works even better than Mermin's method.

## II. BELL'S INEQUALITY

Consider the gedanken experiment with the spin-$\frac{1}{2}$ singlet pair. If we introduce the observables $A(\hat{n})=2\vec{s}\cdot\hat{n}$, where $\hat{n}$ is a unit vector, the following holds: We can measure two families of observables $A_1(\hat{n})$ and $A_2(\hat{n})$ such that

$$|A_1(\hat{n})|=|A_2(\hat{n})|=1\ , \tag{1a}$$

$$A_1(\hat{n})A_2(\hat{n})=-1\ , \tag{1b}$$

$$\langle A_1(\hat{n}_1)A_2(\hat{n}_2)\rangle=f(\theta)\ , \tag{1c}$$

where $\cos\theta=\hat{n}_1\cdot\hat{n}_2$.

$A_1(\hat{n}_1)$ and $A_2(\hat{n}_2)$ are, of course, bearing on the two particles, respectively. If we now suppose[10] that $A_1(\hat{n}_1)$ is independent of $\hat{n}_2$ and vice versa, and that the $A(\hat{n})$'s have definite values for all $\hat{n}$'s, regardless of the directions chosen in the actual experiment, it is straightforward to derive Bell's inequality. Choose three directions $\hat{a}$, $\hat{b}$, and $\hat{c}$. For each pair, the assumptions allow us to form the quantity

$$A_1(\hat{a})A_2(\hat{b})+A_1(\hat{a})A_2(\hat{c}) .$$

The absolute value of this can be rewritten using (1a) and (1b),

$$\begin{aligned}|A_1(\hat{a})A_2(\hat{b})+A_1(\hat{a})A_2(\hat{c})| &= |A_1(\hat{a})A_2(\hat{b})|\,[1+A_2(\hat{b})A_2(\hat{c})] \\ &= 1-A_1(\hat{b})A_2(\hat{c}) .\end{aligned}$$

Upon averaging, the equality changes to an inequality,

$$|\langle A_1(\hat{a})A_2(\hat{b})\rangle+\langle A_1(\hat{a})A_2(\hat{c})\rangle| \le 1-\langle A_1(\hat{b})A_2(\hat{c})\rangle \quad (2)$$

or, in terms of $f(\theta)$,

$$|f(\theta_{ab})+f(\theta_{ac})|+f(\theta_{bc})\le 1 . \quad (3)$$

A suitable choice[11] is to let $\theta_{ab}=\theta_{ac}=\theta$ and $\theta_{bc}=2\theta$, which gives the inequality we are going to use,

$$f(2\theta)-2f(\theta)\le 1 . \quad (4)$$

Note that this holds as an equality for $\theta=0$. If locality and realism are assumed the derivation of Bell's inequality is valid for every $\{A_i(\hat{n}_i)\}$, $i=1,2$ that satisfies (1). We shall focus our attention on experiments where a spin singlet decays into two spin-$s$ particles. Here the observables $A_1(\hat{n})$ and $A_2(\hat{n})$ will be functions of the spin $\vec{s}$ of the two particles, respectively. The function $f(\theta)$ will be analytic.[12] That is important, since it means that the inequality (4) will be violated at least for angles near $\theta=0$. Indeed, for small $\theta$, $f(\theta)$ can be expanded in a MacLaurin series:

$$f(\theta)=-1+c\theta^{2n}+O(\theta^{2n+1}) , \quad c>0 , \; n\in z^+$$

and the inequality (4) becomes

$$c(2^{2n}-2)\theta^{2n}+O(\theta^{2n+1})\le 0 ,$$

which is clearly violated. The maximal angle of violation will be called $\theta_m$. Before specifying $A_1$ and $A_2$ we will so discuss Mermin's method.

