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Optimal inequalities for the subtraction functions of proton-Compton-scattering dispersion theory

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Upper and lower bounds upon the subtraction functions required in the dispersion theory of the proton Compton process are derived in a framework which optimally exploits the gauge invariance, the fixed-t analyticity, and the s-u crossing properties of the scattering amplitudes, together with the consequences of the s- and u-channel unitarity. The bounds, which are expressed only in terms of measurable s- and u-channel physical quantities, without any reference to model-dependent annihilation-channel contributions, appear to be quite restrictive for some values of the momentum transfer t. The results are significant for removing the sign ambiguity of the pion decay constant F_π and for the estimation of the electromagnetic polarizabilities of the proton.

I. INTRODUCTION

It is known that the scattering amplitudes describing the proton Compton process cannot be fully determined in terms of the photoproduction matrix elements, according to the s- and u-channel unitarity condition, since some of the invariant amplitudes require subtractions in the fixed-momentum-transfer dispersion relations.[1,2] The subtraction terms, which are functions of t (s,t,u are the Mandelstam variables), are usually evaluated by exploiting the unitarity relation for the annihilation channel $N\bar{N}\rightarrow\gamma\gamma$. However, the poor knowledge of the reaction $\gamma\gamma\rightarrow\pi\pi$, which is required for the evaluation of the t-channel absorbtive parts in the two-particle approximation of the unitarity sum, seriously affects such calculations, making the determination of the subtraction functions and, consequently, the entire theoretical description of the low-energy Compton scattering strongly model-dependent.[1–3]

Since a clarification of the situation along these lines has to wait until more reliable information about the reaction $\gamma\gamma\rightarrow\pi\pi$ will become available, a different approach to the dispersion theory of hadron Compton scattering, proposed recently,[4–8] developing some ideas previously formulated in Ref. 9–11, seems to be of much interest. Instead of pursuing the delicate line of building models for the annihilation-channel contributions, the efforts have been now concentrated on deriving rigorous restrictions upon the magnitude of this contribution, by exploiting the available s-channel physical quantities, in a framework which optimally incorporates the gauge invariance, the fixed-t analyticity, the s-u crossing properties of the amplitudes, and the s-channel unitarity condition.

Actually, up to now this general program was accomplished only partially. The starting point of this approach was represented by the construction[4] of a new set of six amplitudes, having the same analyticity properties in the ν^2 variable [$\nu=(s-u)/4$] at fixed t as the crossing-even invariant amplitudes, and being connected to the helicity amplitudes by a matrix unitary on the s- and u-channel cuts. The unpolarized differential cross section (UDCS) of the elastic γ-nucleon scattering is written therefore as a sum of moduli squared of such objects and this allows one to resort to the powerful mathematical tools of the interpolation theory for vector-valued analytic functions[5,12] in order to constrain the values of the amplitudes (or their derivatives) at low energies below the pion photoproduction threshold, in terms of the UDCS of the γ-nucleon scattering above this threshold. Particularly, in Ref. 7, upper and lower bounds on the subtraction functions of the fixed-t dispersion relations were derived and computed numerically in terms of this physical input. However, in spite of their optimality, these bounds turned out to be disappointingly weak, and therefore not of much interest from the physical point of view. A possible way to improve these bounds, indicated in Ref. 7, is to take into account the main dynamical information about the proton Compton scattering not included in the above formalism, consisting of the s-channel unitarity condition. A first attempt to use this information in a model-independent way, i.e., with no need to resort to t-channel contributions, was undertaken in Ref. 8. In this work, the absorptive parts of the scattering amplitudes, expressed by the unitarity condition in terms of photoproduction matrix elements in the low- and intermediate-energy region, supplemented by the knowledge of the UDCS of the γ-nucleon scattering along the remaining part of the unitarity cut, were optimally exploited by means of appropriate mathematical techniques, providing model-independent restrictions upon the values of the amplitudes at low and intermediate energies, below and above the pion photoproduction threshold. This approach proved to be very useful for the analysis of the proton Compton scattering above the first inelastic threshold, and especially in the first resonance region, evidencing some inconsistencies between the pion photoproduction multipoles and the data on the UDCS of γ-nucleon scattering. However, the upper and lower bounds on the subtraction functions of the fixed-t dispersion relations computed in this approach remained still unsatisfactory.[13] Actually, neither the method of Refs. 4–7 nor that of Ref. 8 exploited entirely the physical information available in the s channel of proton-Compton scattering, succeeding to take into account, in the low-energy part of the unitarity cut, either the UDCS or the absorptive parts of the amplitudes separately. The problem of exploiting simultaneously both these quantities, with the aim of improving the bounds on the scattering amplitudes up to a convenient level, remained therefore open. In the present paper we consider this problem and solve it completely. By resorting to more powerful mathematical techniques than in the previous works,[4–8] we were able to derive optimal bounds for the scattering amplitudes and in particular for the subtraction functions of the fixed-t dispersion relations, by taking fully into account the UDCS along the whole unitarity out and the s-channel unitarity condition at low and intermediate energies, in a frame which fully exploits the gauge invariance, the s-u crossing symmetry, and the fixed-t analyticity properties of the scattering amplitudes. The general program, formulated in Refs. 4–8, of finding optimal restrictions upon the magnitude of the unknown t-channel contributions, in terms of the known s-channel contributions, is now completely accomplished. Particularly, the numerical upper and lower bounds on the subtraction functions proved to be quite strong for some values of t, showing the great constraining power of the physical information used as input and leading to interesting conclusions concerning the sign of the pion constant F_π and the nucleon electromagnetic polarizabilities.

The paper is organized as follows. In the next section we formulate the problem and derive explicit optimal bounds on the Compton-scattering amplitudes and particularly upon the subtraction functions. The numerical applications of the method are presented and discussed in Sec. III. Section IV contains some final comments.

II. OPTIMAL BOUNDS FOR THE PROTON-COMPTON-SCATTERING AMPLITUDES

We shall work as in Refs. 1 and 4–8 with the Bardeen-Tung invariant proton-Compton-scattering amplitudes $A_i(\nu,t)$, $i=1,\ldots,6$ and with the dimensionless, crossing-symmetric amplitudes $\bar{A}_i(\nu^2,t)$ obtained by multiplying $A_i(\nu,t)$ with suitable factors containing the nucleon poles.[4] In the lowest order in electromagnetism, at fixed t, the amplitudes $\bar{A}_i(\nu^2,t)$ are real analytic functions in the ν^2 complex plane with the s- and u-channel unitarity cut along the real axis, from the pion photoproduction threshold ${\nu_0}^2$ to ∞ [$\nu_0=\mu(\mu+2m)/2+t/4$, $\mu=$pion mass, $m=$proton mass]. In the point $\nu^2={\nu_B}^2=t^2/16$, the amplitudes $\bar{A}_i$ have values $\bar{A}_i({\nu_B}^2,t)$, completely specified in terms of the charge e, and the anomalous magnetic moment κ of the proton.[4].

