Static plane-symmetric scalar fields with a traceless energy-momentum tensor in general relativity

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The problem of zero-mass scalar fields coupled to the gravitational field in the static plane-symmetric case is completely solved for a traceless energy-momentum tensor.

I. INTRODUCTION

Scalar fields are the simplest classical fields, and there exists an extensive literature containing numerous solutions of the Einstein equations where the scalar field is minimally coupled to the gravitational field and its Lagrangian has the form

$$L=+\tfrac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\,. \tag{1.1}$$

In a recent paper Frøyland[1] studied the problem of a conformally invariant scalar field with its energy-momentum tensor trace-free in the static spherically symmetric case. Space-times with spherical symmetry admit three-parameter groups of transformations with minimum varieties having two-dimensional surfaces of constant positive curvature. Plane-symmetric space-times also admit three-parameter groups with minimum varieties having two-dimensional surfaces of zero curvature. It is therefore interesting to consider the conformally invariant scalar field with a trace-free energy-momentum tensor in the static plane-symmetric case. The planar symmetry has the additional advantage of constructing homogeneous anisotropic cosmological solutions by suitable complex transformations of the static solution.

In this paper we discuss the static plane-symmetric solutions of the Einstein equations corresponding to a conformally invariant scalar field with its energy-momentum tensor trace-free.

The action integral for the system of a scalar field coupled to gravitation is taken to be[2]

$$I=\int d^4x\,(-g)^{1/2}\left[\frac{1}{2\kappa}\left[1-\frac{\kappa\phi^2}{6}\right]R+\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi\right], \tag{1.2}$$

where R is the curvature scalar and ϕ is the scalar field.

The variation of this action gives the gravitational field equations

$$(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)f(\phi)=+\kappa(-\phi_{,\mu}\phi_{,\nu}+\tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha})+g_{\mu\nu}f^{;\alpha}{}_{;\alpha}-f_{;\mu\nu}\,, \tag{1.3}$$

where a comma denotes partial derivative and a semicolon denotes covariant derivative and

$$f(\phi)=1-\frac{\kappa\phi^2}{6}\,. \tag{1.4}$$

The matter field equations are

$$\phi^{;\mu}{}_{;\mu}+\frac{R\phi}{6}=0\,. \tag{1.5}$$

Taking the trace of Eq. (1.3) and using Eq. (1.5) we get

$$R=0\,. \tag{1.6}$$

Defining

$$u=\sqrt{\kappa/6}\phi \tag{1.7}$$

we rewrite the field equations in the form

$$R_{\mu\nu}f(u)=g_{\mu\nu}u_{,\alpha}u^{,\alpha}-4u_{,\mu}u_{,\nu}+2uu_{;\mu\nu}\,, \tag{1.8}$$

$$u^{;\mu}{}_{;\mu}=0\,. \tag{1.9}$$

II. FIELD EQUATIONS

In the static, plane-symmetric case, the line element is given by[3]

$$ds^2=A\,(dt^2-dx^2)-C\,(dy^2+dz^2)\,, \tag{2.1}$$

where A and C are functions of x only.

From (1.9) one gets ($u_{,1}=u_1$)

$$u_1 = -\frac{K_1}{C} . \tag{2.2}$$

The field equations (1.8) for the metric (2.1) can be written explicitly as

$$-fR^0{}_0 \equiv \frac{f}{2AC}\left[\frac{A_1 C}{A}\right]_1 = \frac{u_1{}^2}{A} + \frac{A_1}{A^2} u u_1 , \tag{2.3}$$

$$-fR^1{}_1 = -\frac{f}{2AC}\left[\frac{A_1}{A} + \frac{C_1}{2C}\right] = -\frac{u_1{}^2}{A} + \frac{A_1}{A^2} u u_1 , \tag{2.4}$$

$$-fR^2{}_2 \equiv -fR^3{}_3 \equiv f\frac{C_{11}}{2AC} = \frac{u_1{}^2}{A^2} + \frac{C_1 u u_1}{AC} . \tag{2.5}$$

The set of equations (2.3)–(2.5) are not independent due to the relation $R=0$.

We shall consider Eqs. (2.3), (2.5), and (2.2) to determine the three unknown functions A, C, and u uniquely.

Equation (2.5) can be integrated, using (2.2), to give

$$C = \frac{K_3}{f}\left[\frac{1-u}{1+u}\right]^{\nu} , \tag{2.6}$$

where K_3 and ν are integration constants.

