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General static plane-symmetric solutions of the Einstein-Maxwell equations

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A general form of the metric in a space-time with nonvanishing cosmological constant outside a massive, electrically charged plane of infinite extension is found as a solution of the Einstein-Maxwell field equations. The general solution is new, but it includes several well-known solutions corresponding to special physical cases. Also, the general form of our solution makes it possible easily to identify different metrics corresponding to equivalent space-times. Physical properties of the solution are discussed. The plane-symmetric universe analogous to the spherically symmetric de Sitter universe is found separately, since it is not included in the general solution.

I. INTRODUCTION

The study of plane-symmetric solutions of Einstein's field equations has a long history. The static vacuum solution was found by Levi-Civita in 1918,[1] and describes space-time outside a massive plane. This solution has been rediscovered in different coordinate systems by several authors,[2–4] although the identity of these solutions was not pointed out. Davis and Ray[5–7] have shown that the solution also admits the presence of "ghost neutrinos" outside the plane.

Further investigations have included the generalization to nonstatic solutions,[2,8,9] static solutions for a nonvanishing cosmological constant,[10–12] solutions inside a collapsing dust cloud,[13–17] in space-times filled with self-gravitating fluids,[18–25] in colliding plane gravitational waves,[26–35] inside static, massive plates,[36–44] with radiation in thermodynamical equilibrium,[45] with zero-rest-mass scalar fields,[46–49] with "ghost-free" neutrinos,[50,51] and with a Yang-Mills plane wave.[52] The embedding class behavior of the plane-symmetric line element has been investigated by Pandey and Sharma.[53]

The solution of the Einstein-Maxwell equations outside a charged massive plane was found by Kar.[54] Shortly thereafter McVittie found a solution corresponding to a definite ratio of charge density to mass density of the plane.[55] Kar's solution was later rederived by use of Rainich's equations[56] and generalized to the nonstatic case.[57–59] In subsequent works on this problem we have found no reference to Kar's solution, and the calculations have mostly been made in Taub-type coordinates.[60–63] As shown in Sec. IV below these coordinates are especially unfortunate when the Einstein-Maxwell equations are to be solved. Static perfect-fluid distributions with an electromagnetic field have been investigated by Bronnikov and Kovalchuk.[64,65]

Also plane-symmetric solutions of the Brans-Dicke theory have been deduced.[66–70]

Kinematical aspects (properties that do not depend upon the field equations) of the most general (time dependent) plane-symmetric space-times have been discussed by Carlson and Safko.[71]

Although plane-symmetric solutions of Einstein and Einstein-Maxwell equations are thus well studied, the results in the literature appear rather isolated, with little or no discussions of the relation between the solutions or their physical interpretations. In this paper we find a general form of the solution in a space-time with a nonvanishing cosmological constant outside a massive, electrically charged plane. The general form of the solution also allows coordinate transformations between different metrics describing the same physical situation to be found in a simple way. Such general solutions can be useful both for discussing physical (coordinate-independent) features of space-time, and because it may simplify the problems of joining internal and external solutions for space-time in the presence of localized matter fields. Similar results for the spherically symmetric case were recently reported by Abrams.[72]

The organization of our paper is as follows. In the next section, restricting ourselves to traceless energy-momentum tensors, we write down the Einstein-Maxwell equations, including the cosmological term, for the assumed form of the metric. In Sec. III we give the general solution of these equations. The integration constants are interpreted physically. In Sec. IV we specialize to vacuum solutions with a vanishing cosmological constant. We find two classes of solutions. The first class is simply Minkowski space-time, as seen from a rigidly moving reference frame with hyperbolic acceleration. It was found by Rohrlich[73] (see also Horský,[74] Landsberg and Bishop,[75] Greenberger and Overhauser,[76] and Grøn[77,78]). The second class of solutions describe space-time outside a massive plane. The above-mentioned solutions of Levi-Civita, Kasner, Taub, and Das emerge as special cases, corresponding to certain choices of coordinates. Furthermore, we find that within each class of solutions the coordinate transformation connecting two arbitrary metrics can be found immediately by solving an (in general transcendental) equation.

In Sec. V we specialize to the vacuum solution with nonvanishing cosmological constant, obtaining the general form of Horský and Novotny's solution.[10–12] This solution does not represent a plane-symmetric de Sitter universe, since it exists only in the presence of a massive plane. After a suitable change of the field equations, the plane-symmetric de Sitter solution is found.