We shall consider particularly the subtraction functions defined as[1,7]

$$F_1(t)=(A_1^c-A_3^c)(u=m^2,t)\ ,$$
$$F_2(t)=\left[A_2^c+\frac{1}{m}A_3^c\right](u=m^2,t)\ , \tag{2.1}$$

appearing in the fixed-t dispersion relations for the amplitudes A_i, subtracted at the point $u=m^2$ (A_i^c denotes the "continuum part" of A_i, obtained from it by subtracting the nucleon poles). The model calculation of these functions relies usually on dispersion relations in the t variable.[1,2,7] We recall here that[1,7]

$$F_1(0)=4\pi(\alpha-\beta)\ , \tag{2.2}$$

where α and β are, respectively, the generalized electric and magnetic polarizabilities of the proton, and that in a dispersion representation $F_2(t)$ can be written as[1,7]

$$F_2(t)=F_2^{\pi^0\,\text{pole}}(t)+F_2^{\text{HS}}(t)$$
$$=\frac{2F_\pi g_{\pi NN}}{m(\mu^2-t)}+F_2^{\text{HS}}(t)\ , \tag{2.3}$$

where the first term is the π^0 pole and the second contains the higher states contributions. The evaluation of this part is at present model-dependent.[1,7] In Eq. (2.3) $g_{\pi NN}$ denotes the strong πNN coupling constant, while F_π is the $\pi^0 \to 2\gamma$ decay constant, which is experimentally known up to the sign.[1,7] One of the purposes of the present analysis is an attempt to remove the sign ambiguity of F_π in the context of proton-Compton-dispersion theory. Indeed, as was shown in Ref. 7, at low values of $|t|$ it is the pole term which dominates $F_2(t)$. A model-independent estimate of the subtraction function $F_2(t)$ in this range of t would be therefore of much interest for establishing the sign of F_π. In the case of $F_1(t)$, such estimates would give indications about the proton electromagnetic polarizabilities α and β appearing in (2.2). In what follows we treat this problem in its most general formulation and derive model-independent upper and lower bounds on the subtraction functions $F_i(t)$, at fixed $t<0$, which include completely the s-channel physical information.

We consider first the UDCS of the elastic γ-nucleon scattering in the center-of-mass system $(d\sigma/d\Omega)_{\text{c.m.}}$, which is expressed in terms of the amplitudes $\bar{A}_i$ above the pion photoproduction threshold $\nu_0{}^2$ as[4]

$$\sum_{i,j=1}^{6} M_{ij}(\nu^2,t)\bar{A}_i^*(\nu^2,t)\bar{A}_j(\nu^2,t) = \sigma(\nu^2,t), \quad \nu^2 \geq \nu_0{}^2 , \tag{2.4}$$

where the matrix M is Hermitian and positive definite and[4]

$$\sigma(\nu^2,t) = 128\pi^2 s \left[\frac{d\sigma}{d\Omega}\right]_{\text{c.m.}} . \tag{2.5}$$

The s-channel unitarity yields in addition the dynamical conditions[1,8]

$$\operatorname{Im}\bar{A}_i(\nu^2,t) = \rho_i(\nu^2,t), \quad i=1,\ldots,6 , \quad \nu_0{}^2 \leq \nu^2 \leq \nu_{\text{in}}{}^2 , \tag{2.6}$$

where the functions $\rho_i(\nu^2,t)$ can be computed explicitly in terms of the photoproduction matrix elements by using the kinematical relations given in Refs. 1 and 4. The upper limit $\nu_{\text{in}}{}^2$ depends upon the intermediate states taken into account in the unitarity sum. By inserting into this sum only the single-pion photoproduction multipoles available from Refs. 14 and 15, we have to take $\nu_{\text{in}} = m\omega_{\text{in}} + t/4$, where $\omega_{\text{in}} = 2\mu + 2\mu^2/m$ is the photon laboratory energy corresponding to the double photoproduction threshold $s_{\text{in}} = (m+2\mu)^2$.

As was shown in Refs. 4 and 5, in order to exploit in an optimal way the condition (2.4), one must introduce a new set of six analytic amplitudes which "diagonalize" the bilinear form expressing σ. For our purposes it is useful to recall here that this procedure was realized practically in two steps.[4,6] First, let us define the amplitudes[6]

$$\tilde{\varphi}_i(\nu^2,t) = \sum_{j=1}^{6} \tilde{N}_{ij}(\nu^2,t)\bar{A}_j(\nu^2,t) \quad i=1,\ldots,6 , \tag{2.7}$$

where the 6×6 matrix $\tilde{N}$ has the following decomposition:

$$\tilde{N} = \begin{bmatrix} \tilde{N}_{\text{I}} & 0 \\ 0 & \tilde{N}_{\text{II}} \end{bmatrix}$$

in terms of the 3×3 matrices

$$\tilde{N}_{\text{I}} = \frac{m(-t)^{1/2}}{2\sqrt{2}L_1{}^2} \begin{bmatrix} 0 & 0 & -\dfrac{4L_2}{m}\dfrac{[\nu_0+(\nu_0{}^2-\nu^2)^{1/2}]}{(4m^2-t)^{1/2}} \\ (-t)^{1/2}(4m^2-t)^{1/2} & 0 & -\dfrac{4\nu^2}{m^2}\dfrac{(-t)^{1/2}}{(4m^2-t)^{1/2}} \\ 0 & t & -\dfrac{4\nu^2}{m^2} \end{bmatrix} ,$$
$$\tilde{N}_{\text{II}} = \frac{L_2}{4\sqrt{2}L_1{}^2} \begin{bmatrix} \dfrac{8(-t)^{1/2}}{(4m^2-t)^{1/2}}[\nu_0+(\nu_0{}^2-\nu^2)^{1/2}] & 0 & \dfrac{(-t)^{1/2}}{m}(4m^2-t)^{1/2}[\nu_0+(\nu_0{}^2-\nu^2)^{1/2}] \\ -\dfrac{8mL_2}{(4m^2-t)^{1/2}} & -\dfrac{L_2}{m}(4m^2-t)^{1/2} & 0 \\ -2t & 0 & -\dfrac{4\nu^2}{m^2} \end{bmatrix} , \tag{2.8}$$

and

$$L_1=2[(\nu_0{}^2-\nu_B{}^2)^{1/2}+(\nu_0{}^2-\nu^2)^{1/2}]\ ,\quad L_2=2[(\nu_0{}^2-\nu_{\min}{}^2)^{1/2}+(\nu_0{}^2-\nu^2)^{1/2}]\ ,$$

$$\nu_{\min}=\frac{(-t)^{1/2}}{4}(4m^2-t)^{1/2}\ .$$

The amplitude $\tilde{\varphi}_i$ has the same analyticity properties in ν^2 as $\bar{A}_i$ and transform (2.2) into

$$\sum_{i=1}^{6}|\tilde{\varphi}_i(\nu^2,t)|^2=\sigma(\nu^2,t)\ ,\quad \nu^2>\nu_0{}^2\ . \tag{2.9}$$

