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Modification of Dirac's method of Hamiltonian analysis for constrained systems

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A slight modification of Dirac's method of Hamiltonian analysis for constrained systems is introduced. It leads to a verification of Dirac's conjecture that first-class secondary constraints are always symmetry generators in each of the counterexamples to that conjecture which have appeared in the recent literature. Those counterexamples associated with differentiable Hamiltonians are studied here; the cases involving nondifferentiable Hamiltonians will be considered separately. The relationship between the Lagrangian and Hamiltonian descriptions is studied in some detail and is used to motivate our calling the form of the secondary constraints derived via this modified method the natural form of the secondary constraints. Along the way we distinguish between symmetries of the Lagrangian (or Hamiltonian) and symmetries of the Euler-Lagrange (or Hamilton's) equations; we also distinguish between form and content invariance.

I. INTRODUCTION

The procedure introduced by Dirac[1] to deal with constrained Hamiltonian systems provides a systematic way to obtain a Hamiltonian from a Lagrangian, even when the momenta conjugate to the generalized coordinates are not all independent functions of the velocities. In many such cases, because there are more degrees of freedom appearing in the formal description than the number of actual physical degrees of freedom, some elements of the formal description may be chosen arbitrarily without affecting any physical predictions. This freedom of choice implies the existence of symmetry transformations which change the value of the arbitrary elements without changing the physical state of the system.

Dirac's Hamiltonian formalism is particularly suited to discover the dynamical generators of internal symmetries. In fact, some of these generators (linear combinations of the first-class primary constraints) are automatically found through the determination of a Hamiltonian, H_T, which gives the correct time development of the system. Other symmetry generators often appear in the formalism in the form of first-class secondary constraints. Dirac postulated that these latter always generate symmetry transformations.[2] If this is true, then the first-class secondary constraints can always be added to H_T with arbitrary coefficients to form a new Hamiltonian H_E, the so-called extended Hamiltonian, which will generate a time development physically indistinguishable from that given by H_T.

Several examples which appear to contradict this hypothesis have been found.[3–6] In each of these counterexamples, as analyzed in the recent literature, there are first-class secondary constraints which induce observable changes in the dynamical variables; the extended Hamiltonian generates equations of motion which (unlike those associated with H_T) differ in content from the Euler-Lagrange equations.

Those counterexamples for which the Hamiltonian is differentiable are reexamined here. It is found that there is a method of Hamiltonian analysis which, when systematically applied to each of these cases, yields results in full agreement with Dirac's conjecture. The counterexamples with Hamiltonians which are not differentiable will be considered separately.[7]

Section II is devoted to a discussion of the Hamiltonian analysis to be applied to the examples of Sec. IV. The method used here is, for the most part, consistent with the method described in Dirac's book[2] and in the text by Sudarshan and Mukunda[8]; it is therefore only sketched here. It differs from the standard approach only in that, given in the form of the primary constraints, it explicitly indicates a preferred form for the secondary constraints and in the fact that this preferred form differs from the one emphasized by Dirac in his original exposition.[1] Since his emphasis was closely related to his early definition of weak equality, we explore the concepts of weak and strong equality in some detail and end up with definitions which are somewhat different from the usual ones.

In Sec. III, the types of symmetry which may be manifested by the examples are considered, and the relationship between symmetries in the Lagrangian formulation and symmetries in the Hamiltonian formulation is explored. The concept of content (as opposed to form) invariance is introduced. The examples are analyzed in Sec. IV. First, symmetries of the Lagrangian description are studied and are used to predict the form of the dynamical symmetry generators. Then the Hamiltonian analysis is applied. In each case, it is found that the first-class secondary constraints generate symmetry transformations and that they agree with the generators which have been predicted by the Lagrangian-inspired analysis. Furthermore, if the symmetry transformations are applied to the Lagrangian itself, a new Lagrangian is obtained which leads to equations of motion equivalent to the original Euler-Lagrange equations and to a new canonical Hamiltonian closely related to the extended Hamiltonian. Section V is reserved for some general comments and conclusions. It indicates that the method used in this work may provide a systematic way to understand which first-class secondary constraints generate symmetry transformations. In fact, even this examination of a few particular examples leads to some insight into the connection between the symmetry transformations generated by a constraint ϕ and the symmetry transformations generated by that secondary constraint χ which arises from the condition $d\phi/dt=0$. In the Appendix, the question of the choice of form of the primary constraints is considered.

II. THE METHOD OF HAMILTONIAN ANALYSIS

Consider a system which has a finite number of degrees of freedom.[9] There are N dynamical coordinates q_n and N velocities $\dot{q}_n=dq_n/dt$. The dynamics is determined by a Lagrangian $L(q,\dot{q})$, in terms of which the N momentum variables are defined:

$$p_n=\frac{\partial L}{\partial \dot{q}_n} . \tag{2.1}$$

If the momenta are not all independent functions of the velocities, then (2.1) implies a set of relations

$$\phi_m(q,p)=0 , \quad m=1,\ldots,M . \tag{2.2}$$

The functions $\phi_m(q,p)$ are the primary constraints of the theory. The choice of form of the primary constraints will be considered in the Appendix. For now, we assume that we have chosen some particular form $\phi_m(q,p)$ on which to base the discussion which follows. In the cases to be examined in Sec. IV, the primary constraints arise because there is some coordinate y whose time derivative does not appear in the Lagrangian. Thus, the momentum definitions yield $p_y=0$, which we will take to be the primary constraint.

The Hamiltonian is defined to be $p_i\dot{q}_i-L$. It is a property of the Legendre transformation that, even though the velocities cannot all be expressed in terms of the coordinates and momenta if there are primary constraints, the Hamiltonian depends only on the coordinates and momenta when the momentum definitions (2.1) are used.[2,8]

Thus, we may write

$$H(p,q)=p_i\dot{q}_i-L . \tag{2.3}$$

The calculus of variations applied to this (constrained) Hamiltonian leads to the equations of motion

$$\begin{aligned} \dot{q}_i&=\frac{\partial H}{\partial p_i}+u_m\frac{\partial \phi_m}{\partial p_i} , \\ \dot{p}_i&=-\frac{\partial H}{\partial q_i}-u_m\frac{\partial \phi_m}{\partial q_i} , \end{aligned} \tag{2.4}$$

where the coefficients u_m are unknown. If the Poisson bracket (PB) between two dynamical functions is defined in the usual way, then the time derivative of any dynamical function may be expressed as

$$\frac{df(p,q)}{dt}=\dot{f}(p,q)=[f,H]+u_m[f,\phi_m]+\frac{\partial f}{\partial t} . \tag{2.5}$$

The form of Eqs. (2.4) and (2.5) suggests the following useful definition. H_T, which is given by

$$H_T=H+u_m\phi_m , \tag{2.6}$$

is called the *total Hamiltonian.* Since, at this point, the u_m's are not determined, any terms on the right-hand side (RHS) of (2.3) which are proportional to the primary constraints can be absorbed into the sum $u_m\phi_m$. Thus we will, without loss of generality, take H in Eq. (2.6) to be that part of the Hamiltonian which does not include any terms proportional to the primary constraints. Although the term $\partial f/\partial t$ must be included on the RHS of (2.5) for any f which has an explicit time dependence, its presence is not essential to the discussion which follows. Hence, for the sake of brevity, it will not be written out explicitly when the time derivatives of the constraints are computed; of course, in any theory in which the constraints are explicitly time dependent,[10] such a term will appear, but the results derived below will be unaffected.

The primary constraints ϕ_m must be preserved in time. Thus, for each k and all i, it must be the case that

$$\frac{d^i\phi_k}{dt^i}=0 . \quad (2.7)$$

Before explicitly imposing these consistency conditions, it is essential to reexamine the idea of equality as it applies to relations involving dynamical variables. Let us start with the following simple equation involving the coordinate x,

$$x=0 . \quad (2.8)$$

Clearly the PB of p_x, the momentum conjugate to x, with the left-hand side (LHS) of this equation is different from the PB of p_x with the RHS of the equation. Thus if the value zero were to be substituted for x in any expression before all relevant PB's with the expression were computed, any bracket involving p_x would be computed incorrectly. Since the equality in Eq. (2.8) must be used with care it has been given a special name and designation[1]; it is called a weak equality and is written $x\approx 0$. Any equality between dynamical functions $f(q,p)$ and $g(q,p)$ is a weak equality if there exists any dynamical function $h(q,p)$ whose PB with f is different in value (when evaluated on the hypersurface in phase space on which the constraint equations are satisfied) from its PB with g.

