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Superluminal coordinate transformations: Four-dimensional case

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We investigate, from a group-theoretical point of view, the possibility of implementing the so-called extended principle of relativity. This consists in postulating that the set of all equivalent reference frames contains frames whose relative velocities are larger than c, in addition to those whose coordinates are related by proper orthochronous Lorentz transformations. We show that implementing the extended principle of relativity by means of either real or complex linear transformations results in strong conflicts with experiment and/or intractable problems of interpretation. We then briefly analyze alternative approaches to four-dimensional superluminal transformations, in which the extended principle of relativity is either weakened or completely abandoned.

I. INTRODUCTION

The concept of equivalent reference frames related to each other by well-defined coordinate transformations probably ranks among the most useful ones in theoretical physics. It should come as no surprise then that people have tried to apply it to the largest possible class of frames. In the absence of sizable gravitational fields (or in a sufficiently restricted region of space-time), an experimentally well-established set of coordinate transformations between equivalent reference frames is the restricted Poincaré group, i.e., the group generated by proper orthochronous Lorentz transformations and space-time translations. But could the set be larger? A fascinating extension of the Poincaré group is the conformal group SO(4,2) obtained from the former by adding dilatations and the so-called uniform accelerations. The fact that exact conformal symmetry meets with experimental problems is no barrier to its extensive study, since a broken symmetry often provides as fruitful a framework as an exact one.

A far more speculative extension of the Poincaré or Lorentz group consists in introducing superluminal coordinate transformations.[1–5] These are transformations between reference frames moving with respect to each other with a velocity larger than the speed of light in vacuum. The hypothesis that there exist equivalent reference frames related by superluminal transformations, in addition to those related by Lorentz transformations, has been called the extended principle of relativity. The main purpose of this paper is to investigate the tenability of this principle. It is then very important to be particularly clear in specifying the notion of equivalent reference frames.

A reference frame associates quadruplets of (real) numbers to events in space-time in such a way that, locally, a subset of space-time is one-to-one mapped onto a subset of R^4. The correspondence between events in space-time and points in R^4 can in principle be realized by suitably constructed arrays of clocks and meter sticks. Relative to a given reference frame, one can parametrize the space-time evolution of physical systems by means of physical laws. We shall say that two reference frames are equivalent if the empirically allowed systems are the same in both, and they evolve according to the same laws.[6]

A specific example may help in clarifying the concept. Assume there are, in a reference frame K, particles of mass m only, described by the Klein-Gordon equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \ . \tag{1.1}$$

In any frame K'' related to K by a Poincaré transformation, the particles will be described by the same equation, and therefore K'' and K will be equivalent if this is the only type of physical system allowed. On the other hand, consider a frame K' related to K by a dilatation $(x')^\mu = \lambda x^\mu$. In terms of the coordinates of K', Eq. (1.1) now reads

$$\partial'_\mu(\partial')^\mu\phi+\lambda^2 m^2\phi=0\ , \tag{1.2}$$

and therefore K' and K are not equivalent since similar physical systems evolve according to different laws. Note that it does not help to rewrite the Klein-Gordon equation as

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi+m^2\phi=0 \tag{1.3}$$

and say that $(g')^{\mu\nu}=\lambda^2 g^{\mu\nu}$. Indeed a specific matrix $g^{\mu\nu}$ is then inescapably linked with a specific frame K, the elements of $g^{\mu\nu}$ and those of $(g')^{\mu\nu}$ being different.

There are very few *a priori* restrictions on the possible coordinate transformations between equivalent frames. In particular, they could well be nonlinear. It follows from the definition of equivalent frames, however, that the transformations must form a group. In fact, it is not difficult to see that the law of associativity holds, and the law of closure follows from the observation that if K is equivalent to K' and K' is equivalent to K'', then K is equivalent to K''. The symmetry and reflexivity properties of equivalence relations likewise entail the existence of an inverse to each transformation and of the identity.

Going back to the extended principle of relativity, we now see more clearly what it means to postulate the existence of equivalent reference frames related by superluminal coordinate transformations. This clearly implies that tachyons must be allowed, and that their physical laws are the same in their rest frames as the laws of bradyons are in theirs. Moreover, and most importantly, each transformation in the group generated by the Lorentz group and the superluminal transformations relates the coordinates of equivalent frames. This, we shall see, is a most stringent requirement, which is often overlooked in the literature dealing with the extended principle of relativity. In fact, it is so stringent that it casts serious doubts on the tenability of this principle.

As our main task is to investigate the consequences of the extended principle of relativity, we shall restrict ourselves to frames whose space-time origins coincide. Especially important, among these, are the ones related by linear transformations. Locally, these transformations have the property of preserving inertial motion, or motion in a straight line. Our experimental knowledge of equivalent reference frames is such that given an inertial frame, the only other reference frames known to be equivalent to it are also inertial frames. So, throughout this paper, we shall adopt the restriction to linear coordinate transformations, keeping in mind that it may conceivably be dropped in the future.

The organization of the paper is as follows: In Sec. II, we consider the implementation of the extended principle of relativity by real linear transformations. Recalling earlier results obtained in two dimensions, we show that the simplest four-dimensional generalization implies the equivalence of all frames whose coordinates are related by any linear transformation of determinant ±1. This has devastating experimental consequences. Under the assumption that the full group of coordinate transformations has a Lie structure, the result is generalized to any real enlargement of the Lorentz group. Realizing that the extended principle of relativity cannot be implemented by real linear transformations, we investigate in Sec. III the popular alternative that consists in dropping the requirement of reality. First we show that enlarging the Lorentz group by a single complex transformation that changes the sign of the world interval implies the equivalence of all frames whose coordinates are related by any element of $SO'(3,1;\mathbb{C})$. Here $SO'(3,1;\mathbb{C})$ stands for the group of all complex linear transformations that either preserve the world interval or change its sign. In particular, it includes all three-dimensional complex rotations. We then examine the problems of interpretation raised by complex transformations. Section IV is devoted to a brief analysis of proposals for superluminal transformations which do not postulate, or are best construed as not postulating, the extended principle of relativity. The consequences of our results on the tenability of the extended principle of relativity are summarized in the Conclusion.