The only disadvantage of the matrix $\tilde{N}$ displayed above is[6] that it is not invertible everywhere in the ν^2 plane [$\det\tilde{N}_{\rm II}(\nu^2,t)=0$ for $\nu^2=\nu_{\min}{}^2$] and this could in principle spoil the optimality of the bounds derived on the amplitudes using the condition (2.9). Actually, the unwanted zero of $\det\tilde{N}$ could be eliminated by using the standard Blaschke-Potapov factorization.[16]

Finally, the following amplitudes were defined[6]:

$$\varphi_i(\nu^2,t)=\sum_{j=1}^{6}N_{ij}(\nu^2,t)\bar{A}_j(\nu^2,t),\quad i=1,\ldots,6\ , \tag{2.10}$$

where

$$N=\begin{bmatrix}N_{\rm I} & 0\\ 0 & N_{\rm II}\end{bmatrix}\ ,\qquad N_{\rm I}=\tilde{N}_{\rm I}\ , \tag{2.11}$$

and

$$N_{\rm II}=\frac{L_2}{4\sqrt{2}L_1{}^2(2\nu_0)^{1/2}[\nu_0+(\nu_0{}^2-\nu_{\min}{}^2)^{1/2}]^{1/2}}\times\begin{bmatrix} tL_2 & 0 & \frac{2}{m^2}L_2[\nu_0+(\nu_0{}^2-\nu_{\min}{}^2)^{1/2}]\times[\nu_0+(\nu_0{}^2-\nu^2)^{1/2}] \\ -8mL_2(2\nu_0)^{1/2}[\nu_0+(\nu_0{}^2-\nu^2)^{1/2}]^{1/2} & -\frac{L_2}{m}(4m^2-t)^{1/2}(2\nu_0)^{1/2}\times[\nu_0+(\nu_0{}^2-\nu_{\min}{}^2)^{1/2}]^{1/2} & 0 \\ \frac{-2t}{\nu_{\min}}\{[\nu_0+(\nu_0{}^2-\nu_{\min}{}^2)^{1/2}]\times[\nu_0+(\nu_0{}^2-\nu^2)^{1/2}]+\nu_{\min}{}^2\} & 0 & -\frac{4\nu_{\min}}{m^2}\{[\nu_0+(\nu_0{}^2-\nu_{\min}{}^2)^{1/2}]\times[\nu_0+(\nu_0{}^2-\nu^2)^{1/2}]+\nu^2\} \end{bmatrix}\ . \tag{2.12}$$

For convenience, we shall work in what follows with the variable z:

$$z=\frac{(\nu_0{}^2-\nu_B{}^2)^{1/2}-(\nu_0{}^2-\nu^2)^{1/2}}{(\nu_0{}^2-\nu_B{}^2)^{1/2}+(\nu_0{}^2-\nu^2)^{1/2}}\ , \tag{2.13}$$

which maps the cut ν^2 plane into the disc $|z|<1$, such that $\nu_B{}^2$ becomes $z=0$. We note in particular that the point $\nu_{\rm in}{}^2$ becomes $z_{\rm in}=e^{i\theta_{\rm in}}$, where

$$\theta_{\rm in}=2\arctan\frac{(\nu_{\rm in}{}^2-\nu_0{}^2)^{1/2}}{(\nu_0{}^2-\nu_B{}^2)^{1/2}}\ . \tag{2.14}$$

The amplitudes $\varphi_i(\nu^2,t)$ at fixed t are real analytic functions of z $[\varphi_i(z^*)=\varphi_i^*(z)]$ and satisfy on the frontier of the unit disc ($z=e^{i\theta}$) the following conditions resulting from (2.4) and (2.6), respectively:

$$\frac{1}{|S(\theta)|^2}\sum_{i=1}^{6}|\varphi_i(\theta)|^2=1,\quad \theta\in(-\pi,\pi) \tag{2.15}$$

and

$$\mathrm{Im}\left[\sum_{j=1}^{6}N^{-1}{}_{ij}(\theta)\varphi_j(\theta)\right]=\rho_i(\theta)\,,\quad \theta\in(-\theta_{\rm in},\theta_{\rm in}), i=1,\ldots,6\,. \tag{2.16}$$

In (2.15) $S(z)$ is an outer analytic function in $|z|<1$ (Ref. 12), having on the boundary the modulus equal to $[\sigma(\theta)]^{1/2}=[\sigma(\nu^2,t)]^{1/2}$, defined by

$$S(z)=\exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln\sigma(\theta)d\theta\right]. \tag{2.17}$$

In addition to Eqs. (2.15) and (2.16) we have also the relations

$$\varphi_i(0)=\xi_i(e,\kappa,t),\quad i=1,\ldots,6\,, \tag{2.18}$$

where ξ_i are known expressions, computed by using the relations (2.10) in terms of the values $\bar{A}_i(\nu_B{}^2,t)$.

The subtraction functions $F_i(t)$ defined in (2.1) are related to the amplitudes $\varphi_i(z)$ through the relations[7]

$$F_1(t)=\frac{1}{(4m^2-t)^{1/2}}\left\{\left[\left[\frac{d\varphi_2}{dz}\right]_{z=0}-2\varphi_2(0)\right]\frac{2\sqrt{2}\mu^3}{(-t)}-\frac{2e^2(2\kappa+\kappa^2)}{m(4m^2-t)^{1/2}}+\frac{4m^2\mu t}{(4m^2-t)^{1/2}}\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{\mathrm{Im}A_3(s',t)ds'}{(s'-m^2)(s'-m^2+t)}\right\},$$
$$F_2(t)=\frac{2\sqrt{2}\mu^4}{mt(-t)^{1/2}}\left[\left[\frac{d\varphi_3}{dz}\right]_{z=0}-2\varphi_3(0)\right]+\frac{2e^2(2\kappa+\kappa^2)\mu^4}{m(-t)}\,. \tag{2.19}$$

From (2.19) it follows that in order to find restrictions upon the subtraction functions $F_i(t)$, imposed by the conditions (2.15)–(2.18), we must derive optimal bounds upon the derivatives $(d\varphi_i/dz)_{z=0}$, $i=2,3$, since all the other quantities appearing in (2.19) can be easily computed. In what follows we shall treat this problem.