If we now examine the equation

$$x^2=0 , \quad (2.9)$$

we see that, since the value of x is zero, the value of the PB of any well-behaved function h with the LHS of (2.9) (being simply $2x[h,x]$) is zero, as is its bracket with the RHS. For this reason, Eq. (2.9) would, in some of the literature on this subject, be called a strong equation—but this nomenclature will not be used here. We avoid it because the equality in (2.9) must also be handled with care and it is useful to give it a name that explicitly expresses this. For example $[p_x,[p_x,x^2]]$ is certainly not equal to zero and thus in (2.9), as in (2.8), we may not use the equality before computing *all* relevant PB's.

In this work, two functions $f(q,p)$ and $g(q,p)$ which are equal when evaluated on the constraint hypersurface are said to be *weakly equal* if there exists a set of dynamical functions $h=\{h_j(q,p)\}$ which contains a finite number J of elements (not necessarily all distinct) such that

$$[h_J,\ldots[h_2,[h_1,f]]\ldots] \neq [h_J,\ldots[h_2,[h_1,g]]\ldots] \quad (2.10)$$

on the constraint hypersurface. With this definition, every constraint $\phi(q,p)$ which can be expressed as a polynomial in the coordinates and momenta is weakly zero. Instead of (2.9) then, we should write $x^2\approx 0$.

Any equality which is not a weak equality is a *strong equality.* Numerical equalities are strong equalities as are definitions.

These definitions of weak and strong equality serve to classify equalities on the basis of their qualitative features. A weak equality is one which can be broken by successive PB's; a strong equality can never be broken in this way. It is in this emphasis on qualitative features that the definitions above differ from the more commonly used definitions.[11]

In order to be precise, it is necessary to specify the constraint hypersurface on which a weak equality is valid. In what follows, the symbol $\underset{(0)}{\approx}$ will be used to indicate a weak equality which is valid on that hypersurface in phase space on which the primary constraints vanish. The unadorned symbol $\approx$ denotes an equality that holds on that hypersurface on which *all* of the constraints to which the system is subject are zero.

We can now require that Eqs. (2.7) hold. We start with the first time derivative and indicate that it is sufficient for the equality to be a weak one,

$$[\phi_k,H]+u_m[\phi_k,\phi_m]\approx 0 . \quad (2.11)$$

Equation (2.11) can be viewed as a set of inhomogeneous linear equations to be solved[2] for the coefficients u_m. It may happen that this set of equations cannot be satisfied by simply making an appropriate choice of the u_m's or by virtue of the vanishing of the primary constraints. However, if the system is consistent, it will be possible to find a solution if we can set certain dynamical functions, $\chi^{(1)}$, to zero; such dynamical functions are called secondary constraints.

Because we will eventually be interested in the transformations they generate, we need to be as definite about the form of the secondary constraints as we can be without having a particular theory or type of theory in mind. Equations (2.11) are linear equation; suitable linear methods can be used to solve for the u_m's and, in this work, only linear methods will be used to discover the secondary constraints. To illustrate this point consider a system in which the expression for $\dot{\phi}_k$ (i.e., $[\phi_k,H]+u_m[\phi_k,\phi_m]$) does not explicitly involve any of the u_m's. Then

$$\dot{\phi}_k=[\phi_k,H]=\alpha_{km}\phi_m+\chi_k^{(1)} , \quad (2.12a)$$

that is,

$$\dot{\phi}_k \underset{(0)}{\approx} \chi_k^{(1)} , \quad (2.12b)$$

where we assume that $\chi_k^{(1)}$ is independent of the primary constraints. If $\chi_k^{(1)}$ is not identically zero, then it must be set (weakly) to zero; it is a secondary constraint. No matter what its explicit functional form,

$\chi_k^{(1)}$ itself is the secondary constraint and not its square or square root or any nonlinear modification of it. The symbol $\underset{(1)}{\approx}$ will be used to indicate that an equality is true if both the primary constraints and the $\chi^{(1)}$ vanish. We have $\dot{\phi}_k \underset{(1)}{\approx} 0$.

Like the primary constraints, the secondary constraints must also be preserved in time. In order to ensure that they are, the consistency procedure which was executed for the ϕ's must be repeated for the $\chi^{(1)}$'s. Once again this may lead to definite expressions for some of the u_m's or to further secondary constraints $\chi^{(2)}$. The $\chi^{(2)}$'s must also be preserved in time and so the process continues.

The following notation will be useful. Let $\chi_i^{(q)}$ represent a constraint which is derived during the qth stage of the consistency procedure; that is, the condition $\chi_i^{(q)} \approx 0$ is implied by a relation $(d/dt)(a_{ij}\chi_j^{(q-1)}) \approx 0$. It will sometimes be convenient to write a primary constraint ϕ_j as $\chi_j^{(0)}$. Further, let $\underset{(q)}{\approx}$ denote an equality which is valid on that hypersurface in phase space on which all of the constraints, primary and secondary, which have been found after following the consistency procedure q times are zero; it will be called a qth-stage weak equality.

Consistency has been assured when further applications of the consistency procedure lead only to identities or to constraints which have already been derived. It is important to be clear about which constraints can legitimately be considered to have been "already derived." To make this determination for any particular case we examine the action of the constraints in question as the generators of canonical transformations, since it is this action which will be the focus of our attention. Let C_1 be a constraint which generates transformations $\delta_1 q_k$ and $\delta_1 p_k$ in the dynamical coordinates: $\delta_1 q_k = \alpha_1[q_k, C_1]$, $\delta_1 p_k = \alpha_1[p_k, C_1]$, with α_1 completely arbitrary. If C_2 is a constraint which is derived at a later point in the calculations and if the transformations $\delta_2 q_k = \alpha_2[q_k, C_2]$, $\delta p_k = \alpha_2[p_k, C_2]$ are, for arbitrary α_2, just a special case of $\delta_1 q_k$ and $\delta_1 p_k$, then C_2 may be considered to have been already derived. In general, if C_k' can be expressed as $a_{kj}C_j$, where the C_j are constraints which have already been derived and the a_{kj} are well-behaved coefficients, then C_k' itself may be considered to have been already derived. The a_{kj} may be dynamical functions; however, even if they are, they are never responsible for any observable changes which may be generated by C_k' since, for all dynamical functions $V(q,p)$,

$$\delta_k' V(q,p) = \alpha_k[V(q,p), a_{kj}]C_j + \alpha_k a_{kj}[V(q,p), C_j] ,$$

where k is not summed over. Thus, the only transformations induced by C_k' which are not obviously simply sums of constraints are those which are induced by the action of the C_j's; we are therefore justified in considering these latter as fundamental and the C_k''s as already derived.

For systems in which the consistency analysis is as straightforward as it can be (and this includes many cases of physical interest), the secondary constraints can simply be derived as time derivatives of primary or already derived secondary constraints. However, it is possible for the situation to be more complicated. For example, it may happen that all of the consistency relations involve the u_m but only a subset of these relations is independent, so that there will be secondary constraints which do not emerge automatically. We therefore consider the general case to demonstrate the existence of a definite form for all secondary constraints. (These considerations also provide a convenient algorithm for working out these more complicated applications.[12])

Suppose that the tth application of the consistency procedure (where t is less than or equal to n, the number of applications necessary to ensure consistency) yields M_t distinct secondary constraints $\chi_b^{(t)}$, and a total Hamiltonian

$$H_T = H^{(t)} + v_a^{(t)}\Phi_a^{(t)} , \tag{2.13}$$

where $H^{(t)}$ is completely determined, the $v_a^{(t)}$'s are completely arbitrary, and the $\Phi_a^{(t)}$'s are linear combinations of the primary constraints. For $t=0$, $H^{(0)} = H$, which is in fact completely determined, the $v_a^{(0)}$'s are just the u_a's, which are in fact completely arbitrary at this stage, and the $\Phi_a^{(0)}$'s are simply the primary constraints. The $\chi_b^{(0)}$'s are not secondary constraints, but are instead the primary constraints.