TABLE I. Bell's inequality for a spin-$s$ singlet pair. (a) Mermin's results ($y=\sin\theta$). (b) The results for case I. (c) The results for case II. (d) The results for case III.

| | (a) | |
|---|---|---|
| $s$ | $|\langle m_2(\hat{c})-m_1(\hat{b})\rangle| \ge \frac{2}{3}(s+1)y$ | $\theta_m$ |
| $\frac{1}{2}$ | $y^2\ge y$ | 90° |
| 1 | $\frac{1}{3}(8y^2-4y^4)\ge\frac{4}{3}y$ | 38.17° |
| $\frac{3}{2}$ | $5y^2-6y^4+3y^6\ge\frac{5}{3}y$ | 24.08° |
| 2 | $\frac{1}{5}(40y^2-84y^4+96y^6-40y^8)\ge 2y$ | 17.58° |
| | (b) | |
| $s$ | $f_{\rm QM}(\theta)$ ($x=\cos\theta$) | $\theta_m$ |
| $\frac{1}{2}$ | $-x$ | 90° |
| $\frac{3}{2}$ | $-\frac{1}{2}(x+x^3)$ | 49.81° |
| $\frac{5}{2}$ | $-\frac{1}{4}(3x-2x^3+3x^5)$ | 34.11° |
| $\frac{7}{2}$ | $-\frac{1}{16}(9x+15x^3-33x^5+25x^7)$ | 25.84° |
| | (c) | |
| $s$ | $f_{\rm QM}(\theta)$ ($x=\cos\theta$) | $\theta_m$ |
| 1 | $\frac{1}{3}(1-4x^2)$ | 45° |
| 2 | $\frac{1}{5}(-2+6x^2-9x^4)$ | 26.16° |
| 3 | $\frac{1}{7}(-3-9x^2+30x^4-25x^6)$ | 18.52° |
| 4 | $\frac{1}{9}(-\frac{89}{16}+\frac{45}{4}x^2-\frac{555}{8}x^4+\frac{525}{4}x^6-\frac{1225}{16}x^8)$ | 14.35° |
| | (d) | |
| $s$ | $f_{\rm QM}(\theta)=-1+\frac{2}{s+\frac{1}{2}}\cos^{4s}(\theta/2)$ | $\theta_m$ |
| $\frac{1}{2}$ | | 90° |
| 1 | | 63.86° |
| $\frac{3}{2}$ | | 52.00° |
| 2 | | 44.96° |

## III. MERMIN'S METHOD

Mermin[8] derived a new set of inequalities. If $m_i(\hat{n}_i)$ is the value of $\vec{s}_i\cdot\hat{n}_i$ obtained in a certain measurement on a singlet pair, the following relation holds:

$$m_1(\hat{a})m_2(\hat{b})+m_1(\hat{a})m_2(\hat{c})=m_1(\hat{a})[m_2(\hat{b})+m_2(\hat{c})]=m_1(\hat{a})[m_2(\hat{b})-m_1(\hat{c})]\le s\,|m_2(\hat{b})-m_1(\hat{c})| . \quad (5)$$

The expectation value of $m_1(\hat{a})m_2(\hat{b})$ is[13]

$$\langle m_1(\hat{a})m_2(\hat{b})\rangle = -\tfrac{1}{3}s(s+1)\hat{a}\cdot\hat{b} \,. \tag{6}$$

For large $s$ Mermin considers the case when $\hat{a}$ and $\hat{b}$ are nearly orthogonal and therefore the angle between them is $\pi/2+\theta$. Similarly the third direction $\hat{c}$ is chosen in the same plane as $\hat{a}$ and $\hat{b}$ making an angle of $\pi/2+\theta$ with $\hat{a}$ and an angle of $\pi-2\theta$ with $\hat{b}$. The inequality now becomes

$$\tfrac{2}{3}(s+1)\sin\theta \le \langle\, |m_2(\hat{b})-m_1(\hat{c})|\,\rangle \,. \tag{7}$$

For small $s$ the inequality assumes the forms given in Table I(a). For large $s$, it is convenient to put $\theta=a/s$ and the final result, when $s\to\infty$, is

$$\int_{-\infty}^{\infty}\frac{dx}{2\pi x^2}\left[1-\frac{\sin(4a\sin x)}{4a\sin x}\right]\ge \tfrac{2}{3}a \,, \tag{8}$$

which is violated for $a<0.5659$.