Let us consider more generally n_i derivatives $\varphi_i^{(k)}(0)$, $0\le k\le n_i$ of the amplitudes φ_i (in the particular problem investigated here n_i will take the values 0 or 1) and let us denote by

$$\mathscr{D}=\{\varphi_i^{(k)}(0)\}_{i=1,\ldots,6}^{k=0,\ldots,n_i}$$

the largest domain taken by these values, consistent with the conditions (2.15) and (2.16). It can be shown, using arguments standard in the analytic interpolation theory, that the exact description of the domain $\mathscr{D}$ is related to the solution of a minimum norm problem for vector-valued analytic functions.[12,8] Let us consider the quantity

$$\mu_2(w)=\min_{\{\varphi_i\}}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{w(\theta)}{S(\theta)}\right|^2\times\sum_{i=1}^{6}|\varphi_i(\theta)|^2d\theta\right]^{1/2}, \tag{2.20}$$

where the minimization is performed upon the analytic functions $\varphi_i(z)$ which have prescribed values $\varphi_i^{(k)}(0)$, $0\le k\le n_i$ and satisfy also Eq. (2.16). In (2.20), $w(z)$ is a fixed outer-analytic function in $|z|<1$, normalized on the boundary by the condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|w(\theta)|^2d\theta\le 1\,. \tag{2.21}$$

The quantity $\mu_2(w)$ depends implicitly upon the values $\varphi_i^{(k)}(0)$, besides other known quantities entering the conditions (2.15) and (2.16). It can be shown that the inequality

$$\mu_2(w)\le 1 \tag{2.22}$$

represents, for each fixed analytic function w belonging to the class described by the property (2.21), a necessary condition that must be satisfied by the values $\varphi_i^{(k)}(0)$. By taking in (2.22), the supremum upon the admissible functions w, we obtain the stronger inequality

$$\mu_\infty = \sup_w \mu_2(w) \le 1 \; , \tag{2.23}$$

which can be shown[12,8] to represent the optimal, necessary, and sufficient restriction upon the parameters $\varphi_i^{(k)}(0)$ of interest. In what follows we shall find explicitly the quantity $\mu_2(w)$ for each fixed w, by solving completely the minimization problem (2.20), and we shall perform afterwards an approximate maximization upon w, based on a limited class of suitable functions. This procedure of approaching an optimal inequality by means of a family of nonoptimal but necessary inequalities was applied also in other problems.[8,17] We consider first the minimization problem (2.20) for a fixed w, leaving the discussion referring to the choice of w to the end of this section.

The functional minimization (2.20) with the additional constraint (2.16) upon the analytic functions $\varphi_i(z)$ is actually a standard convex optimization problem which can be treated by applying the general theory of Lagrange multipliers.[18] We first write down the Lagrange functional

$$\mathscr{L}(\varphi_i,\eta_i) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{w(\theta)}{S(\theta)}\right|^2 \sum_{i=1}^{6} |\varphi_i(\theta)|^2 d\theta - \frac{2}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\left|\frac{w(\theta)}{S(\theta)}\right|^2 \sum_{i=1}^{6}\eta_i(\theta)\left[\mathrm{Im}\sum_{j=1}^{6} N_{ij}^{-1}(\theta)\varphi_j(\theta) - \rho_i(\theta)\right] d\theta \; , \tag{2.24}$$

containing six Lagrange multipliers $\eta_i(\theta)$ related to the constraints (2.16) (the factors multiplying η_i were introduced for the simplicity of the subsequent calculations). Since we deal with real analytic functions [i.e., $\mathrm{Im}\varphi_i(-\theta) = -\mathrm{Im}\varphi_i(\theta)$], we can assume without loss of generality that $\eta_i(\theta)$ are odd functions on $(-\theta_{\rm in},\theta_{\rm in})$. Then we notice that the last term of the Lagrangian can be written equivalently as

$$-\frac{2}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\left|\frac{w(\theta)}{S(\theta)}\right|^2 \sum_{i=1}^{6}\eta_i(\theta)\left[i\sum_{j=1}^{6} {N^{-1}}_{ij}^{*}(\theta)\varphi_j^{*}(\theta) - \rho_i(\theta)\right] d\theta \; ,$$

since the real part of the complex function $\sum_{j=1}^{6} {N^{-1}}_{ij}\varphi_j$ brings no contributions due to its parity.

According to the Lagrange theory,[18] we must compute first the minimum of the Lagrangian $\mathscr{L}$ with respect to the functions $\varphi_i(z)$. These functions still satisfy the constraint that the first n_i derivatives are fixed. In order to take into account this condition with no loss of information, it is convenient to develop $\varphi_i(z)$,

$$\varphi_i(z) = \frac{S(z)}{w(z)}\left[\sum_{k=0}^{n_i}\left[\frac{w\varphi_i}{S}\right]_{z=0}^{(k)}\frac{z^k}{k!} + z^{n_i+1}\sum_{n=0}^{\infty} c_n^{(i)} z^n\right], \quad i=1,\ldots,6 \; , \tag{2.25}$$

where the real coefficients $c_n^{(i)}$ are free of constraints. By introducing (2.25) into the Lagrangian (2.24), we obtain

$$\mathscr{L} = \sum_{i=1}^{6}\sum_{k=0}^{n_i}\left[\left[\frac{w\varphi_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\right]^2 + \sum_{i=1}^{6}\sum_{n=0}^{\infty} c_n^{(i)2}$$

$$-\frac{2}{\pi}\sum_{c=1}^{6}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\frac{w(\varphi)}{S(\varphi)}\eta_i(\varphi)\left\{ i\sum_{j=1}^{6} {N^{-1}}_{ij}^{*}(\varphi)\left[\sum_{k=0}^{n_j}\left[\frac{w\varphi_j}{S}\right]_{z=0}^{(k)}\frac{e^{-ik\varphi}}{k!} + e^{i(n_j+1)\varphi}\sum_{n=0}^{\infty} c_n^{(j)} e^{-in\varphi}\right] - \rho_i(\varphi)\right\} d\varphi \; .$$

On this expression it is easy to perform the unconstrained minimization of $\mathscr{L}$ with respect to the free coefficients $c_n^{(i)}$ by setting $\partial\mathscr{L}/\partial c_n^{(i)}=0$, $n=0,\ldots,\infty$, $i=1,\ldots,6$. We obtain therefore the optimal coefficients $c_n^{(i)}$,

$$c_n^{(i)}=\frac{i}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\frac{w(\varphi)}{S(\varphi)}\frac{e^{-in\varphi}}{e^{i(n_i+1)\varphi}}\sum_{j=1}^{6}N^{-1}{}_{ji}{}^*(\varphi)\eta_j(\varphi)d\varphi,\quad n=0,\ldots,\infty,\quad i=1,\ldots,6\,. \tag{2.26}$$