We must impose the conditions[13]

$$[\chi_i^{(t)}, H^{(t)}] + v_a^{(t)}[\chi_i^{(t)}, \Phi_a^{(t)}] \approx 0 . \tag{2.14}$$

Let (2.14) contain R_t linearly independent equations. Consider the homogeneous equation $B_{ca}^{(t)}[\chi_b^{(t)}, \Phi_a^{(t)}] \underset{(t)}{\approx} 0$ and define $\Phi_c^{(t+1)}$ to be $B_{ca}^{(t)}\Phi_a^{(t)}$; there are $M_t - R_t$ such linear combinations and each has the property that $[\chi_b^{(t)}, \Phi_c^{(t+1)}] \underset{(t)}{\approx} 0$. Write the total Hamiltonian as $H^{(t)} + u_i^{(t+1)}\psi_i^{(t+1)} + v_a^{(t+1)}\Phi_a^{(t+1)}$, where the $\psi_i^{(t+1)}$ are R_t linear combinations of the $\Phi_a^{(t)}$ which are independent of the $\Phi_c^{(t+1)}$. When the total Hamiltonian is expressed in this way it is clear that the conditions $\dot{\chi}_b^{(t)} \approx 0$ cannot restrict the $v_a^{(t+1)}$ and so these coefficients remain arbitrary.

We will now show that there are as many as R_t linear combinations of the $\chi_b^{(t)}$'s for which the condition that their time derivatives vanish serves to fix the $u_i^{(t+1)}$'s. We will also show that there are at least $M_t - R_t$ combinations whose time derivatives do not (weakly) involve the $u_i^{(t+1)}$'s at all; these combinations may possibly lead to further secondary constraints.

Define the $M_t \times R_t$ matrix A_{bi} to be $[\chi_b^{(t)}, \Psi_i^{(t+1)}]$. There are nonsingular matrices S (an $M_t \times M_t$ matrix) and T (an $R_t \times R_t$ matrix), such that

$$SAT = A' \approx \begin{bmatrix} \mathbb{1}_{R_t} \\ 0 \end{bmatrix} , \tag{2.15}$$

where $\mathbb{1}_{R_t}$ denotes the $R_t \times R_t$ identity matrix. Now consider the combinations $S_{cb}\chi_b^{(t)}$, and define $u_j^{(t+1)\prime}$ to be $T_{ji}^{-1} u_i^{(t+1)}$,

$$(S_{cb}\chi_b^{(t)})^{\cdot} \underset{(t)}{\approx} S_{cb}[\chi_b^{(t)}, H^{(t)}] + S_{cb}[\chi_b^{(t)}, \Psi_m^{(t+1)}] u_m^{(t+1)} , \tag{2.16a}$$

$$(S_{cb}\chi_b^{(t)})^{\cdot} \underset{(t)}{\approx} S_{cb}[\chi_b^{(t)}, H^{(t)}] + S_{cb} A_{bm} T_{mj} u_j^{(t+1)\prime} , \tag{2.16b}$$

$$(S_{cb}\chi_b^{(t)})^{\cdot} \underset{(t)}{\approx} S_{cb}[\chi_b^{(t)}, H^{(t)}] + \left[\begin{bmatrix} \mathbb{1}_{R_t} \\ 0 \end{bmatrix} u^{(t+1)\prime} \right]_c . \tag{2.16c}$$

The last $M_t - R_t$ equations do not involve the coefficients u' at all and may possibly lead to further secondary constraints $\chi_c^{(t+1)} \underset{(t)}{\approx} S_{cb}[\chi_b^{(t)}, H^{(t)}]$. If the first R_t equations are still linearly independent after the $\chi_c^{(t+1)}$ have been set to zero, then they determine the first R_t u''s. If only R_t' of these R_t equations are independent at this point, then there will be $R_t - R_t'$ linear combinations of the $\Psi_i^{(t+1)}$, which we will denote by $\Phi_d^{(t+1)}$ with d running from $M_t - R_t$ to $M_t - R_t'$, which have the property $[\chi^{(t)}, \Phi_d^{(t+1)}] \underset{(t+1)}{\approx} 0$. There will also be R_t' independent combinations of the $\Psi_i^{(t+1)}$'s which we will denote by $\Psi_m^{(t+1)\prime}$ with m running from 1 to R_t'. The total Hamiltonian may now be rewritten in terms of the $\Psi_m^{(t+1)\prime}$'s and the complete set of $\Phi_a^{(t+1)}$'s, and the procedure outlined above for the $\Phi_a^{(t+1)}$'s and the $\psi_i^{(t+1)}$'s can again be followed.

Let us assume that n applications of the consistency procedure are sufficient to ensure that the constraints are preserved in time. The total Hamiltonian can be written as

$$H_T = H + U_m \varphi_m + v_a^{(n)} \Phi_a^{(n)} , \tag{2.17}$$

where the U_m's are fixed, the $v_a^{(n)}$'s are arbitrary, and the $\Phi_a^{(n)}$'s are linear combinations of the primary constraints which have weakly vanishing PB with every constraint, primary or secondary. Now that the consistency procedure has been completed there are K secondary constraints. They may be added to the primary constraints to form the set $C = \{\phi_t(p,q) : \phi_t \approx 0,\ t = 1,\ M+K\}$. A dynamical function is said to be first class if it has weakly vanishing PB with every element of C, i.e., with all of the constraints, primary and secondary. Any dynamical function which is not first class is second class.[2]

The $\Phi_a^{(n)}$ are first class. Because they enter into the total Hamiltonian with arbitrary coefficients it is possible to show that they produce changes in the dynamical variables which do not change the physical state when they act as the generators of infinitesimal canonical transformations.[2]

In addition to the $\Phi_a^{(n)}$ there may be other linear combinations of the constraints, some involving just the secondary constraints and some involving both secondary and primary constraints, which are also first class. Dirac has conjectured[2] that these may also act as symmetry generators.[14] This would mean that the extended Hamiltonian,

$$H_E = H_T + \omega_b \chi_b , \tag{2.18}$$

where the ω_b are arbitrary and the summation is over *all* first-class constraints χ_b, would always generate the same physical time development as H_T.

The examples considered here seemed to deny the validity of this conjecture. Before going on to study these counterexamples, it is useful, both to see how this method actually works and to compare it with the one more commonly used, to study the following simple system:

$$L = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 y . \tag{2.19}$$

The momenta are $p_x = \dot{x}$, $p_y = 0$, and the Euler-Lagrange equations are $\ddot{x} = xy$, $\frac{1}{2}x^2 = 0$. The system described by this Lagrangian has an x coordinate which is fixed to be zero and a y coordinate which is arbitrary. The Hamiltonian is $H = \frac{1}{2}p_x^2 - \frac{1}{2}x^2 y$, and the total Hamiltonian is $H_T = \frac{1}{2}p_x^2 - \frac{1}{2}x^2 y + \alpha p_y$. The consistency conditions are

$$\dot{p}_y = \tfrac{1}{2}x^2 \approx 0 , \tag{2.20a}$$

$$\ddot{p}_y = x\dot{x} = x p_x \approx 0 , \tag{2.20b}$$

$$\dddot{p}_y = \dot{x} p_x + x \dot{p}_x = p_x^2 + x^2 y \approx 0 \Longrightarrow p_x^2 \approx 0 , \tag{2.20c}$$

$$\ddddot{p}_y \approx \frac{d}{dt} p_x^2 = 2 p_x x y \approx 0 . \tag{2.20d}$$

From the viewpoint of ensuring consistency we are finished, since (2.20d) is satisfied as a consequence of (2.20b) and it is obvious that all further time derivatives will involve at most linear combinations of the constraints x^2, xp_x, and $p_x{}^2$. Thus by setting these three dynamical functions (weakly) to zero, we guarantee that $d^n p_y/dt^n \approx 0$ for all n. The full set of constraints is $\{p_y, x^2, xp_x, p_x{}^2\}$. Each of its elements is first class, and each is a symmetry generator. To see this, consider the action of the generator $\alpha p_x{}^2 + \beta x^2 + \gamma x p_x + \sigma p_y$,

$$\begin{aligned} &x \rightarrow x + 2\alpha p_x + \gamma x \ , \\ &p_x \rightarrow p_x - 2\beta x - \gamma p_x \ , \\ &y \rightarrow y + \sigma \ , \\ &p_y \rightarrow p_y \ . \end{aligned} \tag{2.21}$$

Since none of these transformations changes the physical state, each is a symmetry transformation, and the extended Hamiltonian,

$$H_E = H_T + \alpha p_x{}^2 + \beta x^2 + \gamma x p_x + \sigma p_y \ ,$$

should (and does) generate equations of motion which have the same physical content as those generated by H_T, as long as α is not $-\frac{1}{2}$. (The concept of content invariance will be discussed in Sec. III.)