## IV. METHOD OF DICHOTOMIC OBSERVABLES

We will now construct the observables $A_1(\hat{n})$ and $A_2(\hat{n})$. First divide the possible outcomes of a measurement of $\hat{n}_1\cdot\vec{s}_1$ into two sets: $M_1^+$ and $M_1^-$. If a measurement of $\hat{n}_1\cdot\vec{s}_1$ yields the value $m_1$, then $A_1$ is defined by

$$A_1=\begin{cases}+1 & \text{if } m_1\in M_1^+ \,,\\ -1 & \text{if } m_1\in M_1^- \,,\end{cases} \tag{9a}$$

and similarly

$$A_2=\begin{cases}+1 & \text{if } m_2\in M_2^+ \,,\\ -1 & \text{if } m_2\in M_2^- \,.\end{cases} \tag{9b}$$

In order to get perfect anticorrelation for $\theta=0$ it is necessary to choose $M_2^+=-M_1^-$ and $M_2^-=-M_1^+$, because in that case $m_1=-m_2$. The quantum-mechanical correlation function is obviously given by

$$f_{\rm QM}(\theta)=\left[\sum_{\substack{m_1\in M_1^+\\ m_2\in M_2^+}}+\sum_{\substack{m_1\in M_1^-\\ m_2\in M_2^-}}-\sum_{\substack{m_1\in M_1^-\\ m_2\in M_2^+}}-\sum_{\substack{m_1\in M_1^+\\ m_2\in M_2^-}}\right]p(m_1,m_2)\,, \tag{10}$$

where $p(m_1,m_2)$ is the joint probability of finding $\hat{n}_1\cdot\vec{s}_1=m_1$ and $\hat{n}_2\cdot\vec{s}_2=m_2$. Mathematically

$$p(m_1,m_2)=|\langle\psi_s\,|\,m_1,m_2\rangle_{\hat{n}_1\hat{n}_2}|^2\,, \tag{11}$$

where $\psi_s$ is the singlet wave function[14]

$$\psi_s=\frac{1}{(2s+1)^{1/2}}\sum_{m=-s}^{s}(-1)^{m+s}\,|\,m,-m\rangle_{\hat{n}\hat{n}} \tag{12}$$

and $\{\,|\,m_1,m_2\rangle_{\hat{n}_1\hat{n}_2}\}$ are the common eigenvectors of the commuting spin operators $\vec{s}_1\cdot\hat{n}_1$ and $\vec{s}_2\cdot\hat{n}_2$. If we insert this $\psi_s$ in the expression for $p(m_1,m_2)$ we will find that

$$p(m_1,m_2)=\frac{1}{2s+1}\,|\langle -m_1\,|\,m_2\rangle_{\hat{n}_1\hat{n}_2}|^2 = \frac{1}{2s+1}(d^s_{-m_1,m_2})^2 \,. \tag{13}$$

(The explicit form of $d^s_{-m_1m_2}$ is given in Ref. 12.) It remains to decide how to partition the $m$ values into two sets. This can be done in very many ways. The three choices made here are not necessarily the most important ones, but their symmetry gives them a few technical advantages, and perhaps an aesthetic appeal.

I (half-integer spins only) $M_1^+=\{\tfrac{1}{2},\tfrac{3}{2},\ldots,s\}=M_2^+$ , (14a)

$$M_1^-=\{-\tfrac{1}{2},-\tfrac{3}{2},\ldots,-s\}=M_2^- \,. \tag{14b}$$

II (integer spins only) $M_1^+=\{0\}=M_2^-$ , (15a)

$$M_1^-=\{-s,-s+1,\ldots,-1,+1,\ldots,s\}=M_2^+ \,. \tag{15b}$$

III (all spins) $M_1^+=\{s\},\ \ M_2^-=\{-s\}$ , (16a)

$$M_1^-=\{-s,-s+1,\ldots,s-1\}=-M_2^+ \,. \tag{16b}$$

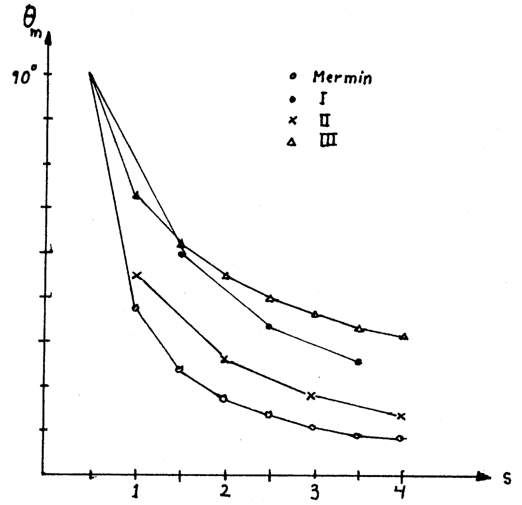


FIG. 1. Comparison of the results for small $s$ in the four cases.