By introducing these expressions in (2.25) we obtain the optimal functions $\varphi_i(z)$:

$$\varphi_i(z)=\frac{S(z)}{w(z)}\sum_{k=0}^{n_i}\left[\frac{w\varphi_i}{S}\right]^{(k)}\frac{z^k}{k!}+\frac{S(z)}{w(z)}z^{n_i+1}\frac{i}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\frac{w(\varphi)}{S(\varphi)}\frac{e^{-i(n_i+1)\varphi}}{1-ze^{-i\varphi}}\sum_{j=1}^{6}N^{-1}{}_{ji}{}^*(\varphi)\eta_j(\varphi)d\varphi\,,$$

$$i=1,\ldots,6,\quad |z|<1\,. \tag{2.27}$$

We determine now the yet unknown Lagrange multipliers $\eta_i(\theta)$ by demanding that these functions fulfill the additional constraints (2.16). Actually we have to set in (2.27) $z=re^{i\theta}$ and take the limit $r\to1$. Then we obtain from (2.16) the conditions

$$\mathrm{Im}\left[\sum_{j=1}^{6}\sum_{k=0}^{n_i}\left[\frac{w\varphi_j}{S}\right]_{z=0}^{(k)}N^{-1}{}_{ij}(\theta)\frac{S(\theta)}{w(\theta)}\frac{e^{ik\theta}}{k!}+\lim_{r\to1}\frac{i}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\frac{w(\varphi)S(\theta)}{w(\theta)S(\varphi)}\sum_{j,l=1}^{6}e^{i(n_j+1)(\theta-\varphi)}N^{-1}{}_{ij}(\theta)N^{-1}{}_{lj}{}^*(\varphi)\frac{d\varphi}{1-re^{i(\theta-\varphi)}}\right]=\rho_i(\theta)\,,$$

$$\theta\in(-\theta_{\rm in},\theta_{\rm in}),\quad i=1,\ldots,6\,,$$

which can be written in the equivalent form

$$\mathrm{Re}\left\{\lim_{r\to1}\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\frac{w(\varphi)S(\theta)}{w(\theta)S(\varphi)}\left[\sum_{j,l=1}^{6}e^{i(n_j+1)(\theta-\varphi)}N^{-1}{}_{ij}(\theta)N^{-1}{}_{lj}(\varphi)\eta_l(\varphi)\right]\frac{d\varphi}{1-re^{i(\theta-\varphi)}}\right\}$$

$$=\rho_i(\theta)-\sum_{j=1}^{6}\sum_{k=0}^{n_j}\left[\frac{w\varphi_j}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\mathrm{Im}\left[\frac{S(\theta)}{w(\theta)}N^{-1}{}_{ij}(\theta)e^{ik\theta}\right],\quad i=1,\ldots,6,\quad \theta\in(-\theta_{\rm in},\theta_{\rm in})\,. \tag{2.28}$$

Actually (2.28) represents a system of six singular integral equations for the determination of the real and odd functions $\eta_i(\theta)$ defined on $(-\theta_{\rm in},\theta_{\rm in})$. We can cast this system in a standard form, by applying the Plemelj relations[19]

$$\lim_{r\to1}\frac{1}{\pi}\int\frac{f(\varphi)d\varphi}{1-re^{i(\theta-\varphi)}}=f(\theta)+\frac{1}{\pi}\int\frac{f(\varphi)d\varphi}{1-e^{i(\theta-\varphi)}}\,,$$

where the last integral is taken as a principal part, and also by writing

$$\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\frac{f(\varphi)d\varphi}{1-e^{i(\theta-\varphi)}}=\frac{1}{\pi i}\int_{\Gamma_{\rm in}}\frac{f(\zeta)}{\zeta-z}d\zeta,\quad \zeta=e^{i\varphi},\quad z=e^{i\theta}\,.$$

Then we obtain from (2.28), after some straightforward manipulations, the Cauchy singular system written in the standard form [19]

$$\sum_{j=1}^{6}A_{ij}(\theta)\eta_j(\theta)+\sum_{j=1}^{6}B_{ij}(\theta)\frac{1}{\pi_i}\int_{\Gamma_{\rm in}}\frac{\eta_j(\zeta)}{\zeta-z}d\zeta+\sum_{j=1}^{6}\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\mathscr{K}_{ij}(\theta,\varphi)\eta_j(\varphi)d\varphi$$

$$=\rho_i(\theta)-\sum_{j=1}^{6}\sum_{k=0}^{n_j}\left[\frac{w\varphi_j}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\mathrm{Im}\left[N^{-1}{}_{ij}(\theta)\frac{S(\theta)}{w(\theta)}e^{ik\theta}\right],\quad i=1,\ldots,6,\quad \theta\in(-\theta_{\rm in},\theta_{\rm in}) \tag{2.29}$$

where the matrices A and B are

$$A=\mathrm{Re}(N^{\dagger}N)^{-1},\quad B=i\,\mathrm{Im}(N^{\dagger}N)^{-1}\,, \tag{2.30}$$

and $\mathscr{K}_{ij}(\theta,\varphi)$ is the Fredholm kernel defined as

$$\mathscr{K}_{ij}(\theta,\varphi)=\mathrm{Re}\left\{\sum_{l=1}^{6}\left[N^{-1}{}_{il}(\theta)N^{-1}{}_{jl}{}^{*}(\varphi)\frac{w(\varphi)S(\theta)}{w(\theta)S(\varphi)}-N^{-1}{}_{il}(\theta)N^{-1}{}_{jl}{}^{*}(\theta)\right]\frac{1}{1-e^{i(\theta-\varphi)}}\right\}, \tag{2.31}$$

where the functions $N^{-1}{}_{ij}(\theta)$, $S(\theta)$ as well as $w(\theta)$ (with the choice made below) are Hölder continuous.

According to the general theory,[19] the system of singular integral equations (2.29) can be regularized if

$$\det(A+B)\neq 0\,.$$

In the present case, the condition is satisfied, since

$$\det(A+B)=\det(N^{\dagger}N)^{-1} =\det N^{-1}\det(N^{\dagger})^{-1}\neq 0\,,\qquad \theta\in(-\theta_{\mathrm{in}},\theta_{\mathrm{in}})$$

as follows from the explicit expression of N given in (2.12). The system (2.29) can be effectively regularized by reducing it to a Hilbert-Riemann boundary value problem for piecewise analytic functions.[19] It is interesting to mention that, since the elements of the matrix N are rational functions of $\zeta=e^{i\theta}$, the regularization of the above system reduces to quadratures in the particular case $\theta_{\mathrm{in}}=\pi$, i.e., if we would know the absorptive parts $\mathrm{Im}\bar{A}_i$ along the whole unitarity cut. Actually, due to the particular form (2.11) of the matrix N, the system (2.29) splits into two separate subsystems, each of them containing three equations. We notice moreover that the first submatrix N_{I} has the property that, except for a global complex factor in front of each line, its elements are functions real on the unitarity cut, i.e., we can write

$$(N_{\mathrm{I}})_{ij}=\frac{1}{G_i(z)}n_{ij}(z),\quad i,j=1,\ldots,3\,, \tag{2.32}$$

where the explicit form of the complex nonzero functions $G_i(z)$ and of the real functions $n_{ij}(z)$ can be easily extracted from (2.8)–(2.11). As we shall prove below, due to the property (2.32), the first system of Eqs. (2.29) can be reduced to three independent Fredholm equations. This property is unfortunately not valid for the submatrix N_{II} given in (2.12). However, by looking at the expression (2.8) of the matrix $\tilde{N}$, from which N was obtained by means of the Blaschke-Potapov factorization of the zero of $\det\tilde{N}_{\mathrm{II}}$, we notice that a property similar to (2.32) holds for the whole $\tilde{N}$, namely

$$\tilde{N}_{ij}(z)=\frac{1}{G_i(z)}\tilde{n}_{ij}(z),\quad i,j=1,\ldots,6\,. \tag{2.33}$$