It is possible to follow another path[3,5] which diverges from the one sketched above during the derivation of the secondary constraints

$$\dot{p}_y = \tfrac{1}{2} x^2 \approx 0 \Longrightarrow x \approx 0 \ . \tag{2.22a}$$

The secondary constraint $x^2 \approx 0$ would be linearized. This leads to

$$\dot{x} = p_x \approx 0 \tag{2.22b}$$

and

$$\dot{p}_x = xy \approx 0 \ . \tag{2.22c}$$

Equation (2.22c) is satisfied by virtue of (2.22a); the consistency procedure is complete. The full set of constraints is $\{p_y, x, p_x\}$. The secondary constraints x and p_x are second class and, from the point of view of Dirac's conjecture, this is just as well since neither generates symmetry transformations.

The two treatments lead to the same physics. The difference between them lies in the form of the secondary constraints and so naturally also in the form of the transformations generated by the secondary constraints. This example has helped to illustrate that even though each secondary constraint χ has an explicit, well-defined form, it may be possible to obtain correct dynamical results by using another function $\psi(\chi)$ in its place to ensure consistency. In particular, if $\psi(\chi) \approx \chi \approx 0$ and $(d/dt)\psi(\chi) \approx d\chi/dt$, then imposition of $(d/dt)\psi(\chi) \approx 0$ will ensure the necessary condition $d\chi/dt \approx 0$. However, for an arbitrary function $f(q,p)$, the PB $[f(q,p), \psi(\chi)]$ may not be even weakly equal to $[f(q,p), \chi]$. This means that in general $\psi(\chi)$ and χ generate completely different transformations of the dynamical variables. Hence if χ is a symmetry generator, $\psi(\chi)$ may very well not be and vice versa.

III. THE SYMMETRIES: THE RELATIONSHIP BETWEEN SYMMETRIES OF THE LAGRANGIAN FORMULATION AND THOSE OF THE HAMILTONIAN FORMULATION

It is reasonable to expect that the symmetries of the Hamiltonian formulation will be closely related to those of the Lagrangian formulation. We should therefore be able to learn about one by studying the other. We will start with the Lagrangian description, study its symmetries, and use them to predict the symmetries of the Hamiltonian description. In order to make as close a contact with the Hamiltonian formulation as possible, we will allow q and $\dot{q}$ to transform independently. All transformations will be viewed as active transformations.

In this work consideration has been restricted to those symmetry transformations for which the transformation parameter can be chosen arbitrarily (except perhaps for isolated values). For example, transformations which, by the following definitions, are symmetry transformations only for constant values of the transformation parameter have been omitted.

A. Symmetry transformations of the physical state

In order to describe how the symmetries of the Hamiltonian description will be derived from those of the Lagrangian description, it will be useful to discuss symmetries of the equations of motion and symmetries of the Lagrangian and of the Hamiltonian as well as symmetries of the physical state. Of course, from a physicist's point of view, it is the symmetries of the physical state which are the most fundamental, and that is what is meant by the word "symmetry" as it is used, unmodified, in this paper.

In using the symmetries of the Lagrangian description to predict those of the Hamiltonian description, we begin by examining the symmetry transformations

$$\begin{aligned} &q \rightarrow q + \delta q(q,\dot{q}) \ , \\ &\dot{q} \rightarrow \dot{q} + \delta \dot{q}(q,\dot{q}) \end{aligned} \tag{3.1}$$

[where in general $\delta\dot{q} \neq (d/dt)\delta q$], which do not alter the physical state. Each of these will be translated into Hamiltonian language by replacing the velocities by their expression in terms of the coordinates and momenta,

$$q \to q + \delta q(q,\dot{q}(q,p)) ,$$
$$\dot{q}(q,p) \to \dot{q}(q,p) + \delta\dot{q}(q,\dot{q}(q,p)) . \tag{3.2}$$

Now each of these transformations can be thought of as having been generated by some generator[15] G with $\delta q = \epsilon[q,G]$ and $\delta p = \epsilon[p,G]$. For each transformation we will construct the associated G. Of course, in general G will also effect transformations other than the one being considered. For example, the generator $G = p_x p_y$ which generates $x \to x + \alpha p_y$ also necessarily generates the transformation $y \to y + \alpha p_x$. All other such transformations associated with G are found and are translated into Lagrangian language, which is to say that p is replaced by $p(q,\dot{q})$ wherever it appears,

$$q \to q + \delta q(q,p(q,\dot{q})) ,$$
$$p(q,\dot{q}) \to p(q,\dot{q}) + \delta p(q,p(q,\dot{q})) . \tag{3.3}$$

The full set of transformations of the coordinates and velocities which are associated with G consists of the original transformation and all of those others which are necessitated by the form G must take if it is to generate the original transformation. We ask whether or not each of the transformations in this set is a symmetry transformation. If the answer to this question is yes, then G should be a generator of symmetries within the Hamiltonian description.

Consider the system introduced in the preceding section. The transformation $x \to x' = x + \alpha x$ is a symmetry transformation. This transformation is generated by $G = xp_x$, a generator which also generates the transformation $p_x \to p_x - \alpha p_x$. In Lagrangian language, this additional transformation translates to $\dot{x} \to \dot{x} - \alpha\dot{x}$, which is also a symmetry transformation. xp_x is a symmetry generator.

B. Symmetry transformations of the equations of motion

If a transformation transforms the equations of motion into equations which have the same solutions as the original equations, then we will say that the equations of motion are *content* invariant under the transformation and the transformation will be called a symmetry transformation of the equations of motion.

Certainly if the equations of motion are form invariant under a particular transformation, then they are also content invariant under it and that transformation is a symmetry transformation of the equations of motion. The concept of content invariance is more general than that of form invariance though, because it is possible for a transformation to preserve the content of the equations of motion and yet to change their form.

Simple examples in which equations of the form

$$F = 0 ,$$
$$G = 0 , \tag{3.4}$$

are transformed into equations

$$F = 0 ,$$
$$G + \alpha F + \beta\dot{F} = 0 , \tag{3.5}$$

which describe the same physical system are examined in Sec. IV. There are other theories (with more physical content) for which the equivalence between the set of transformed equations and the original set is not as easily established. One such, which is still relatively easy, is the Maxwell theory.

Expressed in Hamiltonian form,[2] with the metric $(+,-,-,-)$, the equations of motion are

$$\pi^0 = 0 , \quad \partial_k \pi^k = 0 ,$$
$$\partial_0 A_0 = \text{arbitrary} , \quad \partial_0 A_k = \pi^k + \partial_k A_0 , \tag{3.6}$$
$$\partial_0 \pi^k = \partial_j(\partial_j A_k - \partial_k A_j) .$$

If $A_k \to A_k + \partial_k \Lambda$, where Λ is arbitrary, then all of the equations but one are unaffected. The equation

$$\partial_0 A_k + \partial_0 \partial_k \Lambda = \pi^k + \partial_k A_0 \tag{3.7}$$

is not of the same form as the original. But another of the equations which is preserved indicates that $\partial_0 A_0$, and hence A_0, is arbitrary. Thus, since (3.7) may be rewritten as

$$\partial_0 A_k = \pi^k + \partial_k(A_0 - \partial_0\Lambda) , \tag{3.8}$$

it is clear that $A_k \to A_k + \partial_k\Lambda$ is a symmetry transformation when taken *by itself*, even though it does not manifestly preserve the form of the equations of motion.

These examples provide illustrations of transformations which leave the physical content of the equations of motion unchanged even though they alter their form. The examples to be studied in the next section also have symmetries of this type. Thus even though they are obvious and easily recognizable symmetries, they are missed by an algorithm to find symmetry generators which requires form invariance[16] of the equations of motion.