### A. Results for case I

For small values of $s$ the summations in (10) can be performed.[15] The results can be found in Table I(b). For large $s$ we may, as Mermin did, put $\theta = a/s$. Then one can perform many summations, just keeping the two leading powers of $s$, by replacing them by integrations. The result is

$$f_{\mathrm{QM}}(\theta) = -1 + \frac{1}{s}\sum_{n=1}^{\infty} c_n \left[\frac{a}{2}\right]^{2n} + O\left[\frac{1}{s^2}\right],$$
$$c_n = \frac{1}{(n!)^2}\sum_{r,p=0}^{n} (-1)^{r+p} \binom{n}{r}\binom{n}{p} |r-p| \quad (17)$$

or, with an integral representation,

$$f_{\mathrm{QM}} = -1 + \frac{2}{\pi s}\int_{-\infty}^{\infty} \frac{\cot x}{x}[J_0(a\sin x) - 1]dx + O\left[\frac{1}{s^2}\right]. \quad (18)$$

When $s$ is large, $\theta_m \approx 1.8248/s$, which is more than three times larger than Mermin's value.

### B. Results for case II

The first few correlation functions are found in Table I(c). For large $s$ we can again let $\theta = a/s$. Here we need but one element of the rotation matrix:

$$f_{\mathrm{QM}}(\theta) = -1 + \frac{2}{s+1/2}[1 - (d^s_{00})^2]. \quad (19)$$

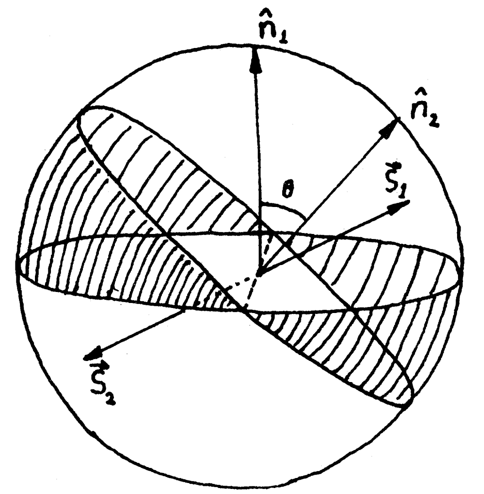


FIG. 2. The classical model for case I.

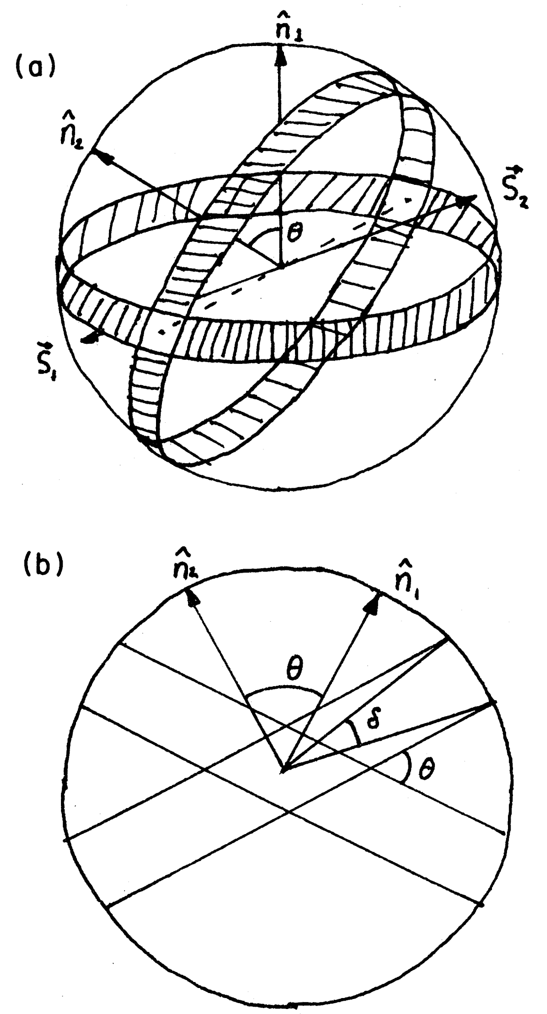


FIG. 3. (a) The classical model for case II. (b) Projection of (a).