In Sec. VI we specialize to space-time with a vanishing cosmological constant outside a massive electrically charged plane. The solutions of Kar[54] and Patnaik[60] represent special cases corresponding to certain choices of coordinates, while McVittie's[55] solution, and a solution given by Banerjee and Chakrabarty,[63] represent a special physical case.

Our results are summarized in Sec. VII.

II. GENERAL EQUATIONS

We shall solve the Einstein-Maxwell equations with purely electromagnetic sources. In this case $T \equiv T^{\mu}{}_{\mu}=0$, and the field equations can be written (we use units with $c=4\pi\gamma=1$, where γ is the gravitational constant)

$$R_{\mu\nu}=-2T_{\mu\nu}+\lambda g_{\mu\nu} \ , \tag{1}$$

$$T_{\mu\nu}=F_{\mu}{}^{\alpha}F_{\alpha\nu}+\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \ , \tag{2}$$

$$F^{\mu\nu}{}_{;\nu}=0 \ , \tag{3}$$

$$F_{[\mu\nu,\alpha]}=0 \ , \tag{4}$$

where all symbols have the usual meanings. The orthogonal plane-symmetric line element has the form

$$ds^2=E\,dt^2-G(dx^2+dy^2)-F\,dz^2 \ , \tag{5}$$

where E, F, and G are functions of z and t.

We will consider only static situations, where all quantities are independent of t. Then the field equations give only two independent equations, so without loss of generality we may assume, for example, a spatial isotropic form of the line element. However, it will be convenient to retain all three functions E, F, and G. Inspection of the field equations will then reveal the most suitable coordinate condition in different situations.

With the metric (5), and in the static case, the gravitational field equations (1) take the form

$$E''-E'^2/2E-E'F'/2F+E'G'/G=4FT_{00}-2\lambda EF \ , \tag{6}$$

$$G''+E'G'/2E-F'G'/2F=-4FT_{11}-2\lambda FG \ , \tag{7}$$

$$E''-E'^2/2E-E'F'/2F+(2E/G)(G''-G'^2/2G-F'G'/2F)=-4ET_{33}-2\lambda EF \ , \tag{8}$$

where a prime denotes differentiation with respect to $x_3=z$. Equation (6) can be written as

$$[(EF)^{1/2}/G][GE'/(EF)^{1/2}]'=4FT_{00}-2\lambda EF \ . \tag{9}$$

Equation (7) can be transformed to

$$(F/EG')(EG'^2/F)'=-8FT_{11}-4\lambda FG \ . \tag{10}$$

Subtracting (6) from (8) gives an equation that can be written as

$$(EFG)^{1/2}[G'/(EFG)^{1/2}]'=-2G[T_{33}+(F/E)T_{00}] \ . \tag{11}$$

III. THE GENERAL SOLUTION WITH AN ELECTROSTATIC FIELD

The energy-momentum density tensor of a parallel electrostatic field in a metric of the form (5) has been calculated by McVittie[55] as

$$T^{\mu}{}_{\nu}=\frac{\sigma_q{}^2}{8G^2}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix} \ , \tag{12}$$

where the charge per unit area of the plane, σ_q, is assumed to be positive for definiteness.

In this case the field equations (9), (10), and (11) take the forms

$$[2/G(EF)^{1/2}][GE'/(EF)^{1/2}]'=\sigma_q{}^2G^{-2}-4\lambda \ , \tag{13}$$

$$(1/EGG')(G'^2E/F)'=-\sigma_q{}^2G^{-2}-4\lambda \ , \tag{14}$$

$$(EFG)^{1/2}[G'/(EFG)^{1/2}]'=0 \ . \tag{15}$$

Integration of Eq. (15) gives

$$(EF)^{1/2}=-(2\sigma)^{-1}G^{-1/2}G', \ \ G'\neq 0 \ . \tag{16}$$

The constant σ is later identified with the mass density of the plane by going to the Newtonian limit. Substituting Eq. (16) into Eq. (13) and integrating twice gives

$$E=(\sigma_q{}^2/4\sigma^2)G^{-1}-(\lambda/3\sigma^2)G+bG^{-1/2}+k \ . \tag{17}$$

Demanding consistency with Eq. (14) gives $k=0$. $E(z=0)=1$ gives $b=1-\sigma_q{}^2/4\sigma^2+\lambda/3\sigma^2$. Thus

$$E=aG^{-1}+bG^{-1/2}+cG, \ \ a=\sigma_q{}^2/4\sigma^2 \ , \quad c=-\lambda/3\sigma^2, b=1-a-c \ . \tag{18}$$