Starting with a symmetry transformation of the equations of motion expressed in the language of the Lagrangian (i.e., in terms of the q's and $\dot{q}$'s), we may translate it into Hamiltonian language and proceed exactly as for symmetry transformation of the physical state to obtain symmetry generators G_i.

C. Symmetry transformations of the Lagrangian and Hamiltonian

Let us apply a transformation $T(q,\dot{q},\alpha)$ to the Lagrangian $L(q,\dot{q})$, where q is *not* assumed to satisfy the equations of motion. The transformed Lagrangian, $L'(q,\dot{q},\alpha)$, is a function of the coordinates, the velocities, and the transformation parameters α. The variational principle can now be applied to L', treating the α as numerical rather than dynamical variables. If and only if the equations of motion obtained in this way are content equivalent to those obtained directly from L, then $T(q,\dot{q},\alpha)$ will be called a symmetry transformation of the Lagrangian.

Similarly, if

$$H(q,p) \underset{T(q,p,\alpha)}{\longrightarrow} H''(q,p,\alpha) ,$$

and if H'' leads to equations of motion which are content equivalent to those generated by H, then $T(q,p,\alpha)$ will be called a symmetry transformation of the Hamiltonian.

The next step in learning about symmetries of the Hamiltonian description through studying the Lagrangian is taken by applying a symmetry transformation to the Lagrangian itself,

$$L(q,\dot{q}) \underset{T(q,\dot{q},\alpha)}{\longrightarrow} L'(q,\dot{q},\alpha) ,$$

where the equations of motion associated with L' are content equivalent to those associated with L. We can now go one step further by obtaining from L', via the Dirac formalism, the total Hamiltonian H_T'. H_T' will generate equations of motion equivalent to those generated by H_T, the total Hamiltonian derived from L, because L and L' yield equivalent equations of motion.

H_T' and H_T can now be compared. If their difference contains a sum of terms $\beta_i G_i$ where the β_i are arbitrary, then an argument of Dirac's indicates that each of the G_i's is a symmetry generator.

Consider, for example, the transformation $y \rightarrow y+\beta$, where β is completely arbitrary, applied to the Lagrangian of the preceding section,

$$L=\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}yx^2 \underset{y\rightarrow y+\beta}{\longrightarrow} L'=\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}yx^2+\tfrac{1}{2}\beta x^2$$
$$\rightarrow H'=\tfrac{1}{2}p_x^2-\tfrac{1}{2}yx^2-\tfrac{1}{2}\beta x^2 . \quad (3.9)$$

It is easily verified that L' and H' generate equations of motion which are equivalent to the original equations. Since x^2 appears in H' multiplied by an arbitrary coefficient, it must be a symmetry generator.

We can also apply transformations directly to the Hamiltonian itself. For example,

$$H_T \approx \tfrac{1}{2}p_x^2-\tfrac{1}{2}yx^2+\alpha p_y$$
$$\rightarrow H_T'' \equiv H_T+\delta H_T , \quad (3.10a)$$

where

$$\delta H_T=[H_T,\sigma p_y+\beta x^2+\gamma x p_x] ,$$
$$H_T''=\tfrac{1}{2}p_x^2-\tfrac{1}{2}yx^2+\alpha p_y-\tfrac{1}{2}\sigma x^2 \quad (3.10b)$$
$$-2\beta x p_x-\gamma p_x^2-\gamma y x^2 .$$

H_T'' generates a dynamical development which is physically equivalent to that generated by H_T; therefore x^2, xp_x, p_x^2, and p_y are symmetry generators.

We note that the set of symmetry transformations of the Hamiltonian (or Lagrangian) is not necessarily the same as the set of symmetry transformations of the physical state.

IV. THE EXAMPLES

A. Cawley's first counterexample

The first Lagrangian to be considered here is one which was introduced by Cawley,[3]

$$L=\dot{x}\dot{z}+\tfrac{1}{2}yz^2 . \quad (4.1)$$

This Lagrangian leads to the momenta

$$p_x=\dot{z} , \; p_z=\dot{x} , \; p_y=0 , \quad (4.2)$$

and to the equations of motion

$$z^2=0 , \ddot{z}=0 , \; \ddot{x}=yz . \quad (4.3)$$

Their solution sets z to be zero, $\dot{x}$ to be constant, and leaves y arbitrary. Note the presence of the primary constraint p_y.

The following transformations leave the content of Eqs. (4.3) unchanged:

$$\begin{aligned} &y \rightarrow y+\beta , \\ &z \rightarrow z+\lambda z \quad (\lambda \neq -1) , \\ &\dot{x} \rightarrow \dot{x}+\Sigma z+\eta\dot{z} , \\ &\dot{z} \rightarrow \dot{z}+\phi\dot{z}+\psi z , \\ &x \rightarrow x+\sigma z+\xi\dot{z} . \end{aligned} \quad (4.4)$$

Here Greek letters have been used to denote arbitrary functions of the time.

These transformations are listed separately, in the first column of Table I. Each transformation $T(q,\dot{q})$ is translated into its Hamiltonian equivalent

TABLE I. This table refers to Cawley's first Lagrangian (4.1). The symmetry transformations of the Euler-Lagrange equations, displayed in the first column, are translated into Hamiltonian language (column 2) so that possible symmetry generators $G(q,p)$ (column 3) can be identified. Any other transformations associated with $G(q,p)$ (column 4) are translated back to Lagrangian language (column 5) to determine whether or not they, too, leave the Euler-Lagrangian equations content invariant. This table indicated that p_y, z^2, zp_x, and $p_x{}^2$ are good candidates for symmetry generators.

Transformation $T(q,\dot q)$	$T(q,p)$	$G(q,p)$ Generator of $T(q,p)$	$T'(q,p)$ Transformations generated by G	$T'(q,\dot q)$	Are all transformations generated by G symmetry transformations?
$y\to y+\beta$	$y\to y+\beta$	p_y	None		Yes
$z\to z+\lambda z$	$z\to z+\lambda z$	zp_z	$p_z\to p_z-\lambda p_z$	$\dot x\to \dot x-\lambda \dot x$	No
$\dot x\to \dot x+\Sigma z$	$p_z\to p_z+\Sigma z$	z^2	None		Yes
$\dot x\to \dot x+\eta \dot z$	$p_z\to p_z+\eta p_x$	zp_x	$x\to x-\eta z$	$x\to x-\eta z$	Yes
$\dot z\to \dot z+\phi \dot z$	$p_x\to p_x+\phi p_x$	xp_x	$x\to x-\phi x$	$x\to x-\phi x$	No
$\dot z\to \dot z+\psi z$	$p_x\to p_x+\psi z$	xz	$p_z\to p_z+\psi x$	$\dot x\to \dot x+\psi x$	No
$x\to x+\xi \dot z$	$x\to x+\xi p_x$	$p_x{}^2$	None		Yes
$x\to x+\sigma z$	$x\to x+\sigma z$	zp_x	$p_z\to p_z-\sigma p_x$	$\dot x\to \dot x-\sigma \dot z$	Yes

$T(q,p)$ in column two. In column three the generator $G(q,p)$ which generates $T(q,p)$ appears and it is followed (in column four) by any other transformations, $T'(q,p)$, also generated by it. Finally, in column five, the $T'(q,p)$ are translated into their Lagrangian equivalents and, by asking whether all of the transformations $T'(q,\dot q)$ are [with $T(q,\dot q)$] symmetry transformations of the Euler-Lagrange equations, we identify the special set of generators $\{p_y,z^2,zp_x,p_x{}^2\}$. Each of the elements of this set should, in the Hamiltonian formalism, generate symmetry transformations.

p_y is the primary constraint while all of the other elements of the set are set to zero by the equations of motion. Furthermore, each element of the set has weakly vanishing PB's with all of the others. It is tempting to predict that z^2, zp_x, and $p_x{}^2$ will be secondary constraints in the Hamiltonian formulation; such a prediction would be accurate.