But for large $s$, $d^s_{00}(a/s)$ can be "identified" with $J_0(a)$ by inspection. So

$$f_{QM}(\theta)=-1+\frac{2}{s}[1-J_0{}^2(a)]+O\left[\frac{1}{s^2}\right] \quad (20)$$

and when $s$ is large $\theta_m \approx 1.1207/s$.

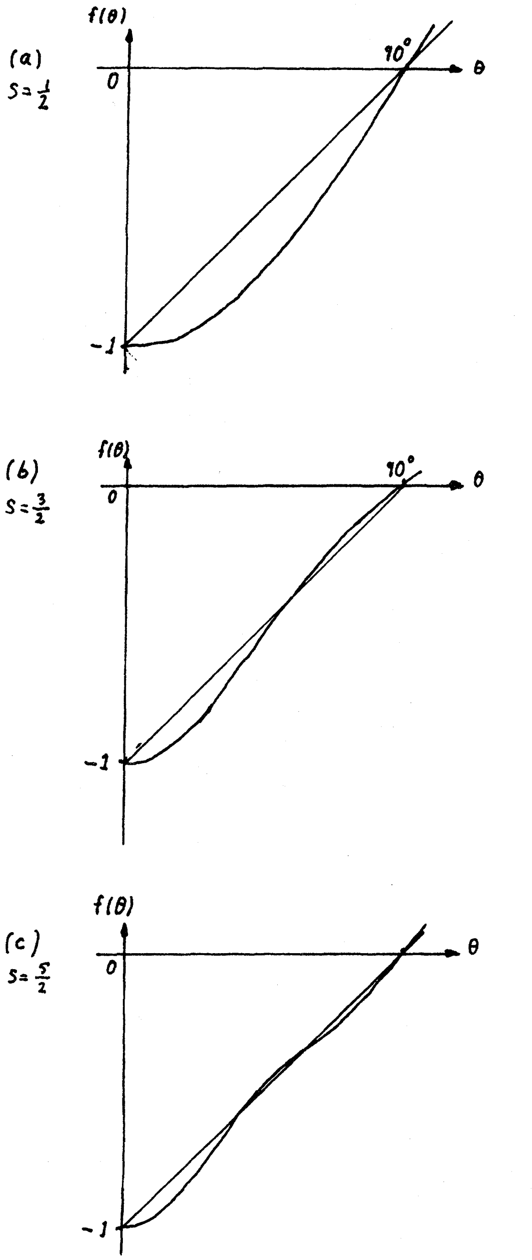


FIG. 4. (a)–(c) Correlation functions for case I (Ref. 15). The classical case is linear and the quantum-mechanical case is curved.

### C. Results for case III

In this case the correlation function is simply

$$f_{QM}(\theta)=-1+\frac{2}{s+\frac{1}{2}}[1-(d^s_{ss})^2]$$

$$=-1+\frac{2}{s+\frac{1}{2}}\left[1-\cos^{4s}\frac{\theta}{2}\right] . \quad (21)$$