The solution may now be written

$$ds^2=(aG^{-1}+bG^{-1/2}+cG)dt^2-G(dx^2+dy^2) -(G'^2/4\sigma^2)(a+bG^{1/2}+cG^2)^{-1}dz^2 \ . \tag{19}$$

We note that under a coordinate transformation $z\rightarrow z_1=z_1(z)$. G transforms according to $G(z)\rightarrow G_1(z_1)$ where

$$G_1(z_1)=G[z(z_1)] \ . \tag{20}$$

Thus we have a simple way of finding the coordinate tranformation which connects two different metrics.

In general we will normalize the coordinates so that the metric has Minkowski form at the plane. Thus we impose the boundary condition $E=F=G=1$ at $z=0$.

Particular solutions corresponding to certain choices of coordinates are easily generated by assuming functional relationships between E, F, and G. Equation (16) is immediately integrated with, for example, $EF=1$, which gives $G=(1-\sigma z)^2$.

The general form of the solution makes it possible to deduce coordinate-independent properties of the metric. Equation (16) shows that G is a positive monotonic function of z wherever it is continuous.

If we disregard the boundary conditions associated with the massive plane, Eqs. (5), (16), and (18) represent a solution for all z such that $z_- \leq z \leq z_+$, where $G(z_-)=\infty$ and $G(z_+)=0$. Taking into account the boundary of the plane, the z_- singularity may be neglected.

The Ricci curvature invariant is given by

$$S=R^\mu{}_\nu R^\nu{}_\mu=\sigma_q{}^4/4G^4+4\lambda \ , \tag{21}$$

showing that z_+ represents a space-time singularity. Equation (21) also shows that the other singularities appearing in the metric $(E\rightarrow 0)$ and $(E\rightarrow\infty)$ are only coordinate singularities.

The proper distance (as measured by standard measuring rods) from the massive plane to the space-time singularity at z_+ is

$$\hat{z}_+=-\int_0^{z_+}F^{1/2}dz =\frac{1}{2\sigma}\int_0^1(a+bG^{1/2}+cG^2)^{-1/2}dG \ . \tag{22}$$

Thus the space-time singularity is at a finite proper distance from the plane. Note that Eq. (22) gives $\hat{z}_+$ without introduction of specific coordinates.

The nonvanishing charge and cosmological constant changes the distance to this plane, but introduce no qualitative modifications of space-time. A further discussion of the physical properties of this space-time, considering geodesics, is therefore treated in the case of vanishing charge and cosmological constant (next section) where the equations are integrated analytically in terms of elementary functions.

IV. VACUUM SOLUTIONS WITH VANISHING COSMOLOGICAL CONSTANT

In this case the field equations permit two classes of solutions.

(I): $G=1$. Equation (13), with $\sigma_q=\lambda=0$, now gives

$$F=(1/4g^2)E'^2/E, \ \ E'\neq 0 \ , \tag{23}$$

where $-g$ is the acceleration of a free particle instantaneously at rest at the plane, $z=0$. This is the solution of Rohrlich,[73] which describes flat space-time. In particular, if one chooses $F=1$, the line element becomes that of a uniformly accelerated rigid reference frame in Minkowski space-time as expressed in Møller coordinates[79]

$$ds^2=(1+gz)^2dt^2-dx^2-dy^2-dz^2 \ . \tag{24}$$

(II): $G'\neq 0$. The general solution now reduces to

$$ds^2=G^{-1/2}dt^2-G(dx^2+dy^2) -(G'^2/4\sigma^2G^{1/2})dz^2 \ . \tag{25}$$

The physical character of the space-time described by this line element can be investigated by considering the motion of test particles. This can be done by finding the Killing vectors. The present situation is so simple, however, that the equations of motion can be integrated. For a null geodesic in the

z direction one finds

$$t-t_0=\pm\frac{1}{2\sigma}\int_{z_0}^{z}G'\,dz=\pm\frac{1}{2\sigma}[G(z)-G(z_0)] \ . \tag{26}$$

Due to the monotonicity of G this equation is always invertible.