Before going on to verify this last statement it is interesting to study the transformations (4.4) from another point of view. They may be applied directly to the Lagrangian L to yield L',

$$L'=(\dot x+\Sigma z+\eta\dot z)(\dot z+\phi\dot z+\psi z)+z^2[\tfrac{1}{2}(y+\beta)(1+\lambda)^2]\ . \tag{4.5}$$

(Note that we have not assumed that the transformation parameters are infinitesimal.)

Taken as a new Lagrangian, L' implies equations of motion which are equivalent to those derived from L as long as neither λ nor ϕ have the value -1 (so we exclude these values), and if and only if $\psi=\dot\phi$. With this substitution, L' leads to the total Hamiltonian H_T':

$$H_T'=\alpha p_y+\frac{p_zp_x}{1+\phi}+zp_x\left[\frac{\eta\dot\phi}{1+\phi}-\Sigma\right]-\frac{\eta}{1+\phi}p_x{}^2-\frac{\dot\phi}{1+\phi}zp_z-\tfrac{1}{2}(1+\lambda)^2z^2(y+\beta)\ . \tag{4.6}$$

The only dynamical functions in H_T' which multiply coefficients which are arbitrary and independent (independent in that they are not necessarily related to any of the other coefficients) are p_y, which multiplies α; zp_x, because of the presence of Σ; $p_x{}^2$, because of η; and z^2, because of the presence of β. Thus, H_T' can, without loss of generality, be written as

$$H_T'=\alpha p_y+\beta zp_x+\gamma p_x{}^2+\sigma z^2+-\tfrac{1}{2}(1+\lambda)^2yz^2+\frac{p_zp_x}{1+\phi}-\frac{\dot\phi}{1+\phi}zp_z\ . \tag{4.7}$$

As expected, H_T' generates equations of motion equivalent to the original Euler-Lagrange equations if $\phi\neq-1$ and $\lambda\neq-1$. The secondary constraints are the same as before and the equations of motion describe the same system. Since p_y, zp_x, $p_x{}^2$, and z^2 all multiply completely arbitrary coefficients in H_T', they are all symmetry generators. Nothing is learned about the symmetry properties of either p_xp_z

or zp_z since their coefficients are related—once the coefficient of p_xp_z, $1/[1+\phi(t)]$, is fixed, so is $-\phi(t)/[1+\phi(t)]$, the coefficient of zp_z. However, by examining the transformations they effect on the dynamical variables, it becomes clear that each of these generators generates symmetry transformations only for constant values of the transformation parameter. Therefore, neither p_xp_z nor zp_z could appear multiplied by independent arbitrary coefficients in a dynamically correct Hamiltonian. Thus, although their value is zero when the equations of motion are satisfied, if they themselves were to appear as secondary constraints in the consistency procedure they would negate Dirac's conjecture. This indicates how sensitive the validity of the conjecture is to the precise form of the constraints.

Now for the Hamiltonian analysis. p_y is the only primary constraint. The total Hamiltonian is $H_T=\alpha p_y+p_zp_x-\frac{1}{2}yz^2$. Using H_T to calculate $\dot{p}_y$, we obtain $\dot{p}_y=\frac{1}{2}z^2$. Since $\dot{p}_y$ must be weakly zero, $\frac{1}{2}z^2$ must also be weakly zero, as must its time derivative $(d/dt)(\frac{1}{2}z^2)=zp_x\approx 0$. Similarly, $(d/dt)(zp_x)=p_x{}^2\approx 0$. But $(d/dt)(p_x{}^2)$ is identically zero since H_T is independent of x. Thus the complete set of constraints is $\{p_y,z^2,zp_x,p_x{}^2\}$ as predicted.

The dynamical equations derived from H_T are identical in content to the Euler-Lagrange equations, as is easily seen from the fact that H_T is just a special case of H_T'. y is arbitrary, thus p_y is a symmetry generator. Each of the secondary constraints also generates unobservable changes in every dynamical function and is therefore also a symmetry generator.

The extended Hamiltonian H_E,

$$H_E=\alpha p_y+\beta zp_x+\gamma p_x{}^2+\sigma z^2+p_zp_x-\tfrac{1}{2}yz^2 ,$$

also generates the correct dynamical and constraint equations. In fact, H_E is really just a special case of H_T'. This indicates that the extended Hamiltonian is not necessarily the most general Hamiltonian associated with the system. Nevertheless, H_E is a generalization of H_T and may be derived from it as follows:

$$H_E=H_T+[H_T,-2\sigma p_y-\tfrac{1}{2}\beta z^2-\gamma zp_x+\epsilon p_x{}^2] . \tag{4.8}$$

This treatment differs from that given by Cawley in that the secondary constraint (z^2) which has been used to ensure consistency is the PB of the primary constraint p_y with the Hamiltonian and not a modification of it. (Cawley uses z.) Similarly, all further secondary constraints are taken to be the PB of other constraints with the Hamiltonian and are not altered before they are used to derive more consistency conditions.

Of course, if the value of z^2 is zero, then the value of z is also zero and if, in addition, d^nz^2/dt^n is zero for all n, then the value of dz/dt is zero as well. Thus the conditions that Cawley derives are certain to be consistent with those derived here. In fact, his set of secondary constraints $\{z,p_x\}$ consists of functions of the dynamical variables which are zero when the equations of motion are satisfied. But even though z and p_x have zero PB with each other and each has zero PB with p_y, neither is a symmetry generator. The change induced in x by αp_x is α, and for arbitrary α this leads to an arbitrary change in x which is inconsistent with the most general solution of (4.3). Equations (4.3) would also be contradicted by the arbitrary change in p_z (i.e., $\dot{x}$) which is generated by βz. So the modification of the secondary constraints has led to a set of first-class secondary constraints which are not symmetry generators.

In his letter, Cawley indicates that his choice of z in preference to z^2 was motivated by two considerations. First of all, he wanted to derive those consistency conditions which were not trivial consequences of constraints which were already known. We have noted though that the content of these new consistency conditions would have been obtained in any case by examining enough successive time derivatives.

Secondly, he points out that, within the framework of Dirac's original definitions, the equality $z\approx 0$ uniquely presents itself as the weak equality and thus seems to occupy a special place in the formalism.

B. Cawley's second counterexample

Next we will study another example proposed by Cawley,[3]

$$L=\dot{x}\dot{z}+\tfrac{1}{2}y\dot{z}^2 . \tag{4.9}$$

The momenta are $p_x=\dot{z}$, $p_y=0$, $p_z=\dot{x}+y\dot{z}$. The Euler-Lagrange equations are $\ddot{z}=0$, $\dot{z}^2=0$, and $\ddot{x}+(d/dt)(y\dot{z})=0$. The z coordinate is constant as is the x velocity. y is arbitrary so that the primary constraint p_y generates a symmetry transformation.

The study of the following symmetry transformations of the Euler-Lagrange equations and of the physical state (Table II) indicates that $p_x{}^2$ is also a symmetry generator and this is verified by the study of the symmetries of the Lagrangian,

$$\begin{aligned} &y\to y+\alpha ,\\ &\dot{z}\to\dot{z}+\beta\dot{z} ,\\ &\dot{x}\to\dot{x}+\sigma\dot{z} ,\\ &z\to z+\lambda\dot{z} ,\\ &x\to x+\gamma\dot{z} . \end{aligned} \tag{4.10}$$

TABLE II. This table refers to Cawley's second Lagrangian (4.9).