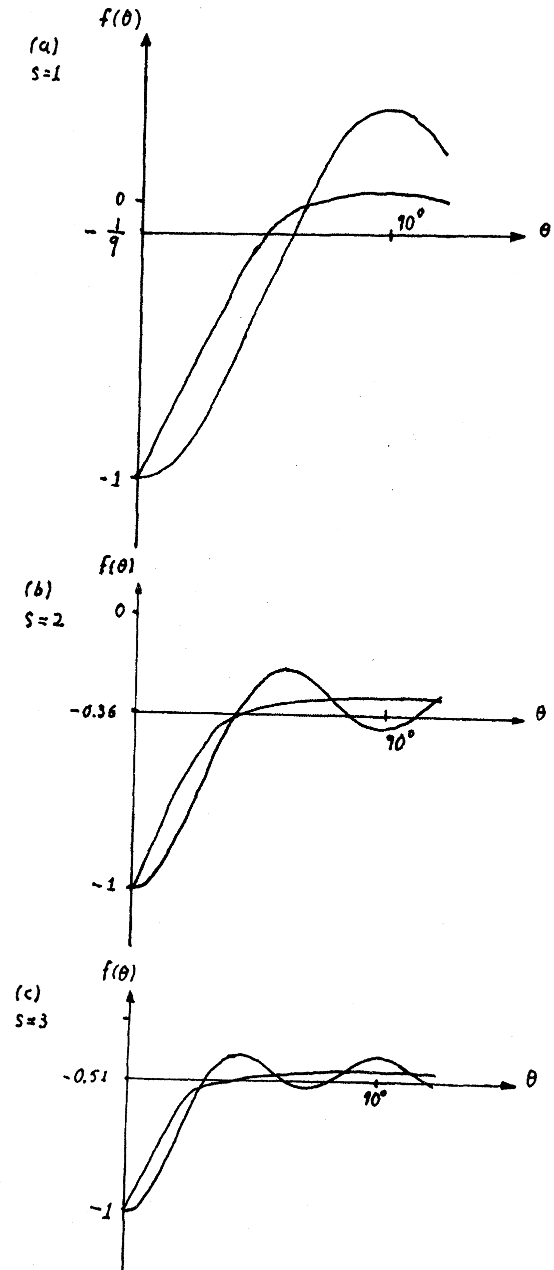


FIG. 5. (a)–(c) Correlation functions for case II. The classical functions are convex and the quantum-mechanical functions are sinusoidal. Two independent particles will have a correlation $f=-(2s-1)^2/(2s+1)^2$.

Now something new occurs when $s$ is large. Instead of $\theta = a/s$, we get $\theta = a/\sqrt{s}$ and

$$f_{\mathrm{QM}}(\theta) = -1 + \frac{2}{s}(1 - e^{-a^2/2}) + O\left[\frac{1}{s^2}\right] . \tag{22}$$

When $s$ is large $\theta_m \approx 1.1040/\sqrt{s}$, which is substantially "better" than all previous results, yet this is the simplest case.

We compare the results for the four cases in Fig. 1.

### V. Classical limits

It might be instructive to compare the quantum-mechanical results with those from classical mechanics.

In Mermin's case we want to calculate $\langle |\vec{s}\cdot\hat{a} + \vec{s}\cdot\hat{b}| \rangle$, but this is easy when $\vec{s}$ is just a vector of magnitude $s$:

$$\begin{aligned} |\vec{s}\cdot\hat{a} + \vec{s}\cdot\hat{b}| &= |s_z||\hat{a}+\hat{b}| \\ &= |s_z|\, 2\sin\frac{2\theta}{2} = 2|s_z|\sin\theta , \end{aligned} \tag{23}$$

where the $z$ axis is taken along the direction of $\hat{a}+\hat{b}$. The average of $|s_z|$ is $\frac{1}{2}s$, so $\langle |\vec{s}\cdot\hat{a} + \vec{s}\cdot\hat{b}| \rangle = s\sin\theta$, and the inequality is trivially satisfied:

$$s\sin\theta \geq \tfrac{2}{3}s\sin\theta . \tag{24}$$

The classical counterpart to case I is also obtained when $\vec{s}$ is considered to be an ordinary vector. Then $A(\hat{n}) = \mathrm{sign}(\hat{n}\cdot\vec{s})$ and from Fig. 2 it is readily seen that

$$f_{\mathrm{cl}}(\theta) \equiv \langle A_1(\hat{a})A_2(\hat{b}) \rangle = -1 + \frac{2\theta}{\pi} , \quad 0 \leq \theta \leq \pi . \tag{25}$$

Test this function in Bell's inequality (3):

$$|\theta_{ab} + \theta_{ac} - \pi| + \theta_{bc} \leq \pi \Longleftrightarrow \tag{26}$$

$$\begin{cases} \theta_{ab} + \theta_{ac} + \theta_{bc} \leq 2\pi , \\ \theta_{ac} \leq \theta_{ab} + \theta_{bc} . \end{cases} \tag{27}$$

These relations are automatically satisfied for every choice of $\hat{a}$, $\hat{b}$, and $\hat{c}$, so Bell's inequality does hold.

Case II on the other hand requires more work. A natural suggestion is to represent the spin with a vector of magnitude $s+\frac{1}{2}$ and to let $\vec{s}\cdot\hat{n}$ be rounded off to the nearest integer (for integer spin $s$). With this procedure, every value of $m$ will have the same probability.