For a massive particle the geodesic equation can be written

$$F\ddot{z}+\tfrac{1}{2}F'\dot{z}^2+\gamma^2E'/2E^2=0 \ , \tag{27}$$

where the overdots denote differentiation of the proper time of the particle, and

$$\gamma=E\dot{t}=G^{-1/2}\dot{t} \tag{28}$$

is the constant total energy of the particle. The local acceleration of a free particle instantaneously at rest at the plane becomes

$$\ddot{z}(0)\equiv -g=-\sigma/2 \ . \tag{29}$$

Calculation of this acceleration, using Gauss's theorem, as in Newtonian dynamics, shows that σ is indeed the mass density of the plane.

That the space-time described by the present special case is curved can be seen by calculating the Kretschmann curvature invariant

$$S=R^{\lambda\mu\nu\kappa}R_{\lambda\mu\nu\kappa}=12\sigma^4[G(z)]^{-3} \ . \tag{30}$$

In this case the finite invariant distance to the space-time singularity z_+, given by Eq. (22), is $\hat{z}_+=(3g)^{-1}$. This singularity must be given a physical interpretation. Now the use of coordinates with standard measuring rods in the z direction is particularly advantageous, both for reasons of interpretation and because the geodesic equation (27) is easily integrated when $F=1$. In this case Eq. (25) gives the metric[80–82]

$$ds^2=(1-3g\hat{z})^{-2/3}dt^2 -(1-3g\hat{z})^{4/3}(dx^2+dy^2)-d\hat{z}^2 \ , \tag{31}$$

where $g=\sigma/2$ is the acceleration scalar of a fixed coordinate point at the position of the plane. The geodesic equation (27) now takes the form

$$\ddot{\hat{z}}+g\gamma^2(1-3g\hat{z})^{-1/3}=0 \ , \tag{32}$$

which gives

$$\dot{\hat{z}}=[\gamma^2(1-3g\hat{z})^{2/3}-1]^{1/2} \ . \tag{33}$$

It follows that the maximum height a massive particle can reach is

$$\hat{z}_M=(1-\gamma^{-3})/3g \ , \tag{34}$$

showing that $\hat{z}_M<\hat{z}_+$ for all finite (positive) values of γ. A massive particle cannot reach the space-time singularity at $\hat{z}_+=(3g)^{-1}$.[83]

Integrating Eq. (33) we find that the proper time, as measured on a clock following the particle, taken to reach $\hat{z}_M$ is

$$\tau(\hat{z}_M)=(2\gamma^3g)^{-1}(\gamma^2v_0+\text{arccosh}\gamma) \ , \quad \gamma=(1-v_0{}^2)^{-1/2} \ , \tag{35}$$

where v_0 is the velocity of the particle at $z=0$, giving $\lim_{\gamma\to\infty}\tau(\hat{z}_M)=0$. This may be understood by comparing the coordinate velocity of a massive particle, v, with that of a photon v_p. The metric (31) with $ds=dx=dy=0$ immediately gives

$$v_p=(1-3g\hat{z})^{-1/3} \ . \tag{36}$$

Thus

$$v=\dot{\hat{z}}/\dot{t}=v_p(1-\gamma^{-2}v_p)^{1/2} \ , \tag{37}$$

showing that $\lim_{\gamma\to\infty}v=v_p$. A massive particle shot up from the plane with a velocity arbitrarily near that of light, follows a photon right up to the space-time singularity, without being measurably slowed down.

The geodesics for photons moving in the z direction are given by

$$t=\pm[1-(1-3g\hat{z})^{3/4}](4g)^{-1} \ . \tag{38}$$

Since the metric is static, t may be found by measuring on a standard clock at the plane, the time $2t$ that a photon takes in traveling to $\hat{z}$, where it is reflected, and back again. Equation (38) shows that photons are able to reach the space-time singularity in a finite time $t_M=(4g)^{-1}$, as measured with standard clocks at $\hat{z}=0$.

Using the equation for the gravitational Doppler effect, the frequency of light measured locally at a position $\hat{z}$, is

$$\omega=E^{-1/2}\omega_0=(1-3g\hat{z})^{1/3}\omega_0 \ , \tag{39}$$

where ω_0 is the frequency of the light as measured locally at the plane. The frequency becomes zero as the light reaches the singularity of space-time.