Transformation $T(q,\dot{q})$	$T(q,p)$	$G(q,p)$ Generator of $T(q,p)$	$T'(q,p)$ Other transformations generated by G	$T'(q,\dot{q})$	Are all transformations generated by G symmetry transformations?
$y \to y+\alpha$	$y \to y+\alpha$	p_y	None		Yes
$\dot{z} \to \dot{z}+\beta\dot{z}$	$p_x \to p_x+\beta p_x$	xp_x	$x \to x-\beta x$	$x \to x-\beta x$	No
	$p_z \to p_z+\beta y p_x$	yzp_x	$p_y \to p_y+\beta z p_x$	$p_y \to p_y+\beta z\dot{z}$ but no equivalent for p_y	No
			$x \to x-\beta yz$	$x \to x-\beta yz$	
$\dot{x} \to \dot{x}+\sigma\dot{z}$	$p_z \to p_z+\sigma p_x$	zp_x	$x \to x-\sigma z$	$x \to x-\sigma z$	No
$z \to z+\lambda\dot{z}$	$z \to z+\lambda p_x$	$p_z p_x$	$x \to x+\lambda p_z$	$x \to x+\lambda\dot{x}+\lambda y\dot{z}$	No
$x \to x+\gamma\dot{z}$	$x \to x+\gamma p_x$	${p_x}^2$	None		Yes

Applying (4.10) to L yields

$$L'=(\dot{x}+\sigma\dot{z})\dot{z}(1+\beta)+\tfrac{1}{2}(y+\alpha)\dot{z}^2(1+\beta)^2 \; . \tag{4.11}$$

L' leads to equations of motion which are content equivalent to the original Euler-Lagrange equations only if β is a constant which is not (-1). Eliminating the (not entirely arbitrary) transformation $\dot{z} \to \dot{z}+\beta\dot{z}$, L' becomes

$$L'=(\dot{x}+\sigma\dot{z})\dot{z}+\tfrac{1}{2}\dot{z}^2(y+\alpha) \; . \tag{4.12}$$

This L' leads to the total Hamiltonian H_T',

$$H_T'=\alpha p_y+\lambda {p_x}^2+p_z p_x-\tfrac{1}{2}y{p_x}^2 \; , \tag{4.13}$$

where the arbitrariness of the coefficients has been used. As expected, p_y is multiplied by an arbitrary coefficient and so is ${p_x}^2$, confirming that it too is a symmetry generator.

The Hamiltonian analysis starts with the derivation of the total Hamiltonian H_T from L,

$$H_T=\alpha p_y+p_z p_x-\tfrac{1}{2}y{p_x}^2 \; . \tag{4.14}$$

Since $p_y \approx 0$, we must have $(d/dt)p_y=\frac{1}{2}{p_x}^2 \approx 0$. But $(d/dt){p_x}^2$ is identically zero, so the full set of constraints is just $\{p_y,{p_x}^2\}$. The extended Hamiltonian, H_E, is identical with H_T' in (4.13) and may also be derived by transforming H_T:

$$H_E=H_T+[H_T,-2\lambda p_y+\phi {p_x}^2] \; .$$

H_E and H_T generate equivalent equations of motion.

C. Allcock's counterexample

In 1975, Allcock studied a Lagrangian which turns out to be an abbreviated version of the preceding one,

$$L=y\dot{z}^2 \; . \tag{4.15}$$

The momenta are $p_y=0$, $p_z=2y\dot{z}$, and the equations of motion are $\dot{z}^2=0$, $(d/dt)(y\dot{z})=0$. Here too z is a constant, y is arbitrary, p_y is a symmetry generator. The symmetry transformations of the Euler-Lagrange equations and of the state (Table III) show that ${p_z}^2/2y$ is also a symmetry generator. Furthermore, when the symmetry transformations shown in

TABLE III. This table refers to Allcock's Lagrangian (4.15).

Transformation $T(q,\dot{q})$	$T(q,p)$	$G(q,p)$ Generator of $T(q,p)$	$T'(q,p)$ Other transformations generated by G	$T'(q,\dot{q})$	Are all transformations generated by symmetry transformations?
$y \to y+\alpha$	$y \to y+\alpha$	p_y	None		Yes
$\dot{z} \to \dot{z}+\beta\dot{z}$	$p_z \to p_z+\beta p_z$	zp_z	$z \to z-\beta z$	$z \to z-\beta z$	No
$z \to z+\sigma\dot{z}$	$z \to z+\sigma p_z/2y$	${p_z}^2/2y$	$p_y \to p_y-\sigma {p_z}^2/2y^2$	$p_y \to p_y-2\sigma\dot{z}^2$ (no equivalent for p_y)	Yes

the first column of Table III are applied to the Lagrangian (4.15), the result is a new Lagrangian L',

$$L'=\dot{z}^2(y+\alpha)(1+\beta)^2 , \tag{4.16}$$

which, for $\beta \neq -1$ leads to equations of motion equivalent to the original equation and to a total Hamiltonian

$$H_T'=\lambda p_y+\frac{1}{4}\frac{{p_z}^2}{(y+\alpha)(1+\beta)^2} , \tag{4.17}$$

which again suggests a symmetry generator proportional to ${p_z}^2/y$. The Hamiltonian analysis agrees,

$$\begin{aligned} H_T&=\lambda p_y+\frac{{p_z}^2}{4y} ,\\ \dot{p}_y&=-\frac{{p_z}^2}{4y^2}\approx 0 , \\ \frac{d}{dt}\left[\frac{{p_z}^2}{y^2}\right]&=\frac{-2\lambda}{y}\frac{{p_z}^2}{y^2}\approx 0 . \end{aligned} \tag{4.18}$$

The set of constraints $\{p_y, {p_z}^2/y^2\}$ is first class. Both of its elements are symmetry generators (as long as $y\neq 0$), as may be seen from Hamilton's equations. The extended Hamiltonian is

$$H_E=\lambda p_y+\frac{{p_z}^2}{4y}+\sigma\frac{{p_z}^2}{y^2} . \tag{4.19}$$

Under the transformations induced by the constraints, H_T becomes H_T'',

$$\begin{aligned} H_T''&=H_T+\left[H_T,\beta p_y+\sigma\frac{{p_z}^2}{y^2}\right] \\ &=\lambda p_y+\frac{{p_z}^2}{4y}\left[\frac{2\lambda\sigma}{y}-\frac{\beta}{4}\right] , \end{aligned} \tag{4.20}$$

H_T, H_E, and H_T'' all generate equivalent equations of motion.

In his analysis, Allcock notes that the bracket of p_y with H *is* a symmetry generator, but indicates that it is not this bracket (which is proportional to ${p_z}^2$) which is the constraint, but rather p_z itself. Presumably, this latter choice is made in order to use that equality which is a weak equality of the first kind. In any case he raises an interesting point for the comments that, in this theory, a generator proportional to ${p_z}^2$ generates trivial symmetry transformations. Of course this is true of its action on dynamical quantities when the equations of motion are satisfied. However, Eq. (4.20) shows that ${p_z}^2$ generates a transformation of the Hamiltonian which is a symmetry transformation of sorts (since the new Hamiltonian generates equations of motion which are equivalent to the original equations) even when the equations of motion are not assumed.

V. CONCLUSIONS AND IMPLICATIONS

The validity of Dirac's conjecture that all first-class secondary constraints generate transformations which do not change the physical state is crucially dependent upon the form chosen for those constraints. What we have proposed here is a prescription for the form of the secondary constraints. This prescription supplements the method of Hamiltonian analysis which was originally suggested by Dirac and which has been described and expanded upon elsewhere. It is consistent with the main features of the formalism outlined in Dirac's book,[2] but (in its definitions of weak and strong equalities) not with his original exposition.[1] In that work, quantitative features of an equality (whether it was broken by an amount of order ϵ or ϵ^2, for example, under a variation of the dynamical valuables of order ϵ) served to define it as either weak or strong. The constraints were always written in such a way that their variations were of order ϵ; the constraint equations were modified if necessary so that they could satisfy the requirement of being weakly valid. We have noted, however, that it is possible to distinguish between weak and strong equalities on the basis of qualitative features and have chosen to do so. Within the framework set by these new definitions, there is no reason to modify the constraints which are derived directly from the consistency conditions. In the absence of qualitative guidelines, we have chosen this unmodified form to be the form of the secondary constraints.

The advantage of this approach is that the secondary constraints can be expressed in a way which is more theory independent than is otherwise possible. We have

$$\chi_i^{(n)}\approx a_{ij}\frac{d^n\phi}{dt^{nj}} .$$

With such an expression in hand we can systematically explore the properties of the secondary constraints and the transformations they generate. In particular, we can explore the general question of the validity of Dirac's conjecture.

Although it is useful to have a neat general form for the secondary constraints, we must ask whether the form we have chosen has any particular merit. In an attempt to answer this question we have examined some of the simplest symmetry transformations which are apparent in the Lagrangian description of the systems we have studied and have shown that they correspond to or imply symmetry transformations which are generated by the first-class constraints of the Hamiltonian description. Along the way to this result we have seen that the extended Hamiltonian is not always the most general Hamiltonian to generate the correct dynamical equations.