Look at Fig. 3. If $\vec{s}$ points through a shadowed surface, $A_1(\hat{a})A_2(\hat{b}) = +1$ (remember the complementary definitions of $A_1$ and $A_2$), and if it does not, $A_1(\hat{a})A_2(\hat{b}) = -1$. We need the solid angle of a spherical "lune" $L_\delta(\theta)$, and of a ring, $B_\delta$:

$$L_\delta(\theta) = 2\left[2\arccos\frac{\rho}{d} + \sin\frac{\delta}{2}\left[\arcsin\frac{d^2-\rho}{d(1-\rho)} - \arcsin\frac{d^2+\rho}{d(1+\rho)} - 2\arctan\frac{\rho\cos(\theta/2)}{d}\right]\right] \quad \text{for } \theta \geq \delta , \tag{28a}$$

$$L_\delta(\theta) = 0 \quad \text{for } \theta < \delta , \tag{28b}$$

where $\sin(\delta/2) = 1/(2s+1)$, $d = \cos(\delta/2)$, and $\rho = [\cos^2(\delta/2) - \cos^2(\theta/2)]^{1/2}$ have been introduced;

$$B_\delta = 4\pi\sin\frac{\delta}{2} = \frac{4\pi}{2s+1} . \tag{29}$$

The correlation function can then be written

$$f_{\mathrm{cl}}(\theta) = -\left[3 - \frac{4}{2s+1}\right] + \frac{1}{\pi}[L_\delta(\theta) + L_\delta(\pi-\theta)] . \tag{30}$$

The correlations functions for cases I and II are given in Figs. 4 and 5, respectively.

The third case is similar to the second. Here we have $\sin(\delta/2) = 1 - 1/(2s+1)$ and

$$f_{\mathrm{cl}} = -1 + \frac{4}{2s+1} - \frac{1}{\pi}L_\delta(\pi-\theta) , \tag{31}$$

where $s$ can be both integer or half integer.

It is not immediately obvious that the two last correlation functions satisfy Bell's inequality (3), but they are derived from models which are both local and realistic, so we can safely conclude that they do.

In connection with Fig. 4 we should mention a remark due to Bell.[9] If $f(\theta)$ is subject to the restrictions (i) $f(\pi-\theta) = -f(\theta)$, (ii) $f(0) = -1$; (iii) $f(\theta)$ satisfies Bell's inequality (3), then $f(\theta)$ cannot

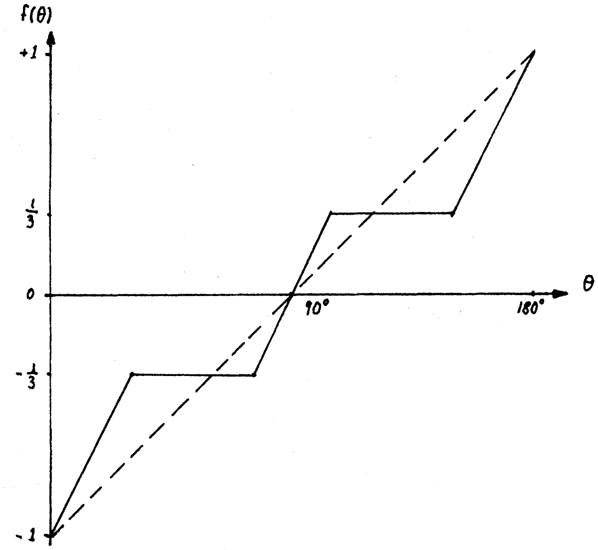


FIG. 6. A function which does not violate Bell's inequality. The slope is $k > 2/\pi$ or 0.

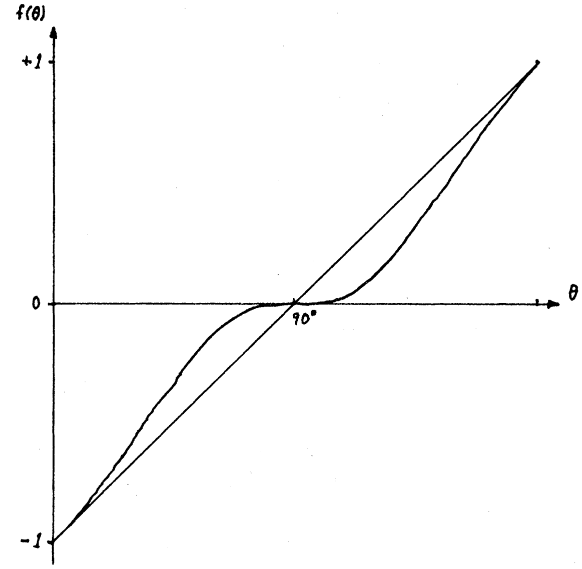


FIG. 7. A function which does violate Bell's inequality. $f(\theta) = -\cos\theta + (2\theta/\pi)[1-(2\theta/\pi)^2]$, $\theta_m = 31°$.