The question is now what is happening to light that reaches $\hat{z}_+$. There seem to be two possibilities. One is that it is possible to continue the metric to $\hat{z}>\hat{z}_+$. From Eq. (26) is seen that when $\hat{z}$ is increased so that $\hat{z}>\hat{z}_+$, one must change the sign in front of $(2\sigma)^{-1}$ in order to make the t coordinate one-valued. Thus the acceleration of gravity is directed away from the wall on both sides of it. This motivated Liang[80] to associate the singular wall with the position of a plane with *negative* mass. A similar interpretation had been given earlier by Gautreau and Hoffman.[84] In order to produce the singularity, the negative mass density of the plane would have to be infinite.

Another, and in our view more acceptable solution, is that even light when it reaches $\hat{z}=\hat{z}_+$, and has lost all its intrinsic energy [Eq. (39)], will fall back. This interpretation is still consistent with Eq. (38) as now the negative sign gives the null geodesic curve for $t>t_M$.

According to this interpretation there is no massive plane at the singularity, but a plane with positive mass at $\hat{z}=0$. This produces a gravitational field. Space-time is curved outside the plane. The energy of the gravitational field contributes to the gravitational acceleration of test particles. This interpretation ensures that the accessible universe is closed not only for massive particles (even in the case of nongeodetic motion, due to the divergence of $\ddot{\hat{z}}_+$), but also for radiation. The physical universe outside the plane has a finite extension, reaching only out to $\hat{z}_+=(3g)^{-1}=\frac{2}{3}\sigma$. (A similar interpretation is hinted at by Avakyan and Horsky.[39])

One can also consider the acceleration of a particle at rest at any positive z. Equations (32) and (34) give

$$-(\ddot{\hat{z}})_M=g(1-3g\hat{z}_M)^{-1} \quad . \tag{40}$$

Since this quantity is measured by standard clocks and standard measuring rods, it is invariant against a time-independent coordinate transformation. Thus we can impose the boundary condition of a massive plane at any z simply by adjusting the mass density of the plane so that $\sigma=2g(1-3g\hat{z}_0)^{-1}$, where $\hat{z}_0$ is the position of the plane. As a limiting case, if the massive plane is removed, we have a solution with a "universe" consisting solely of a homogeneous gravitational field for all z; $z_-<z<z_+$. This universe is asymptotically flat for $z\to z_-$.

Some previous solutions that have been announced in the literature can be identified with particular cases of Eq. (25). The harmonic coordinate condition $g^{\alpha\beta}\Gamma^{\mu}{}_{\alpha\beta}=0$ gives $F=EG^2$. In these coordinates the line element takes the form[45]

$$ds^2=e^{2gz}dt^2-e^{-4gz}(dx^2+dy^2)-e^{-6gz}dz^2 \quad . \tag{41}$$

This metric is singularity free for all finite z. However the space-time singularity is only hidden because of the coordinates used. The transformation (20) between the metrices (31) and (41) is

$$1-3g\hat{z}=e^{-3gz} \quad . \tag{42}$$

Approaching z_+ the length of the coordinate measuring rods approaches zero, and the coordinate region $0<z<\infty$ only covers the physical domain $0<\hat{z}<(3g)^{-1}$.

It is also possible to choose coordinates so that the metric is spatially isotropic, $F=G$. This shows that the spatial component of space-time is conformally flat. In such coordinates Eq. (25) gives

$$ds^2=(1-gz)^{-2}dt^2 -(1-gz)^4(dx^2+dy^2+dz^2) \quad . \tag{43}$$

This form of the solution was given by Das,[4] but in corresponding Rindler-type coordinates, $G=z^4$, it was found by Kasner already in 1925.[2]

Taub's form of the solution has $E=F$ which gives

$$ds^2=(1-4gz)^{-1/2}(dt^2-dz^2) -(1-4gz)(dx^2+dy^2) \quad . \tag{44}$$

This metric has also been found in corresponding Rindler-type coordinates by Liang.[80]

Finally the metric with $EF=1$ was found by Novotný,[12] and has $G=(1-2gz)^2$.

V. VACUUM SOLUTIONS WITH NONVANISHING COSMOLOGICAL CONSTANT

Putting $\sigma_q=0$ in Eq. (19) gives

$$ds^2=(bG^{-1/2}+cG)dt^2-G(dx^2+dy^2) -(G'^2/4\sigma^2)(bG^{1/2}+cG^2)^{-1}dz^2 \tag{45}$$

with $b=1+\lambda/3\sigma^2$.