The method introduced here is, in a sense, the minimal method. Consistency demands that $d^n\phi/dt^n$ vanish for all n, so we simply impose these conditions directly (allowing for addition of equations and multiplication by nonsingular functions). However, although it is neat in concept it is not always so in practice. For it is clear that in general this method requires more than the minimal number of steps necessary to ensure consistency. In fact, by working with a form of the secondary constraints (the form suggested by Dirac) different from the one which is a direct result of the consistency conditions, it is sometimes possible to streamline the procedure considerably and still obtain all of the correct dynamical results. Of course this can be an advantage. But it seems to be an advantage which is bought at the price of knowing the natural form of the secondary constraints and the transformations which they generate. In many cases this information is not needed and for them the usual streamlined approach is preferable. However, when the symmetry properties associated with the secondary constraints are important, then use of the method introduced here seems appropriate.

In order to show that our results are not in contradiction with previously derived results,[15] but instead apply to the more general situation, it has been necessary to distinguish between form and content invariance. It has also been useful to distinguish between symmetries of the Hamiltonian (or Lagrangian) and symmetries of the physical state.

This last distinction is necessary in the following case. Let $\chi_k^{(j)}$ be a constraint which leads to a further constraint via the relation $[\chi^{(j)},H^{(j)}]\approx\chi_k^{(j+1)}$. For $j<n$ (where n is the number of applications of the consistency procedure necessary to ensure consistency), $\chi_k^{(j+1)}$ is not necessarily first class even if $\chi_k^{(j)}$ is, since $H^{(j)}$ is guaranteed to be first class only for $n=j$. Now let $\chi_k^{(j)}$ be a first-class constraint which generates transformations which do not change the physical state. $\chi_k^{(j)}$ induces a change in the total Hamiltonian which is weakly equal to $\alpha\chi_k^{(j+1)}$, with α arbitrary. Clearly, if $\chi_k^{(j+1)}$ is second class, then $H_T'=H_T+\alpha[H_T,\chi_k^{(j)}]$ generates equations of motion which are different in content from the equations generated by H_T alone. In this case, $\chi_k^{(j)}$ is a generator which generates symmetry transformations of the state but not of the Hamiltonian.

On the other hand, let $\chi_m^{(j)}$ be a first-class constraint which generates symmetry transformations of the Hamiltonian. If $\chi_m^{(j+1)}=[\chi_m^{(j)},H^{(j)}]$, then $\chi_m^{(j+1)}$ generates transformations which are symmetries of the physical state since $H_T'=H_T+\alpha\chi_m^{(j+1)}$ generates equations of motion which are content equivalent to those generated by H_T'. In this case, $\chi_m^{(j+1)}$ can be placed on the same footing as the primary constraints right from the beginning, even before the variational principle is applied.

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APPENDIX

We have proposed a form for the secondary constraints which can be determined (up to linear combinations) in any particular case once one is given the form of the primary constraints. In the cases considered so far, the first-class secondary constraints, so determined, generate symmetry transformations. It is tempting to conjecture that this is always the case. However, the validity of this conjecture cannot be tested until another point is resolved. This is because the form of the secondary constraints is, given the method for finding them suggested here, sensitive to the form of the primary constraints; so we must address the question of how to choose the primary constraints.

Consider the most general Lagrangian quadratic in the velocities

$$L=\tfrac{1}{2}\dot{q}_i a_{ij}\dot{q}_j+b_i\dot{q}_i+c\ , \tag{A1}$$

where a_{ij}, b_i, and c may be functions of the n independent coordinates. The momentum definitions give

$$p_i=a_{ij}\dot{q}_j+b_i\ , \tag{A2}$$

where we have utilized the symmetry of a_{ij} with respect to the interchange of its indices. There are nonsingular $n\times n$ matrices S and T such that

$$S_{mi}a_{ij}T_{jn}=\begin{bmatrix}\mathbb{1} & \\ & 0\end{bmatrix}_{mn}\ , \tag{A3}$$

where the dimension of the unit matrix is equal to r, the rank of the matrix a. Thus, (A2) can be written as

$$S_{mi}(p_i-b_i)=\begin{bmatrix}\mathbb{1} & \\ & 0\end{bmatrix}_{mi}T^{-1}{}_{ij}\dot{q}_j\ . \tag{A4}$$

Define π_m to be $S_{mi}(p_i - b_i)$ and λ_m to be $T^{-1}{}_{mi}\dot{q}_i$. Then (A4) is

$$\pi_m = \begin{bmatrix} \mathbb{1} & \\ & 0 \end{bmatrix}_{mi} \lambda_i \; . \tag{A5}$$

The last $n-r$ equations given in (A5) may be taken to be the primary constraints. With the first r indices denoted by upper case latin letters and the remaining $n-r$ indices denoted by greek letters, we have

$$\begin{aligned} \pi_M &= \lambda_M \; , \\ \pi_\alpha &= 0 \; . \end{aligned} \tag{A6}$$

In this case, it is possible to find what seems to be a reasonable form for the primary constraints by taking linear combinations of the momentum definitions. This approach is analogous to the one applied earlier to discover the secondary constraints.

Although the quadratic Lagrangian (A1) is obviously not the most general one possible, it does cover many cases of physical interest. Furthermore, even if the Lagrangian is not quadratic, it may happen that that part of the Lagrangian which leads to constraints is. Even so, if one is to use the method presented here as a general method to find the secondary constraints, then one must also have a general method to find the primary constraints. This question will not be considered further here. It seems more appropriate to first consider the question of whether or not the approach suggested here for quadratic Lagrangians generally leads to first-class secondary constraints which are symmetry generators. Depending on what the answer is, it may be interesting to try to generalize the approach used here for quadratic Lagrangians, or perhaps to try some other approach.

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[9]The formal structure of the method is the same for a system with an infinite number of degrees of freedom.

[10]See, for example, A. Kihlberg *et al.*, Phys. Rev. D 23, 2201 (1981); N. Mukunda *et al.*, *ibid.* 23, 2189 (1981).

[11]It is possible to refine the definitions given here in a way that may be useful for certain applications. If $f \approx g$ is a weak equality, then there is some set (or sets each of) which contains the minimum number of elements necessary to force the inequality (2.10). Let W be this minimum number; we say that the equality f and g is a weak equality of the Wth kind.

[12]The results concerning systems of linear equations which are cited in this section are derived in standard texts on linear algebra.

[13]$H^{(t)}$ is $H + \bar{u}_i\phi_i$ with the $\bar{u}$'s completely determined. Thus $\dot{\chi}_b$ is actually strongly equal to $[\chi_b^{(t)},H]+\bar{u}_i[\chi_b^{(t)},\phi_i] \; + v_a^{(t)}[\chi_b^{(t)},\Phi_a^{(t)}]$, and not to $[\chi_b,H^{(t)}]+v_a^{(t)}\,[\chi_b^{(t)},\Phi_a^{(t)}]$. However, in our case, it is legitimate to use this latter expression for $\dot{\chi}_b^{(t)}$ for two reasons. First of all, the extra terms it entails $([\chi_b^{(t)},\bar{u}_i]\phi_i)$ are all weakly zero. Second, in this work we are only interested in the time derivatives of dynamical variables—that is, in the action of only the Hamiltonian as the generator of canonical transformations. It is easily verified that d^ng/dt^n as computed by successive applications of $[g,H]+[g,u_m\phi_m]$ is always weakly equal to the actual expression for d^ng/dt^n. However, if we were interested in the action of a generator other than the Hamiltonian on some time derivative d^mg/dt^m, we would have to take care to use the correct expression, unless the generator of interest happened to be one which preserved the constraints.

[14]Although in Ref. 2 Dirac mentions explicitly only the first-class secondary constraints (along with the first-class primary constraints) as potential symmetry generators, it is a natural extension to assume that *all* first-class linear combinations of constraints should play the same role.

[15]It may be the case that $\dot{q}_i$ cannot be expressed in terms of the coordinates and momenta. In this case, the translation procedure outlined here cannot be applied; there is no generator G which can be directly associated with the transformation of $\dot{q}_i$. Similarly, not all momenta can be expressed as functions of the coordinates and velocities.

[16]L. Castellani, Report No. ITP-SB-81-5 (unpublished).