II. REAL TRANSFORMATIONS BETWEEN EQUIVALENT FRAMES

In a previous paper,[7] we analyzed in detail the problem of defining superluminal coordinate transformations between equivalent frames in a universe with one space and one time dimension. The main result of Ref. 7 was the following: Let a reference frame K' move with respect to an equivalent frame K with a velocity v, and let the coordinate transformations between frames satisfy the following requirements:

(i) They are real.
(ii) They are linear.
(iii) They leave the speed of light invariant ($c=1$).
(iv) They form a group.
(v) The group contains the proper orthochronous two-dimensional Lorentz group.

Then the coordinate transformations (both subluminal and superluminal) between K and K' are essentially[8] uniquely determined, and are given by

$$\Delta t = \mu(v) \,|\, 1-v^2 \,|^{-1/2}(\Delta t' + v\Delta x') ,$$
$$\Delta x = \mu(v) \,|\, 1-v^2 \,|^{-1/2}(v\Delta t' + \Delta x') , \tag{2.1}$$

where

$$\mu(v) = \begin{cases} 1 \text{ if } |v| < 1 , \\ v/|v| \text{ if } |v| > 1 . \end{cases} \tag{2.2}$$

So it seems that, kinematically at least, the extended principle of relativity can be implemented rather nicely in two dimensions. It is widely believed that, in four dimensions, this is no longer the case. For instance, it is seen at once that the five requirements satisfied by the two-dimensional transformations cannot, when restated in four dimensions, accommodate superluminal transformations. Indeed the Lorentz group (together with parity and time reversal) is the largest real linear group that preserves the $(1,-1,-1,-1)$ metric, and there are no real linear transformations that just change the overall sign of the metric.[9] Dilatations leave the speed of light invariant, but they represent transformations between frames relatively at rest. So any real linear four-dimensional superluminal transformation is bound to violate requirement (iii). It is important, however, to realize that this consequence is ruled out neither by the extended principle of relativity, nor by experiment. The former simply means that if there are lightlike particles in one frame, there must be in any other equivalent frame, but the ones do not have to be the transforms of the others. And empirically, we have no easy handle on how light behaves as seen from the point of view of a superluminal frame.

To investigate the consequences of the extended principle of relativity, we shall look for a set of four-dimensional transformations satisfying requirements (i), (ii), (iv), and (v′) [where (v′) refers to the proper orthochronous four-dimensional Lorentz group, hereafter denoted by $L_+^\uparrow$] and containing at least one superluminal transformation. But what does a four-dimensional superluminal transformation look like? The simplest thing to do at this point is to take Eqs. (2.1) and (2.2) with $|v| > 1$, together with $y=y'$ and $z=z'$. (We drop the Δ's, since space-time origins coincide.) We now introduce some additional notation. Let $R(\hat{n};\theta)$ denote a (passive) right-handed rotation by an angle θ around a spatial axis $\hat{n}$, and let $S_{\hat{x}}$ denote the superluminal transformation given by

$$t = x', \quad y = y' ,$$
$$x = t', \quad z = z' . \tag{2.3}$$

It is easy to show that by applying rotations on $S_{\hat{x}}$, one can generate any $S_{\hat{n}}$ with $S_{\hat{n}}$ denoting the superluminal transformation, whereby t goes to the primed coordinate along $\hat{n}$, the coordinate along $\hat{n}$ goes to t', and the coordinates perpendicular to $\hat{n}$ remain unchanged. Finally, let $\overline{\text{SL}}(4;\text{R})$ denote the group of all real linear four-dimensional transformations with determinant equal to ± 1. We are now ready to prove the following theorem.

Theorem 1. The smallest group that contains $L_+^\uparrow$ and $S_{\hat{x}}$ is $\overline{\text{SL}}(4;\text{R})$.

Proof. Let G denote the group in question. We have seen that rotations and $S_{\hat{x}}$ generate $S_{\hat{n}}$ for any $\hat{n}$, and therefore $S_{\hat{n}}$ belongs to G. It is easy to check that $S_{\hat{x}} R(\hat{z};-\pi/2) S_{\hat{x}} S_{\hat{y}} = T$, the time-reversal transformation. But any element of $\overline{\text{SL}}(4;\text{R})$ either belongs to SL(4;R) (i.e., has determinant $+1$), or is the product of T times an element of SL(4;R). Hence it is enough to show that G contains SL(4;R).

Any matrix M in SL(4;R) can be written as the product of an orthogonal times a symmetric matrix, both of unit determinant. (Explicitly, $M = [M(M^T M)^{-1/2}](M^T M)^{1/2}$.) Furthermore, any symmetric matrix can be diagonalized by a unimodular orthogonal transformation. Thus we can write $M = O_1 \Delta O_2$, where O_1 and O_2 belong to SO(4) and Δ is diagonal. Without loss of generality we can take Δ positive definite; it simply amounts to a redefinition of O_1. So it is enough to show that G contains SO(4) and the set of all diagonal positive-definite matrices with unit determinant.

To show that G contains SO(4), we first observe that a basis for the Lie algebra of SO(4) is provided by the six antisymmetric matrices $A_{\alpha\beta}$ $(0 \le \alpha < \beta \le 3)$, with components $(A_{\alpha\beta})_{\mu\nu}$ given by[10]

$$(A_{\alpha\beta})_{\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu} \tag{2.4}$$

and satisfying the relations $(A_{\beta\alpha} = -A_{\alpha\beta})$

$$[A_{\alpha\beta}, A_{\gamma\delta}] = \delta_{\alpha\delta} A_{\beta\gamma} + \delta_{\beta\gamma} A_{\alpha\delta} - \delta_{\alpha\gamma} A_{\beta\delta} - \delta_{\beta\delta} A_{\alpha\gamma} . \tag{2.5}$$

An arbitrary element O in SO(4) can be written as $O = R_1 K R_2$, where R_1 and R_2 are spatial rotations and K is a rotation in a plane including the t axis, say the (02) plane. (This is analogous to the decomposition of the Lorentz group in terms of spatial rotations and boosts along a given axis.) Since spatial rotations belong to G, G will include SO(4) if it contains all rotations in the (02) plane. Now consider the product $R(\hat{z};-\pi/2) S_{\hat{x}} S_{\hat{y}}$. Clearly, it belongs to G, and one easily checks that

$$R(\hat{z};-\pi/2)S_{\hat{x}}S_{\hat{y}}=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$=\exp\left[\frac{\pi}{2}A_{01}\right]. \tag{2.6}$$

On the other hand, we also have, for any θ

$$\exp\left[\frac{\pi}{2}A_{01}\right]R(\hat{z};\theta)\exp\left[\frac{-\pi}{2}A_{01}\right]=\exp(\theta A_{02}), \tag{2.7}$$

which shows that all rotations in the (02) plane indeed belong to G.