In this case Eq. (22) gives for the invariant distance to the space-time singularity at z_+,

$$\hat{z}_+=\begin{cases} 2(3\lambda)^{-1/2}\arcsin(1+3\sigma^2/\lambda)^{-1/2}, \ \lambda>0 \ , & (46a)\\ 2(-3\lambda)^{-1/2}\operatorname{arcsinh}(-1-3\sigma^2/\lambda)^{-1/2}, \ -3\sigma^2<\lambda<0 \ , & (46b)\\ \infty, \ \lambda=-3\sigma^2 \ , & (46c)\end{cases}$$

which for $\lambda \ll \sigma^2$ gives

$$\hat{z}_+\cong(3g)^{-1}(1-\lambda/6\sigma^2) \quad . \tag{47}$$

This equation shows that, interpreted as a gravitational source, the λ term gives a positive contribution to the gravitational mass for $\lambda>0$, and a negative one for $\lambda<0$.

Particular cases of the solution (45) have been considered by Horský and Novotný.[10–12] The metric (45) with $EF=1$, giving $G=(1-\sigma z)^2$ has been identified as a generalized "Taub solution." Permitting $\lambda<0$, the special physical case $\lambda=-3\sigma^2$ gives $b=0, c=1$, so that in coordinates with $EF=1$,

$$ds^2=(1-\sigma z)^2(dt^2-dx^2-dy^2) -(1-\sigma z)^{-2}dz^2 \ . \tag{48}$$

Horský and Novotný[10,11] have interpreted this metric, expressed in Rindler-type coordinates, with $G=\lambda z^2/3$, as describing a plane-symmetric de Sitter universe. With this interpretation the integration constant introduced in Eq. (16) cannot be interpreted any longer as the mass density of a massive plane, since there is no massive plane in a de Sitter universe. Thus the massive plane is "interpreted away."

If the massive plane is still considered as being present, the line element (48) describes an infinitely extended singularity-free universe in front of a massive plane, with an isotropic cosmological repulsion.

The spherically symmetric line element corresponding to (45) is the generalized Schwarzschild solution with $g_{00}=1-2m/r-\lambda r^{2/3}$, where m is the mass of the particle. One obtains the static form of de Sitter's solution by putting $m=0$, not $m=m(\lambda)$.

Correspondingly we would like to find a plane-symmetric solution of Einstein's equations with $\lambda\neq 0, \sigma=0$, and interpret this as the plane-symmetric analog of the de Sitter universe. Such a solution is not included as a special case of our general metric equation (19).

In order to describe a plane-symmetric de Sitter universe, we introduce a cosmological repulsion ($\lambda>0$) in the z direction only. The field equations for this case are

$$R_{\mu\nu}=\lambda(g_{00}\delta^0{}_\mu\delta^0{}_\nu+g_{33}\delta^3{}_\mu\delta^3{}_\nu) \ . \tag{49}$$

Equations (9) and (11) are still valid, while Eq. (10) is changed to

$$(EG'^2/F)'=0 \ , \tag{50}$$

giving

$$G'=k(F/E)^{1/2} \ , \tag{51}$$

where k is a constant. One may easily verify that the field equations permit no solution with $k\neq 0$. Thus $G=1$. Equation (13) with $\sigma_q=0$ now gives

$$F=E'^2/2E(k_I-2\lambda E) \ , \tag{52}$$

where k_I is an integration constant. The geodesic equation (27) then gives, for a particle instantaneously at rest at the origin $E'(z=0)=-2g$. The constant k_I is now determined by the normalization $E(z=0)=1$, giving $k_I=2(\lambda+g^2)$, and the metric becomes

$$ds^2=E\,dt^2-dx^2-dy^2 -[E'^2/4E(g^2+\lambda-\lambda E)]dz^2 \ . \tag{53}$$

Equation (53) describes a plane symmetric de Sitter universe.

In coordinates with $EF=1$ the metric becomes

$$ds^2=(1+2gz-\lambda z^2)dt^2-dx^2-dy^2 -(1+2gz-\lambda z^2)^{-1}dz^2 \ , \tag{54}$$

which reduces to the Kottler-Whittaker line element[73] describing Minkowski space-time from a uniformly accelerated reference frame, for $\lambda=0$. Equations (53) and (54) describe the plane-symmetric de Sitter universe from an accelerated reference frame.

This space-time has no singularities. The ones appearing in the metric are coordinate singularities.