From (2.2) and (2.3) one obtains

$$\left[\frac{A_1 C}{A}\right]_1 f = -2K_1 u_1 - f_1\left[\frac{A_1 C}{A}\right] . \tag{2.7}$$

Equation (2.7), on integration, yields

$$A = \frac{K_2}{f}\left[\frac{1-u}{1+u}\right]^{\mu} , \tag{2.8}$$

where K_2 and μ are constants of integration. However, the constants μ and ν are not quite arbitrary due to the relation $R=0$. From $R=0$ one obtains

$$\left[\frac{A_1}{A} + \frac{2C_1}{C}\right]_1 + \frac{3}{2}\frac{C_1{}^2}{C^2} = 0 . \tag{2.9}$$

Substituting (2.6) and (2.8) in (2.9), after some simple manipulation, one finds

$$\nu(\nu+2\mu) = 3 . \tag{2.10}$$

With (2.6) and (2.2) one obtains

$$u_1 = -\frac{K_1}{K_3}(1+u)^{1+\nu}(1-u)^{1-\nu} . \tag{2.11}$$

Equation (2.11) can be integrated to give

$$\left[\frac{1-u}{1+u}\right]^{\nu} = 2\nu\left[\frac{K_1}{K_3}x + K_4\right] , \tag{2.12}$$

where K_4 is another integration constant.

Unlike the static spherically symmetric case, u can be expressed explicitly in terms of a simple function of x. From (2.12) one can write

$$u = \frac{1-\beta(\lambda x + K_4)^{1/\nu}}{1+\beta(\lambda x + K_4)^{1/\nu}} . \tag{2.13}$$

From (2.13) it is evident that as $\nu \to \infty$, $\beta \to 1$ and $u \to 0$, and as $\nu \to 0$, $\beta \to 0$ and $u \to 1$. Thus one finds that the constant of integration ν determines the strength of the scalar field.

From (2.6) and (2.8), using (2.13), one can reexpress the metric elements as

$$A = \frac{K_2 \beta^{3/2\nu^2 - 1/2}}{1-u^2}(\lambda x + K_4)^{3/2\nu^2 - 1/2} , \tag{2.14}$$

and

$$C = \frac{K_3 \beta^{\nu}}{1-u^2}(\lambda x + K_4) . \tag{2.15}$$

As $u \to 0$, we recover the static plane-symmetric solutions given by Taub[2]:

$$A \to (\lambda x + K_4)^{-1/2} , \qquad C \to (\lambda x + K_4) . \tag{2.16}$$

The case $\nu < \sqrt{3}$ is quite interesting. The disklike singularity of Taub's solutions disappears. The metric elements are monotonically increasing functions of x in all directions.

III. PLANE-SYMMETRIC COSMOLOGICAL SOLUTION

From the static solutions (2.13)–(2.15), one can construct the homogeneous anisotropic cosmological solutions by the following complex transformations:

$$t \to ix , \quad x \to it , \text{ and } \lambda \to -i\lambda . \tag{3.1}$$

The line element (2.1) then takes the form

$$ds^2 = \frac{K_2 \beta^{3/2\nu^2 - 1/2}}{1-u^2}(\lambda t + K_4)^{3/2\nu^2 - 1/2} dt^2 - dx^2 - \frac{K_3 \beta^{\nu}}{1-u^2}(\lambda t + K_4)(dy^2 + dz^2) \tag{3.2}$$

with

$$u = \frac{1-\beta(\lambda t + K_4)^{1/\nu}}{1+\beta(\lambda t + K_4)^{1/\nu}} . \tag{3.3}$$

In the absence of the scalar field, the line element (3.2) reduces to the plane-symmetric Kasner universe

$$ds^2=K_2(\lambda t+K_4)^{-1/2}(dt^2-dx^2)-K_3(\lambda t+K_4)(dy^2+dz^2)\ . \tag{3.4}$$

It is evident from (3.2) that one has two types of universe depending on the parameter which determines the strength of the scalar field. If $\nu>\sqrt{3}$, the universe is a Kasner type with contraction along the x axis and expansion along the y and z axes. In the case of $\nu>\sqrt{3}$ there is either collapse or explosion in all three spatial directions.

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[1]J. Frøyland, Phys. Rev. D 25, 1470 (1982).
[2]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).
[3]A. H. Taub, Ann. Math. 53, 472 (1951).