The only thing left to show is that G contains all unimodular diagonal positive-definite matrices. Define two matrices S_{01} and Δ_{01} as

$$S_{01}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \Delta_{01}=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}. \tag{2.8}$$

Recognizing S_{01} as the generator of a boost along the x axis, one easily checks that

$$\exp\left[\frac{\pi}{4}A_{01}\right]\exp(\eta_1 S_{01})\exp\left[\frac{-\pi}{4}A_{01}\right]=\exp(\eta_1\Delta_{01}). \tag{2.9}$$

Thus $\exp(\eta_1\Delta_{01})$ belongs to G for any η_1. Similarly, one shows that $\exp(\eta_2\Delta_{02})$ and $\exp(\eta_3\Delta_{03})$, in obvious notation, also belong to G. But the product of these three elements represents an arbitrary unimodular diagonal positive-definite matrix. Hence G contains them all. Q.E.D.

What is the meaning of the theorem we have just proved? We have shown that, when implementing the extended principle of relativity by means of a group of real linear transformations that includes $L_+^\uparrow$ and one superluminal transformation of the form (2.1), the full group $\overline{\mathrm{SL}}(4;\mathrm{R})$ necessarily results. That is, there has to be equivalent reference frames whose coordinates are related by any linear transformation with determinant equal to ±1. Clearly, such a strong result clashes violently with experiment and the laws of physics as we presently know them. To give a specific example, the following transformation, which represents a direction-dependent dilatation, is part of $\overline{\mathrm{SL}}(4;\mathrm{R})$:

$$\begin{bmatrix}(\lambda_1\lambda_2\lambda_3)^{-1}&0&0&0\\0&\lambda_1&0&0\\0&0&\lambda_2&0\\0&0&0&\lambda_3\end{bmatrix}. \tag{2.10}$$

This kind of dilatation is certainly not a symmetry of the known physical laws. Nor are, in general, the parity and time-reversal transformations, which also belong to $\overline{\mathrm{SL}}(4;\mathrm{R})$. The upshot is that a superluminal transformation of the form (2.1) and the proper orthochronous Lorentz group $L_+^\uparrow$ are mutually exclusive as coordinate transformations between equivalent frames.

Theorem 1 can be made stronger in several ways. We have just made use of a trivial generalization that consists in replacing $S_{\hat{x}}$ by any superluminal transformation of the form (2.1). [Call it $S(\vec{v})$, $\vec{v}=v\vec{x}$, $|v|>1$.] $S_{\hat{x}}$ is then easily generated by multiplying $S(\vec{v})$ by a Lorentz boost of velocity $-v^{-1}\hat{x}$. More significantly, the theorem can also be proved if, instead of assuming $L_+^\uparrow$ and $S(\vec{v})$, one just starts with the rotation group in three-space and $S(\vec{v})$, for finite v. Indeed, the product $S(\vec{v})S(\vec{v})$ is then a Lorentz boost with velocity $(1+v^2)^{-1}2\vec{v}\neq0$. But one can show that the rotation group together with any Lorentz boost generates the full $L_+^\uparrow$. So the hypotheses of theorem 1 are recovered.

In theorem 1, we have assumed a rather specific form for the superluminal transformation adjoined to $L_+^\uparrow$. The justification for using that particular form lies in the fact that it is the simplest generalization of the two-dimensional transformations (2.1) and (2.2), which were obtained in Ref. 7. However, one can prove a much more general result if one assumes a Lie-group structure, which concept we take to include both connected and disconnected groups. Let g denote the Lorentz metric diag$(1,-1,-1,-1)$. The full Lorentz group, O(3,1), is a subgroup of $\overline{\mathrm{SL}}(4;\mathrm{R})$. Clearly, an element S in $\overline{\mathrm{SL}}(4;\mathrm{R})$ will not belong to O(3,1) if and only if $S^TgS\neq g$. We shall now prove that adjoining any such S to $L_+^\uparrow$ generates the full $\overline{\mathrm{SL}}(4;\mathrm{R})$ [or SL(4;R)]. More precisely, we have the following.

Theorem 2. Let S be an arbitrary element of $\overline{\mathrm{SL}}(4;\mathrm{R})$ that does not preserve the Lorentz metric, i.e., $S^TgS\neq g$. The smallest Lie group that contains S and $L_+^\uparrow$ is either SL(4;R) or $\overline{\mathrm{SL}}(4;\mathrm{R})$, depending on whether det(S) is $+1$ or -1.

Proof. We first show that there is at least one element $\mathscr{L}$ in the Lie algebra of $L^{\uparrow}_{+}$ such that $S\mathscr{L}S^{-1}$ is not in the Lie algebra of $L^{\uparrow}_{+}$. To see this, we suppose the contrary, i.e., we assume that $(S\mathscr{L}S^{-1})^T g = -g(S\mathscr{L}S^{-1})$ for every $\mathscr{L}$ in the Lie algebra. This is equivalent to

$$\mathscr{L}^T(S^T g S) = -(S^T g S)\mathscr{L} \tag{2.11}$$

for every $\mathscr{L}$. Now it is easy to show that any matrix g' which satisfies $\mathscr{L}^T g' = -g'\mathscr{L}$, for all $\mathscr{L}$ in the Lie algebra of $L^{\uparrow}_{+}$, is a multiple of g. Hence $S^T g S = \lambda g$. Taking determinants on both sides and remembering that S belongs to $\overline{\mathrm{SL}}(4;\mathrm{R})$ yields $\lambda = \pm 1$. On the other hand, it is well known[9] that there are no real matrices that just change the sign of the Lorentz metric. Therefore $\lambda = 1$, and $S^T g S = g$, which contradicts the hypothesis of the theorem. Given an S so that $S^T g S \neq g$ then, there must exist an $\mathscr{L}_0$ in the Lie algebra of $L^{\uparrow}_{+}$ such that $S\mathscr{L}_0 S^{-1}$ (and therefore $\alpha S\mathscr{L}_0 S^{-1}$ for any real $\alpha \neq 0$) is not in the Lie algebra of $L^{\uparrow}_{+}$.