We show now that the factorization (2.33) leads to a considerable simplification of the system of equations (2.29), in which we replace the matrix N with the matrix $\tilde{N}$. We start from the relations (2.29) and introduce there the expressions

$$\tilde{N}^{-1}{}_{ij}(\theta)=G_j(\theta)\tilde{n}^{-1}{}_{ij}(\theta)\,,\quad i,j=1,\ldots,6\,,$$

which follows from (2.33). It is convenient to define a new set of six Lagrange multipliers δ_i, related to the previous ones $\tilde{\eta}_i$ (corresponding to $\tilde{N}$) by the relations

$$\tilde{\eta}_i(\theta)=\sum_{j=1}^{6}\frac{1}{|G_j(\theta)|^2}\tilde{n}_{ji}(\theta)\delta_j(\theta)\,,\qquad i=1,\ldots,6\,. \tag{2.34}$$

Then we obtain from (2.28)

$$\sum_{j=1}^{6}\tilde{n}^{-1}{}_{lj}(\theta)\delta_j(\theta)+\mathrm{Re}\frac{1}{\pi}\int_{-\theta_{\mathrm{in}}}^{\theta_{\mathrm{in}}}\frac{w(\varphi)S(\theta)}{w(\theta)S(\varphi)}\sum_{j=1}^{6}\frac{G_j(\theta)}{G_j(\varphi)}\frac{e^{i(n_j+1)(\theta-\varphi)}}{1-e^{i(\theta-\varphi)}}\tilde{n}^{-1}{}_{lj}(\theta)\delta_j(\varphi)d\varphi$$
$$=\rho_l(\theta)-\sum_{j=1}^{6}\sum_{k=0}^{n_j}\left[\frac{w\varphi_j}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\tilde{n}^{-1}{}_{lj}(\theta)\mathrm{Im}\left[\frac{G_j(\theta)S(\theta)}{w(\theta)}e^{ik\theta}\right],\quad l=1,\ldots,6\,.$$

We multiply each of the above equalities by $\tilde{n}_{il}(\theta)$ and sum upon l. Then the system splits in six independent equations, one for each unknown function $\delta_i(\theta)$. By introducing the notations

$$g_i(\theta)\equiv |g_i(\theta)|e^{i\phi_i(\theta)}=\frac{G_i(\theta)S(\theta)}{w(\theta)},\quad i=1,\ldots,6 \tag{2.35}$$

and

$$\tilde{\rho}_i(\theta)=\sum_{j=1}^{6}\tilde{n}_{ij}(\theta)\rho_j(\theta),\quad i=1,\ldots,6\,, \tag{2.36}$$

we obtain the equations

$$\begin{aligned}K_i\delta_i\equiv\delta_i(\theta)-\frac{1}{2\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\left|\frac{g_i(\theta)}{g_i(\varphi)}\right|\frac{\sin[(\theta-\varphi)(n_i+\frac{1}{2})+\phi_i(\theta)-\phi_i(\varphi)]}{\sin[(\theta-\varphi)/2]}\delta_i(\varphi)d\varphi\\ =\tilde{\rho}_i-\sum_{k=0}^{n_i}\left[\frac{w\varphi_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}|g_i(\theta)|\sin[\phi_i(\theta)+k\theta],\quad i=1,\ldots,6;\ \theta\in(-\theta_{\rm in},\theta_{\rm in})\,.\end{aligned} \tag{2.37}$$

From the definition (2.35) it follows that, with a suitable form of the function w, the phases $\phi_i(\theta)$ are Hölder-continuous functions. Hence the integral equations (2.37) are of Fredholm type, and their solutions $\delta_i(\theta)$ can be found easily by applying standard numerical methods. It is convenient to take advantage of this situation, and therefore, in practice, we worked with the amplitudes $\tilde{\varphi}_i$ defined in (2.7) instead of the optimal amplitudes φ_i. As we explained above, this is equivalent to a certain loss of information (actually affecting only the amplitudes φ_4, φ_5, and φ_6 and not the amplitudes φ_2 and φ_3 which are related to the subtraction functions $F_i(t)$, but this leads on the other hand to a considerable simplification of the numerical work. We mention that this simple solution was adopted only for computation reasons, since actually the treatment of the optimal system (2.29) raises no fundamental difficulties. Its solution can be found completely using appropriate techniques, and will in principle improve the results obtained in the simple version adopted here.

Having determined the Lagrange multipliers η_i or δ_i, we return now to the explicit evaluation of the minimum expression $\mu_2(w)$ of interest. By introducing the development (2.25) in the definition (2.20) we can write

$$\mu_2(w)=\left\{\sum_{i=1}^{6}\sum_{k=0}^{n_i}\left[\left[\frac{w\varphi_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\right]^2+\lim_{r\to 1}\sum_{i=1}^{6}\sum_{n=0}^{\infty}r^{2n}c_n^{(i)}\right\}^{1/2}, \tag{2.38}$$

where the real coefficient $c_n^{(i)}$ have the optimal expressions (2.26). By using these expressions in (2.38) we obtain

$$\begin{aligned}\mu_2(w)=\Bigg\{\sum_{i=1}^{6}\sum_{k=0}^{n_i}\left[\left[\frac{w\varphi_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\right]^2\\ +\lim_{r\to 1}\frac{1}{\pi^2}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}d\theta\left|\frac{w(\theta)}{S(\theta)}\right|^2\\ \times\int_{-\theta_{\rm in}}^{\theta_{\rm in}}d\varphi\frac{w(\varphi)S(\theta)}{w(\theta)S(\varphi)}\sum_{i,j,l=1}^{6}\frac{e^{i(n_j+1)(\theta-\varphi)}}{1-e^{i(\theta-\varphi)}}N^{-1}{}_{ij}(\theta)N^{-1}{}_{lj}{}^{*}(\varphi)\eta_i(\theta)\eta_l(\varphi)\Bigg\}^{1/2}.\end{aligned}$$

We take now into account the equation (2.28), which simplify the last relations to

$$\mu_2(w)=\left\{\sum_{i=1}^{6}\sum_{k=0}^{n_i}\left[\left[\frac{w\tilde{\varphi}_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\right]^2\right.$$

$$\left.+\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\left|\frac{w(\theta)}{S(\theta)}\right|^2\sum_{i=1}^{6}\eta_i(\theta)\left[\rho_i(\theta)-\sum_{j=1}^{6}\sum_{k=0}^{n_j}\left[\frac{w\varphi_j}{S}\right]_{z=0}^{(k)}\frac{1}{k!}{\rm Im}\left[\frac{N^{-1}{}_{ij}(\theta)S(\theta)e^{ik\theta}}{w(\theta)}\right]\right]d\theta\right\}^{1/2}. \tag{2.39}$$