give stronger correlation than that given by the classical straight line for arguments $\theta = \pi/n$ (and $\theta = \pi - \pi/n$) $n = 1,2,\ldots$. This does not have to be true for other arguments.

To see this, first note that if all three directions are chosen in the same plane, then, from Bell's inequality (3),

$$f(\theta+\vartheta) \le f(\theta) + f(\vartheta) + 1 \Longrightarrow$$
$$f(n\theta) \le nf(\theta) + n - 1 \Longleftrightarrow \qquad (32)$$
$$f\left[\frac{\theta}{n}\right] \ge -1 + \frac{f(\theta) - (-1)}{n} ,$$

for which $\theta = \pi$ yields the desired result

$$f\left[\frac{\pi}{n}\right] \ge -1 + \frac{2}{n} . \qquad (33)$$

To prove the last part, consider the function in Fig. 6. It is not hard to verify that it satisfies conditions (i), (ii), and (iii), but it is not bounded by the classical limit.

It is also possible to produce functions, which do not exceed the classical limit, but nevertheless violate Bell's inequality (see Fig. 7).

## VI. CONCLUSION

The range of settings for which quantum mechanics violate Bell's inequality has been calculated for three different sequences of dichotomic observables. The results are found to be of the same magnitude, at least for small $s$, and in every case larger than those obtained by Mermin, who used a more sophisticated approach. It has been shown that a violation always occurs, and the choice of observables does not seem to be critical. It is also noted that the violation of Bell's inequality is not primarily due to the strength of the correlation but depends crucially on the shape of the correlation function, although a strong correlation is necessary.

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[10]This is the crucial step, and it can be justified from a

hypothesis of local realism. Several ways to do this have been suggested, see Refs. 4–6.

[11]The choice is obtained by looking for a locally maximal violation when $\theta_{ab}$ is fixed.

[12]$f(\theta)$ will be formed from rotation matrix elements:

$$f(\theta)=\sum d^s_{ij}d^{*s}_{kl}\ ,$$

$$d^s_{m_1m_2}(\theta)=\sum_r(-1)^r\frac{(s+m_1)!(s+m_2)!(s-m_1)!(s-m_2)!}{(s+m_2-r)!(s-m_1-r)!(m_1-m_2+r)!r!}\,i^{m_1-m_2}(\cos\theta/2)^{2s-(2r+m_1-m_2)}(\sin\theta/2)^{2r+m_1-m_2}\ .$$

[13]The quantum-mechanical expectation value of $m_1(\hat{a})m_2(\hat{b})$ is easily calculated using the singlet wave function (12),

$$\begin{aligned}\langle m_1(\hat{a})m_2(\hat{b})\rangle &\equiv\langle\psi_s\,|\,\vec{s}_1\cdot\hat{a}\,\vec{s}_2\cdot\hat{b}\,|\,\psi_s\rangle\\ &=\hat{a}\cdot\hat{b}\langle\psi_s\,|\,\vec{s}_1\cdot\hat{a}\,\vec{s}_2\cdot\hat{a}\,|\,\psi_s\rangle\\ &=-\hat{a}\cdot\hat{b}\frac{1}{2s+1}\sum_{m=-s}^{s}m^2\\ &=-\tfrac{1}{3}s(s+1)\hat{a}\cdot\hat{b}\ .\end{aligned}$$

[14]That this really is the proper wave function can be verified by operating on it with $\vec{s}\cdot\hat{n}$ and the corresponding step operators.

[15]L. S. Bartell (Dept. of Chemistry, Ann Arbor, Michigan) has independently worked out the spin-$\frac{3}{2}$ case (with positive and negative $m$ values lumped together) (private communication to J. S. Bell).