Now let G denote the group we are seeking, i.e., the smallest Lie group that contains S and $L^{\uparrow}_{+}$. Clearly, S^{-1} belongs to G, and so does $S\exp(\alpha\mathscr{L}_0)S^{-1} = \exp(\alpha S\mathscr{L}_0 S^{-1})$. On the other hand, $S\exp(\alpha\mathscr{L}_0)S^{-1}$ is in SL(4;R), and thus there is an element in the Lie algebra of G, namely $S\mathscr{L}_0 S^{-1}$, which belongs to the Lie algebra of SL(4;R), but not to the one of $L^{\uparrow}_{+}$.

We now show that the Lie algebra of G [say $\mathscr{L}(G)$] must include the full Lie algebra of SL(4;R). The latter is made up of all real traceless 4×4 matrices. These constitute a fifteen-dimensional vector space, which we denote by V. It is easy to show that any matrix N in V can be uniquely written as a sum $N_1 + N_2$, where N_1 and N_2 satisfy

$$N_1^T g + g N_1 = 0 ,$$

$$N_2^T g - g N_2 = 0 .$$

In this way, V is decomposed into the direct sum of two vector spaces V_1 and V_2, respectively six- and nine-dimensional. V_1 is just the vector space of the Lie algebra of $L^{\uparrow}_{+}$.

Clearly then, $\mathscr{L}(G)$ contains all matrices in V_1, and we have seen that it also contains one matrix of the form $N_1 + N_2$ with $N_2 \neq 0$ (namely, $S\mathscr{L}_0 S^{-1}$). Being an algebra, i.e., having the structure of a vector space, $\mathscr{L}(G)$ also contains N_2, and αN_2 for any real α. Thus any element of the form $\exp(\alpha N_2)$ belongs to G, and so does any $\Lambda\exp(\alpha N_2)\Lambda^{-1}$ for Λ in $L^{\uparrow}_{+}$. But this is equal to $\exp(\alpha\Lambda N_2\Lambda^{-1})$. On the other hand, one can show that the representation of $L^{\uparrow}_{+}$ defined on the nine-dimensional space V_2 by $N_2 \to \Lambda N_2 \Lambda^{-1}$ is irreducible. In fact, it is the well-known representation of the Lorentz group on a second-rank mixed tensor [the (1,1) representation]. Matrices of the form $\Lambda N_2 \Lambda^{-1}$, when Λ ranges over the full $L^{\uparrow}_{+}$, then contain a basis for V_2. Since $\exp(\alpha\Lambda N_2\Lambda^{-1})$ is in G for any α, $\Lambda N_2 \Lambda^{-1}$ is in $\mathscr{L}(G)$. Hence $\mathscr{L}(G)$ contains V_2, and therefore V since it already includes V_1. That is, $\mathscr{L}(G)$ includes the full Lie algebra of SL(4;R).

If $\det S = +1$, all elements are unimodular, so G is SL(4;R). If $\det S = -1$, all linear matrices with determinant equal to -1 are also generated. In that case G is $\overline{\mathrm{SL}}(4;\mathrm{R})$. Q.E.D.

Summarizing the meaning of theorem 2: Assuming that the group G of coordinate transformations between equivalent frames has a (possibly disconnected) Lie structure, contains $L^{\uparrow}_{+}$, and includes one transformation S outside the full Lorentz group, implies that G contains the full SL(4;R) [or $\overline{\mathrm{SL}}(4;\mathrm{R})$]. This, we have seen, has unacceptable experimental consequences. Therefore, the extended principle of relativity cannot be implemented by means of a Lie group of real linear transformations.

In concluding this section, it is interesting to compare our results, namely, theorems 1 and 2, with those obtained sometime ago by Gorini.[11] Gorini considered the following problem: What are the possible groups of real linear transformations (in space-time with one time and n space dimensions) such that in each case the subgroup of transformations between frames at rest is exactly SO(n), the rotation group in space? He found that for $n \geq 3$, the only solutions are the rotation group itself, the Lorentz group, and the Galilei group. Clearly, Gorini's results and ours are consistent, for we have shown that, loosely speaking, any enlargement of the Lorentz group implies the full linear group, whose subgroup of transformations between frames at rest is much larger than SO(3). In $3+1$ dimensions, Gorini's theorem tells us what happens if one postulates that the group of transformations between frames at rest is exactly SO(3). [He also considers the case when it is SO(3) + parity (P).] It does not say, however, what happens if one postulates a slightly different or enlarged group. For instance, should one start with SO(3) + PT, a solution for the full kinematical group would be SO(4), which is very different from the Lorentz group. Our aims have been quite different. We have shown what happens when one postulates the Lorentz group plus anything else as transformations between equivalent frames (not necessarily at rest). The result is that in any such case there is no choice left. The full linear group results, with unacceptable experimental consequences.

III. COMPLEX TRANSFORMATIONS BETWEEN EQUIVALENT FRAMES

The perceived impossibility (which, to our knowledge, has been rigorously demonstrated here for the first time) of an experimentally viable group of real linear transformations, including both $L_+^\uparrow$ and superluminal transformations, has prompted many people to relax some of these requirements. One of the most popular choices in this context consists in dropping the requirement of reality, i.e., in allowing reference frames related by complex coordinate transformations.[12] Obviously, this immediately raises the question of how to interpret complex coordinates.[13–19] For the moment we shall leave it aside, coming back to it later in this section. What we want to do first is again to investigate the consequences of the group property, which necessarily follows if the transformations relate the coordinates of equivalent reference frames.

The complex coordinate transformations, first introduced by Olkhowsky, Recami and Mignani,[4,14] are essentially[20] given by Eqs. (2.1) and (2.2) with $|v|>1$, and by the equations $y=iy'$ and $z=iz'$. The matrix $CS_{\hat{x}}$ of the transformation obtained by setting $v=\infty$ is given by

$$CS_{\hat{x}}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&i&0\\0&0&0&i\end{bmatrix}. \tag{3.1}$$

Suppose we take $CS_{\hat{x}}$ as a coordinate transformation between equivalent frames, in addition to the group $L_+^\uparrow$. The question is then, what group is generated by repeated products of these transformations? Or, equivalently, what is the smallest group that contains $CS_{\hat{x}}$ and $L_+^\uparrow$?