It is convenient also to express $\mu_2(w)$ in terms of the Lagrange multipliers $\delta_i(\theta)$, defined in (2.34). After some straightforward manipulating we obtain from (2.39)

$$\mu_2(w)=\left\{\sum_{i=1}^{6}\sum_{k=0}^{n_i}\left[\left[\frac{w\varphi_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\right]^2\right.$$

$$\left.+\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\sum_{i=1}^{6}\frac{\delta_i(\theta)}{|g_i(\theta)|^2}\left[\tilde{\rho}_i(\theta)-\sum_{k=0}^{n_i}\left[\frac{w\varphi_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}|g_i(\theta)|\sin(\phi_i(\theta)+k\theta)\right]d\theta\right\}^{1/2}. \tag{2.40}$$

The above expressions of $\mu_2(w)$ are still not very convenient for applications since their dependence upon the parameters $\varphi_i^{(k)}(0)$ of interest is not entirely made explicit. Indeed, as follows from Eqs. (2.29) and (2.37), the Lagrange multipliers $\eta_i(\theta)$ and $\delta_i(\theta)$ themselves depend upon these values. Fortunately, the parameters $\varphi_i^{(k)}(0)$ appear linearly in the right-hand side of the integral equations (2.29) and (2.37), so that the dependence of the solutions upon them can be easily made explicit. Namely, considering for simplicity the equations (2.37) and using the linearity of the corresponding Fredholm operators K_i and their inverses, we can express the solution $\delta_i(\theta)$ as

$$\delta_i(\theta)=\gamma_i(\theta)-\sum_{k=0}^{n_i}\left[\frac{w\varphi_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\gamma_{i,k}(\theta),\quad i=1,\ldots,6\,, \tag{2.41}$$

where the functions $\gamma_i(\theta)$ and $\gamma_{i,k}(\theta)$ satisfy the Fredholm integral equations

$$\begin{aligned}&(K_i\gamma_i)(\theta)=\tilde{\rho}_i(\theta),\quad i=1,\ldots,6,\quad \theta\in(-\theta_{\rm in},\theta_{\rm in})\,,\\&(K_i\gamma_{i,k})(\theta)=|g_i(\theta)|\sin[\phi_i(\theta)+k\theta]\,,\quad k=0,\ldots,n_i\,,\end{aligned} \tag{2.42}$$

the operator K_i being defined in (2.37). It is convenient to write formally the solutions of these equations,

$$\begin{aligned}&\gamma_i=K_i{}^{-1}[\tilde{\rho}_i]\,,\\&\gamma_{i,k}=K_i{}^{-1}[\,|g_i(\theta)|\sin[\phi_i(\theta)+k\theta]],\quad i=1,\ldots,6,\quad k=0,\ldots,n_i\end{aligned} \tag{2.43}$$

and introduce them in (2.40). We obtain, after rearranging the terms ,

$$\mu_2^2(w)=\sum_{i=1}^{6}\left\{\sum_{k=0}^{n_i}\left[\left[\frac{w\tilde{\varphi}_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\right]^2\right.$$

$$+\sum_{k,k'=0}^{n_i}\left[\frac{w\tilde{\varphi}_i}{S}\right]_{z=0}^{(k)}\left[\frac{w\tilde{\varphi}_i}{S}\right]_{z=0}^{(k')}\frac{1}{k!k'!}$$

$$\times\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}K_i^{-1}[\,|g_i|\sin(\phi_i+k'\theta)]\,|g_i(\theta)|\sin(\phi_i+k\theta)\frac{d\theta}{|g_i(\theta)|^2}$$

$$-\sum_{k=0}^{n_i}\left[\frac{w\tilde{\varphi}_i}{S}\right]_{z=0}^{(k)}\frac{1}{k!}\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\{K_i^{-1}[\tilde{\rho}_i]\,|g_i(\theta)|\sin(\phi_i+k\theta)$$

$$+\tilde{\rho}_i(\theta)K_i^{-1}[\,|g_i|\sin(\phi_i+k\theta)]\}\frac{d\theta}{|g_i(\theta)|^2}$$

$$\left.+\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}\tilde{\rho}_i(\theta)K_i^{-1}[\tilde{\rho}_i]\frac{d\theta}{|g_i(\theta)|^2}\right\}. \tag{2.44}$$

The general case (2.39) can be treated in a similar way. Therefore, we have expressed finally $\mu_2^2(w)$ as a quadratic function of the parameters $\tilde{\varphi}_i^{(k)}(0)$, involving calculable coefficients. It can be shown[12] that this is also a convex function of $\tilde{\varphi}_i^{(k)}(0)$. By introducing $\mu_2(w)$ in the inequality (2.22), for different admissible functions w, we obtain necessary restrictions upon the values $\tilde{\varphi}_i^{(k)}(0)$ of interest. Of course, in order to optimize these inequalities we have to perform the additional maximization (2.23) upon the admissible functions w. As we mentioned, we shall treat this problem approximately, using to this end a particular but very suitable class of functions w. As was shown in the previous works,[8,17] such a suitable choice for w proves to be

$$w(z)=\frac{\sqrt{1-\xi^2}}{(1-\xi z)}\,,\quad \xi\in(-1,1)\,, \tag{2.45}$$

the maximization upon w reducing to simply varying the real parameter ξ in its allowed interval. Details about the procedure and its efficiency in closely approaching, with little computational efforts, the exact supremum are given in Refs. 8 and 17.

III. APPLICATIONS

We first apply the present formalism in order to deduce a rigorous sum rule for the Born-pole residua. To this end we consider in (2.44) the particular case $n_i=0$, $i=1,\ldots,6$, which leads to the inequality

$$\mu_2^2(w)=\sum_{i=1}^{6}\left\{\frac{w^2(0)\tilde{\varphi}_i^2(0)}{S^2(0)}\left[1+\frac{1}{\pi}\int_{-\theta_{\rm in}}^{\theta_{\rm in}}K_i^{-1}[\,|g_i|\sin\phi_i]\,|g_i(\theta)|\sin\phi_i(\theta)\frac{d\theta}{|g_i(\theta)|^2}\right]\right. \tag{3.1}$$

$$-\frac{w(0)\tilde{\varphi}_i(0)}{S(0)}\frac{1}{\pi}\int_{-\theta_{in}}^{\theta_{in}}[K_i^{-1}[\tilde{\rho}_i]\,|g_i(\theta)|\sin\phi_i+\tilde{\rho}_i(\theta)K_i^{-1}[\,|g_i|\sin\phi_i]]\frac{d\theta}{|g_i(\theta)|^2}$$

$$+\frac{1}{\pi}\int_{-\theta_{in}}^{\theta_{in}}\tilde{\rho}_i(\theta)K_i^{-1}[\tilde{\rho}_i]\frac{d\theta}{|g_i(\theta)|^2}\Bigg\}\leq 1\,.$$

All the quantities entering in this relation are actually known, being expressed in terms of the physical input of the problem. We have derived therefore a family of rigorous sum rules, relating the static electromagnetic characteristics (e and κ) of the target to the differential cross section of the Compton process above the pion photoproduction threshold ν_0^2 and the absorptive parts of the scattering amplitudes, expressed by the unitarity condition in terms of the photoproduction matrix elements. The present result generalizes the sum rule for Born pole residua obtained in[4] without incorporating the s −channel unitarity, as well as the corresponding sum rule given in,[8] where the knowledge of the differential cross section was exploited only partially.