VI. SPACE-TIME WITH VANISHING COSMOLOGICAL CONSTANT OUTSIDE A MASSIVE CHARGED PLANE

In this case the general solution (19) takes the form

$$ds^2=(aG^{-1}+bG^{-1/2})dt^2-G(dx^2+dy^2) -(G'^2/4\sigma^2)(a+bG^{1/2})dz^2 \ , \tag{55}$$

with $b=1-\sigma_q{}^2/4\sigma^2$.

Equation (22) for the invariant distant to the space-time singularity at z_+ now gives

$$\hat{z}_+=(3g)^{-1}(1+\sigma_q/\sigma)(1+\sigma_q/2\sigma)^{-2} \ . \tag{56}$$

Thus the presence of the electrostatic field makes the singularity approach the plane.

Two special forms of Eq. (55) were found by Kar[54] The first form has $EF=1$, which here gives

$$ds^2=\left[\frac{b}{1-\sigma z}+\frac{a}{(1-\sigma z)^2}\right]dt^2 -(1-\sigma z)^2(dx^2+dy^2) -\left[\frac{b}{1-\sigma z}+\frac{a}{(1-\sigma z)^2}\right]^{-1}dz^2 \ . \tag{57}$$

Kar did not interpret his integration constant corresponding to our b. His metric was written as

$$ds^2 = \left[\frac{m}{z} + \frac{e^2}{z^2}\right] dt^2 - z^2(dx^2 + dt^2) - \left[\frac{m}{z} + \frac{e^2}{z^2}\right]^{-1} dz^2 \qquad (58)$$

in a recent survey of exact solutions of Einstein's field equations,[85] indicating that m is the mass of the plane. However we do not consider the charge and the mass of a plane of infinite extension as well-defined quantities.

The second form of this solution found by Kar has $EF = G^2$ giving

$$ds^2 = (1+\sigma z)(1+a\sigma z)dt^2 - (1-\sigma z)^{-2}(dx^2+dy^2) - (1+\sigma z)^{-5}(1+a\sigma z)^{-1}dz^2 \ . \qquad (59)$$

In these coordinates the electrical field outside the plane is uniform.

In coordinates with $EF = G$ the line element (55) takes the form

$$ds^2 = (ae^{2\sigma z} + be^{\sigma z})dt^2 - e^{-2\sigma z}(dx^2+dy^2) - e^{-4\sigma z}(a + be^{-\sigma z})^{-1}dz^2 \ . \qquad (60)$$

The line element (55) permits a solution with $F = G$, showing that the spatial geometry outside the charged plane is conformally flat. In this case $G = [1 - \sigma z + (1/4)b\sigma^2 z^2]^2$.

In the special case that $b = 0$, which means that $\sigma_q = 2\sigma$, or in SI units $\sigma_q = 2\times 10^{-10}(C/kg)\sigma$, this line element reduces to

$$ds^2 = e^{\sigma_q z}dt^2 - e^{-\sigma_q z}(dx^2+dy^2) - e^{-2\sigma_q z}dz^2 \ , \qquad (61)$$

which was found by McVittie.[55] The general form of the line element for McVittie's special case is

$$ds^2 = G^{-1}dt^2 - G(dx^2+dy^2) - (G'^2/\sigma_q{}^2)dz^2 . \qquad (62)$$

Patnaik[60] attacked the present problem, using Taub-type coordinates, with $E = F$. Unfortunately the field equations cannot be integrated explicitly in these coordinates, except for McVittie's special case. From Eq. (62) with $E = F$ one immediately finds[60,63]

$$ds^2 = (1-3\sigma z)^{-2/3}(dt^2 - dz^2) - (1-3\sigma z)^{2/3}(dx^2+dy^2) \ . \qquad (63)$$

VII. CONCLUSION

We have found two new solutions of Einstein's field equations and studied their physical significance. The first one describes space-time with a nonvanishing cosmological constant outside an infinitely large, massive, electrically charged plane. The second solution describes space-time in a plane-symmetric empty universe with a nonvanishing cosmological repulsion in the direction normal to the symmetry plane.

The solutions are given in general forms permitting analytic integration in terms of elementary functions by suitable choices of coordinates. Previously known solutions emerge as a special case, and by means of our general solution we have identified their physical meaning.

The space-time of the plane-symmetric de Sitter universe is singularity free, while the space-time of the general solution contains a singularity at a finite invariant distance from the massive plane. This singularity acts as a horizon both for massive particles and for photons. The presence of an electrical field and of a cosmological repulsion both reduce the distance to this horizon.

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