Let G denote the group in question. Furthermore, let SO′(3,1;C) denote the group of all complex unimodular 4×4 matrices that either preserve the Lorentz metric or change its sign. It is clear that G is a subgroup of SO′(3,1;C). Indeed, $CS_{\hat{x}}$ as well as all elements of $L_+^\uparrow$ satisfy the defining condition of SO′(3,1;C), and so will arbitrary products of them. In fact, we will show that G is exactly SO′(3,1;C). This forms the content of the next theorem.

Theorem 3. The smallest group that contains $CS_{\hat{x}}$ and $L_+^\uparrow$ is SO′(3,1;C).

Instead of proving theorem 3 directly, we shall first translate it into an equivalent statement, and then prove the latter. Let h denote the following diagonal matrix:

$$h=\begin{bmatrix}i&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}. \tag{3.2}$$

Note that $hgh^{-1}=g$ and $h^Tgh=hgh=I$. We will use h to perform a similarity transformation on $CS_{\hat{x}}$ and on all matrices of the standard real representation of $L_+^\uparrow$. In this way we obtain a matrix $C_{\hat{x}}$ given by

$$C_{\hat{x}}=hCS_{\hat{x}}h^{-1}=\begin{bmatrix}0&i&0&0\\-i&0&0&0\\0&0&i&0\\0&0&0&i\end{bmatrix} \tag{3.3}$$

and a new (equivalent) representation of $L_+^\uparrow$ made up of complex matrices. It is easy to check that a basis for the generators of the new matrices is provided by the six matrices A_{jk} and iA_{0j} ($j,k=1,2,3$), defined in Eq. (2.4). In other words, any matrix obtained by exponentiating a real linear combination of A_{jk} and iA_{0j} is related to the original $L_+^\uparrow$ matrices by the similarity transformation.

Let M be an element of SO′(3,1;C), and let $N=hMh^{-1}$. One sees immediately that a necessary and sufficient condition for $M^TgM=\pm g$ to hold is that $N^TN=\pm I$ be satisfied. That is, M preserves (or changes the sign of) the Lorentz metric if and only if N preserves (or changes the sign of) the Euclidean metric. In fact, this is just an instance of the well-known isomorphism of the groups SO(p,q;C) and SO($p+q$;C) (which of course does not hold over the real numbers).[21] The following statement is then completely equivalent to theorem 3.

Theorem 3′. Let $C_{\hat{x}}=hCS_{\hat{x}}h^{-1}$ and let $h\{L_+^\uparrow\}h^{-1}$ denote the similarity transforms of all matrices of the standard real representation of $L_+^\uparrow$. The smallest group that contains $C_{\hat{x}}$ and $h\{L_+^\uparrow\}h^{-1}$ is SO′(4;C), made up of all unimodular matrices that satisfy $N^TN=\pm I$.

Proof. Let G denote the group in question. $C_{\hat{x}}$ satisfies $C_{\hat{x}}^TC_{\hat{x}}=-I$, and any matrix N' satisfying $(N')^T(N')=-I$ can be written as $N'=C_{\hat{x}}N$, with $N^TN=I$. Explicitly, $N'=C_{\hat{x}}(C_{\hat{x}}^{-1}N')$. Therefore it is enough to show that G contains SO(4;C).

The Lie algebra of SO(4;C) is made up of all complex antisymmetric matrices. A maximal compact subgroup of SO(4;C) is SO(4;R), whose Lie algebra is made up of all real antisymmetric matrices. This means that any element N in SO(4;C) can be written as a product $N=KO$, where O belongs to SO(4;R) and K is a coset representative in the space SO(4;C)/SO(4;R).[21] Since SO(4;R) is a compact connected Lie group, all its elements can be obtained by exponentiation, namely as $\exp(c_{\alpha\beta}A_{\alpha\beta})$ with $c_{\alpha\beta}$ real arbitrary constants. Furthermore, the representative K can always be chosen as an exponential of the remaining generators, i.e., as $\exp(id_{\alpha\beta}A_{\alpha\beta})$, with $d_{\alpha\beta}$ real arbitrary constants. Therefore it is enough to show that G contains SO(4;R) and all coset representatives K of the form $\exp(id_{\alpha\beta}A_{\alpha\beta})$.

We recall that G contains all elements obtained by exponentiating real linear combinations of A_{jk} and iA_{0j}. The following equation, which shows that $\exp(\theta A_{02})$ is in G for any θ, is a little tedious but straightforward to check $[R_\theta=\exp(\theta A_{12})]$

$$C_{\hat{x}}R_\theta C_{\hat{x}}R_\theta{}^{-1}C_{\hat{x}}C_{\hat{x}}R_\theta{}^{-1}=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\exp(\theta A_{02})\ . \qquad (3.4)$$

Elements of the form $\exp(c_{jk}A_{jk})$ span the three-dimensional rotation group SO(3;R), and elements of the form $\exp(\theta A_{02})$ make up all rotations in the (ty) plane. Recalling the discussion of a similar case in the proof of theorem 1, we see that G must include the full SO(4;R).

To show that G contains all elements of the form $\exp(id_{\alpha\beta}A_{\alpha\beta})$, we first recall the well-known decomposition of the Lie algebra of SO(4;R) into two mutually commuting SO(3;R) Lie algebras. Explicitly, defining L_j and L_j' as

$$\begin{aligned}L_j&=-\tfrac{1}{2}(\tfrac{1}{2}\epsilon_{jkl}A_{kl}+A_{0j})\ ,\\ L_j'&=-\tfrac{1}{2}(\tfrac{1}{2}\epsilon_{jkl}A_{kl}-A_{0j})\ ,\end{aligned} \qquad (3.5)$$

one easily checks that $[L_j,L_k]=\epsilon_{jkl}L_l$, similarly for L_j', and that $[L_j,L_k']=0$. From this one concludes that any O can be written as $\exp(c_jL_j)\exp(c_j'L_j')$, and any K as $\exp(id_jL_j)\exp(id_j'L_j')$, where c_j, c_j', d_j, and d_j' are arbitrary real coefficients.

Since elements of the form $\exp(\theta A_{02})$ and $\exp(i\eta A_{01})$ are in G, the following product also belongs to G for any η:

$$\exp\left[\frac{\pi}{2}A_{02}\right]\exp(i\eta A_{01})\exp\left[\frac{-\pi}{2}A_{02}\right]=\exp(i\eta A_{12})\ . \qquad (3.6)$$

On the other hand, $\exp(i\theta A_{03})$ is in G, and A_{03} commutes with A_{12}. Hence G contains $\exp(i\theta A_{03}+i\eta A_{12})$. But

$$\exp(i\theta A_{03}+i\eta A_{12})=\exp[-i(\theta+\eta)L_3]\exp[i(\theta-\eta)L_3']\ . \qquad (3.7)$$

Since θ and η are arbitrary, so are $\theta+\eta$ and $\theta-\eta$. It is not difficult to see that acting on the element $\exp[-i(\theta+\eta)L_3]\exp[i(\theta-\eta)L_3']$ from the left by an arbitrary O and from the right by O^{-1} generates all K's, i.e., all elements of the form $\exp(id_jL_j)\exp(id_j'L_j')$. So G contains all K's, and therefore G includes SO(4;C). Q.E.D.

We have shown that the smallest group that includes $C_{\hat{x}}$ and $h\{L_+^\uparrow\}h^{-1}$ is SO′(4;C). Equivalently, the smallest group that includes $CS_{\hat{x}}$ and $L_+^\uparrow$ is SO′(3,1;C). The two groups are isomorphic. In the representation where matrices belonging to $L_+^\uparrow$ are the standard real ones, SO′(3,1;C) is made up of all unimodular complex matrices that either preserve the Lorentz metric or change its sign. This result is in sharp contrast with a claim often made by Recami and others.[12,22] They have argued that the group generated by $L_+^\uparrow$ and complex transformations of the form ($|v|>1$)

$$\begin{bmatrix}\pm(v^2-1)^{-1/2} & \pm(v^2-1)^{-1/2}v & 0 & 0\\ \pm(v^2-1)^{-1/2}v & \pm(v^2-1)^{-1/2} & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & i\end{bmatrix} \qquad (3.8)$$

is SO(3,1;R) plus a set of matrices obtained by multiplying unimodular Lorentz transformations by i and making a formal substitution $\beta\rightarrow 1/\beta$ (where β is the relative speed). The problem is that such a set of transformations is not closed under multiplication. The smallest group that contains $L_+^\uparrow$ and transformations of the form (3.8) is the full SO′(3,1;C).

On the other hand, our results are consistent with those obtained sometime ago by Imaeda.[23] Working in a quaternionic formulation, Imaeda has shown that the group generated by the Lorentz group and transformations of the form (3.8) is one-to-two homomorphic with a group of linear transformations of complex quaternions. Imaeda's results can easily be translated from complex quaternions to complex 2×2 matrices. Let Z be a complex 2×2 matrix, and let K and L be such matrices with determinants, respectively, equal to $+1$ and ± 1. The transformation $Z\rightarrow Z'=KZL$ then effects a complex linear transformation of the components of Z, which either leaves the determinant invariant or changes its sign. Note that if one writes $Z=z_0I^{(2)}+z_i\sigma_i$, then $\det Z=z_0{}^2-z_iz_i$, the Lorentz bilinear form. Real Lorentz transformations correspond to the choice $K=\pm L^\dagger$, and Imaeda succeeded in expressing (3.8) in terms of appropriate K and L. He then showed that repeated products of Lorentz and (3.8) generate all transformations of the form KZL. These form a group, which has two disconnected components, corresponding to the ± 1 values of det$L$. It is not difficult to compute the Lie algebra of the connected part of this group, and one finds that it coincides with the Lie algebra of SO(3,1;C)$\sim$SO(4;C). Elements of SO′(3,1;C) and pairs (K,L) are in a one-to-two correspondence, since (K,L) and $(-K,-L)$ transform Z in the same way. This illustrates the consistency of our results with Imaeda's.

The fact that the minimal group containing $CS_{\hat{x}}$ and $L^{\uparrow}_{+}$ is SO′(3,1;C) implies a major difficulty, related to the problem of interpreting the complex coordinates. This results from the fact that SO′(3,1;C) contains complex transformations between frames relatively at rest. Consider, for example, the matrix $M(\alpha)$ for arbitrary real α given by

$$M(\alpha)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cosh\alpha & i\sinh\alpha\\ 0 & 0 & -i\sinh\alpha & \cosh\alpha\end{bmatrix} . \qquad (3.9)$$

This matrix satisfies $M^{T}gM=g$ and thus belongs to SO′(3,1;C). Therefore, by theorem 3, it can be obtained from repeated products of $CS_{\hat{x}}$ and elements of $L^{\uparrow}_{+}$. Moreover, since $M(\alpha)$ only acts on space coordinates, if the coordinates in two frames K and K' are related by $M(\alpha)$, the two frames are relatively at rest. One is thus faced with the problem of interpreting complex coordinates not only in superluminal frames but in subluminal ones. Moreover, the transformation equations between two frames are no longer specified, up to questions of the relative orientation of the axes, by the relative velocity $\vec{\beta}$ of the two frames. For, if $T(\vec{\beta})$ is a matrix giving a transformation between two frames with relative velocity $\vec{\beta}$, then so are matrices such as $T(\vec{\beta})M(\alpha)$, which clearly do not differ from $T(\vec{\beta})$ by a (real) rotation of the coordinate axes.

The complex matrices (3.1) and (3.8) are intended by their proponents to represent coordinate transformations between equivalent frames having a superluminal relative velocity directed along the x axis. In trying to interpret them physically, several authors have suggested that if the transverse coordinates of an event in one frame become imaginary in a second frame, they are unobservable in the second frame. Corben,[17] for example, has argued that if a meter stick parallel with the y axis moves with superluminal speed along the x axis, then its length cannot be measured by means of light signals transmitted from one end to the other and hence is unobservable. This argument is not particularly convincing, since there are other ways in which the length of the meter stick can be measured in the second frame, e.g., simply by determining the coordinates of the two ends of the stick at a given time.[18] There is an even more serious problem, however. Since the transformation equations are arbitrary up to one of the complex rotations, such as $M(\alpha)$, contained in SO′(3,1;C), there is no way of specifying unambiguously whether coordinates are real or complex, i.e., whether they are or are not observable.

The necessity, when postulating (3.1), of introducing the notion of a fully complex space-time was realized by Imaeda.[23] Fully complex coordinates had previously been used by Yaccarini[24] and by Cole.[25] To our knowledge, however, no convincing interpretation of it has ever been proposed. It has sometimes been suggested that the complex four-vectors should be associated with twistors. But such an analogy, resting solely on the four-dimensional complex character of both, appears rather shallow. The standard relationship between twistors and points in space-time is such that twistors correspond to lines (in fact, complex ones) more than points, a point being associated with two twistors.[26] The relevant group in twistor space is SU(2,2), the covering group of the conformal group, whereas the relevant group in complex space-time is SO′(3,1;C). So the complex coordinate transformations, if interpreted as holding between points in space-time, do not seem to be related to twistors, in any direct way. Conversely, an interpretation of them in terms of twistors, if at all possible, would spoil the direct space-time interpretation of the four-vectors.

The complex coordinate transformations (3.1) and (3.8), being elements of SO′(3,1;C), transform null intervals into null intervals. This is often interpreted as meaning that they leave the speed of light invariant. Such an interpretation, however, overlooks the fact that the resulting velocities are complex and thus cannot be equated with genuine velocities. Several authors, aware of the difficulties raised by complex coordinates and velocities, have suggested that transformations like (3.1) and (3.8) should be taken as formal devices only, coordinates and velocities being real in all frames.[14,27] But then one could use real transformations from the outset, and this would lead us right back to the analysis of Sec. II. A rather unusual interpretation of (3.8) was also proposed by Recami and Maccarrone.[19] They noticed that the portion of three-space occupied by a bradyon of radius r at rest at the origin of a frame K satisfies the constraint $0\le x^2+y^2+z^2\le r^2$. Transforming the coordinates by means of (3.8), they obtained for the corresponding constraint in K' the inequalities

$$0\le\frac{(x'-vt')^2}{v^2-1}-(y')^2-(z')^2\le r^2 . \qquad (3.10)$$

If (3.8) is to be taken literally, then y' and z' are pure imaginary numbers. But Recami and Maccarrone take (3.8) to apply to real values of the primed coordinates. At a given t' then, the constraint (3.10) defines an infinite portion of three-space limited by an infinite cone and an infinite hyperboloid. This means that a finite portion of three-space in K goes over into an infinite portion in K', and that a point goes over into a cone. Clearly this is no longer a one-to-one mapping of the points in K onto those in K', and thus the interpretation of Recami and Maccarrone falls outside the concept of coordinate transformations between equivalent reference frames.

The complex coordinate transformations were proposed in an attempt to salvage the invariance of the speed of light and the extended principle of relativity applied to Lorentz and superluminal frames. Our analysis shows that they have failed on both counts. On the one hand, they do not preserve the speed of light since they imply a concept of velocity different from the usual one. On the other hand, they entail much too large a set of equivalent frames, some of which are related by complex coordinate transformations even though relatively at rest. Thus complex transformations fare no better than real transformations even on the speed of light and the extended principle of relativity. Since these two requirements must be dropped in any case when introducing linear superluminal transformations, it seems much preferable to avoid the additional problems of complex coordinates, and stick with real ones.

IV. TRANSFORMATIONS BETWEEN NONEQUIVALENT FRAMES

In the preceding two sections, we have shown that the full extended principle of relativity, namely, the adjunction of superluminal transformations to the proper orthochronous Lorentz group as transformations between equivalent reference frames, cannot be maintained. The complete group of transformations generated in each case is much too large, and contains symmetries which are not possessed by known physical laws. Once one eliminates the extended principle of relativity, there are surprisingly few candidates left in the literature for four-dimensional coordinate transformations. We will mainly devote this section to a brief look at them, emphasizing the ways that the extended principle of relativity is weakened or dropped altogether.

A. The tachyon corridor

In an approach proposed by two of us,[28,29] there is still complete equivalence between some frames related by superluminal transformations, but Lorentz and rotational invariance no longer hold exactly. More precisely, there are a preferred direction in three-space, called the tachyon corridor, and a preferred velocity perpendicular to it. The tachyon corridor can be taken parallel to the x axis, and its perpendicular velocity can be set equal to zero. Let K be a reference frame moving along the tachyon corridor. Any frame K' obtained from K either by a Lorentz boost along the x axis or by a superluminal transformation of the form (2.1) and (2.2) (supplemented with $y=y'$ and $z=z'$) is taken to be equivalent to K. Thus the extended principle of relativity is weakened to hold only among frames that move along a preferred direction with a preferred perpendicular velocity.

In addition to these preferred frames, there are nonpreferred ones related to the former by arbitrary elements of $L_+^\uparrow$. The nonpreferred frames are not equivalent to the preferred ones; either their velocity perpendicular to the corridor is different from zero, or their x axis is inclined with respect to the corridor (or both).[30] The set of all frames can be divided into two disjoint subsets, so that the relative velocity of K and K' is superluminal if and only if K and K' belong to different subsets.

Since Lorentz and rotational invariances are broken, the question arises of the experimental consequences of such an approach.[28] What breaks $L_+^\uparrow$ is the presence of the tachyon corridor, and so one expects tachyon phenomena to show large departures from $L_+^\uparrow$ symmetry. Since tachyons have never been observed, however, this much is not experimentally ruled out. On the other hand, strong couplings between tachyons and bradyons will produce departures from $L_+^\uparrow$ symmetry even in phenomena where only bradyons are observed. Experimentally, there are very stringent limits on rotational symmetry breaking. This in turn restricts tachyon-bradyon couplings to small numerical values. In an approach like the tachyon corridor, part of the content of the extended principle of relativity is maintained, but tachyons are particularly difficult to observe.

B. A different interpretation of the tachyon corridor

An alternative interpretation of the tachyon corridor was proposed elsewhere in the literature.[31] It consists in associating the corridor not with a fixed direction in space, but with the instantaneous direction of motion of a tachyon under observation. Specifically, the coordinates of a tachyon with velocity $\vec{u}$ are taken to transform according to equations similar to (2.1) and (2.2), except that the x axis in those equations is replaced by one parallel with $\vec{u}$.

It turns out that such an interpretation is inconsistent.[32] To see this, assume that a tachyon, at rest in a superluminal frame K', is observed from two subluminal frames K and K''. The coordinate transformations between K and K', and the transformations between K'' and K', carried out according to the above prescription, imply coordinate transformations between K and K'' other than Lorentz's. But this cannot be, since by assumption K and K'' move with a subluminal relative velocity. Furthermore, intractable problems will result if one considers several tachyons with different velocity vectors meeting at a given space-time point. Indeed, how will the coordinates of this point transform? Coordinate transformations should depend only on the two frames they relate, and not on what happens at a particular point.

C. Breaking proper Lorentz invariance

Abandoning altogether the concept of equivalence of frames related by superluminal transformations, one of us has proposed an approach to superluminal frames[33] that is rather different from the one discussed in Part IV A. It is characterized by exact rotational invariance, but broken invariance under proper Lorentz boosts. Explicitly, there is a set of preferred frames, all at rest and arbitrarily rotated with respect to each other. The preferred frames are all equivalent, but there are nonpreferred frames moving with arbitrary velocities with respect to the preferred ones and nonequivalent to them. Implementing such an approach by means of explicit coordinate transformations leaves one with considerable freedom, as illustrated in Ref. 33. In general, the coordinate transformations between preferred frames and nonpreferred frames will not have a group structure. And there is nothing wrong with that, because they relate different kinds of frames.

This approach to tachyons and superluminal frames, being less specific than the one based on the tachyon corridor, has less predictive power. But it can accommodate tachyon-bradyon couplings of substantially larger amplitude, and still be consistent with experiment.

D. Goldoni's faster-than-light frames

A rather different approach to superluminal frames was proposed by Goldoni,[34] who first wrote down the following transformation equations ($|v|>1$):

$$t'=(v^2-1)^{-1/2}(t-vx),\quad y'=y\ ,$$
$$x'=(v^2-1)^{-1/2}(-vt+x),\quad z'=z\ . \tag{4.1}$$

These equations look very much like the ones used in the tachyon corridor approach, except for a sign factor [compare with Eqs. (2.1) and (2.2)]. The two sets of equations were proposed independently, and in fact have quite a different meaning. Letting M denote the matrix representing the transformation (4.1), and taking g_x to stand for the diagonal matrix $\mathrm{diag}(-1,1,-1,-1)$, Goldoni points out that

$$M^T g_x M = g\ , \tag{4.2}$$

where g stands for $\mathrm{diag}(1,-1,-1,-1)$. Goldoni then considers the set of all matrices satisfying Eq. (4.2), thereby defining a class C_x of superluminal reference frames. Two more such classes, denoted by C_y and C_z, are obtained in a similar way by substituting g_y and g_z, respectively, for g_x in (4.2). Here $g_y=(-,1-1,1,-1)$ and $g_z=(-1,-1,-1,1)$.

The resulting structure appears rather complicated. Goldoni wants to have an extended principle of relativity holding between all these frames. Clearly, the results of Sec. II immediately rule out such a possibility. Some of the frames must be preferred in some way. A plausible picture is then the following: The way C_x, C_y, and C_z are defined implies a breaking of rotational invariance, since in a rotated frame the three axes are no longer the same. Proper Lorentz invariance is also broken, since the commutator of two boost generators is a rotation. So there is a unique preferred frame to which all superluminal transformations like (4.2) (and the correspoding ones with g_y and g_z) can be applied. Goldoni also suggests ways of making tachyons and bradyons interact. It is outside the scope of this paper to dwell upon that matter, as well as to investigate the viability of his scheme on other than kinematical grounds.

Goldoni's idea of introducing three sets of transformations, respectively, changing g_x, g_y, and g_z into g was also taken up by Lord and Shankara.[35] Their way of implementing the scheme is doomed from the beginning, since they postulate the equivalence of all frames.

E. Six-dimensional transformations

The difficulty of generalizing the Lorentz transformations to $v>c$ results essentially from the mismatch, in the four-dimensional case, between the number of spatial and time dimensions. Thus it was suggested that the problem could be solved by formulating physics in a six-dimensional space with three space and three time dimensions.[36,25] One can then postulate symmetry under the group O(3,3). In order to yield agreement with observation, one can try to suppose that all observable time displacements are along one of the three possible time directions, say the t_1 axis, with the Lorentz group being the subgroup of O(3,3) which leaves t_2 and t_3 invariant.

It is beyond the scope of the present paper to discuss such theories in detail. We note, however, that they do appear to possess a serious problem. It is true that the full group O(3,3) contains transformations in which the parameter v representing the relative speed of the two reference frames satisfies $v > 1$. However, Cole has shown[25] that superluminal transformations necessarily involve transformations on the other two components of the time. In particular, under a superluminal transformation along the x axis, one has $t_2' = y$ and $t_3' = z$, and similarly for $y'(z')$ and $t_2(t_3)$. Hence if superluminal coordinate systems exist in such a theory on an equivalent footing with subluminal frames, the freedom to make spatial displacements in all three directions in a superluminal frame implies the occurrence of displacements in all three temporal directions. Thus the problem of assigning meaning to the other two time directions can no longer be avoided.

V. CONCLUSION

To many of those interested in building theories of faster-than-light particles, the extended principle of relativity has always exerted a strong appeal. Unfortunately, the analysis carried out in this paper casts serious doubts on the tenability of this principle. True, we have restricted ourselves throughout to linear coordinate transformations.[37] Within this rather natural framework, though, we have proved three theorems which essentially exclude the possibility of implementing the extended principle of relativity by means of either real or complex transformations. In both cases it was shown that the slightest extension of the proper orthochronous Lorentz group implies a large number of new symmetries, in violent conflict with what is observed in nature. The best avenues left to study hypothetical faster-than-light particles thus seem to be of two types: (i) Either retain the notion of superluminal frames while weakening or abandoning altogether the extended principle of relativity. This forms the basis of several approaches as were analyzed in Sec. IV. (ii) Or do without any notion of superluminal frames. That was the point of view adopted in some of the original papers on tachyons.[38–41] References 42 and 43 are examples of recent work along this line.

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