The inequality (3.1) can be tested numerically by using the experimental data. In our calculations we have used the pion photoproduction multipoles tabulated in,[14,15] for photon laboratory energies ω below 1210 MeV and a compilation of the data on the UDCS for γ-nucleon elastic scattering.[20–23] Details about the utilization of the experimental material are given in Ref. 8. With these data we calculated the coefficients required in the integral equations (2.42) and solved these equations by applying standard numerical techniques. By using the solutions (2.42) thus found and the known values $\tilde{\varphi}_i(0)$ from (2.18), we evaluated then the expression (3.1) of $\mu_2^2(w)$, choosing, according to the remarks given above, a function w of the particular form (2.45). The calculations were repeated for various values of the parameter ξ entering this form, the largest value of $\mu_2^2(w)$ yielding the optimal sum-rule inequality. It turned out that practically in all the cases considered in the present work, the supremum upon w was obtained by taking the parameter ξ equal to 1, which corresponds to $w(\theta)=1$.

In our analysis we applied the two-particle unitarity condition (2.6) in the region where it is rigorously valid, i.e., below the double pion photoproduction threshold $\omega_{in}=2\mu+2\mu^2/m$. Above this threshold multipion photoproduction matrix elements must be taken into account. For completeness, we investigated the condition (2.6) also for several higher values of ω_{in}, simulating the possible effect of the neglected multipion photoproduction by randomly varying $\mathrm{Im}\bar{A}_i$ by $\pm 10\%$, this being the estimated order of magnitude of this contribution.[1,2] In Table I we give the results of these calculations for several values of t in the range allowed in the present formalism (as shown in Ref. 4 the condition that the entire unitarity cut is in physical region imposes the limitation $t \gtrsim -3.48\mu^2$). For completeness we listed in the first column the values obtained previously (Ref. 7) without incorporating the unitarity condition, which corresponds to setting in the present formalism ω_{in} equal to the single-pion photoproduction threshold $\omega_0=\mu+\mu^2/m$. The errors quoted in the table were obtained by taking into account both the uncertainties upon the absorptive parts discussed above and the experimental errors upon the cross

TABLE I. Test of the sum-rule inequality for the Born-pole residua Eq. (3.1), for several values of ω_{in} and t.

t/μ^2 \ ω_{in}	$\omega_0=\mu+\mu^2/2m$	$2\mu+2\mu^2/m\approx 320$ MeV	400 MeV	450 MeV	1210 MeV
−0.1	0.40±0.04	0.50±0.01	0.56±0.01	0.57±0.01	0.61±0.015
−0.5	0.30±0.03	0.41±0.01	0.51±0.03	0.54±0.04	0.62±0.04
−1.0	0.25±0.02	0.39±0.02	0.52±0.01	0.56±0.04	0.70±0.05
−2.0	0.20±0.02	0.39±0.02	0.57±0.04	0.63±0.05	0.96±0.09
−2.5	0.18±0.02	0.40±0.02	0.60±0.04	0.70±0.06	1.20±0.1
−3.0	0.10±0.01	0.41±0.02	0.67±0.05	0.77±0.07	1.40±0.2
−3.3	0.10±0.01	0.42±0.02	0.69±0.05	0.79±0.07	1.50±0.25

section $\sigma(\nu^2,t)$. Actually, in order to increase the reliability of the results we overestimated these data by 5%, since this increase can be shown (Ref. 8) to weaken the resulting inequalities.

The results given in Table I show that the incorporation of the unitarity condition (2.6) has the effect of bringing the sum rule for the Born-pole residua closer to saturation. This effect is most significant for large values of $|t|$. As expected, the results become gradually stronger by increasing ω_{in}, i.e., by imposing the unitarity constraint along a greater energy interval.

The most remarkable feature of our results is the almost exact saturation and even the slight violation of the sum rule (3.1) which occurs in some cases when $\mu_2{}^2(w)$ exceeds unity. Having in view the general theoretical frame in which the sum rule (3.1) was deduced, we can explain these situations only by invoking the experimental information used as input in the numerical evaluations. We mention that some inconsistencies between pion photoproduction multipole extraction and the data on the elastic γ-nucleon differential cross section were discovered already in the more particular formalism developed in Ref. 8. The results given in Table I confirm these conclusions and suggest that either the multipion photoproduction contribution to the absorptive parts of the Compton-scattering amplitudes up to $\omega=1210$ MeV is more significant than our estimation of 10%, or the cross section has to be increased by a convenient factor, in order to make these data consistent with the rigorous sum rule (3.1).

After the illustration of the formalism in the above particular case we pass now to the problem of major interest for the fixed-t dispersion theory of the proton Compton scattering, i.e., the investigation of the subtraction functions $F_i(t)$. According to the relations (2.19), we have to consider in this case the derivatives $\varphi_2'(0)$ and $\varphi_3'(0)$. Actually, by taking suitable values for the parameters n_i in the general relation (2.44), we can obtain in the present formalism rigorous restrictions upon these derivatives, considered either separately or simultaneously. For simplicity we consider here the first alternative, which will lead to explicit upper and lower bounds upon each of the subtraction functions separately. The simultaneous treatment of $\varphi_2'(0)$ and $\varphi_3'(0)$, which would provide a direct correlation between the values of the subtraction functions $F_1(t)$ and $F_2(t)$, can be performed in a similar way.

We take therefore in (2.44) the parameters $n_i=0$, for $i\neq j$ and $n_j=1$, where j will be set equal either to 2 or to 3. We introduce for simplicity the notation

$$\xi_j=\left[\frac{w\varphi_j}{S}\right]'_{z=0}=\left[\frac{w}{S}\right]'_{z=0}\varphi_j(0)+\frac{w(0)}{S(0)}\varphi_j'(0)\ . \tag{3.2}$$

Then we can write from (2.44) the inequality

$$\mu_2{}^2(w)=a_j\xi_j{}^2-2b_j\xi_j-c\le 0,\quad j=2,3\ , \tag{3.3}$$

where the real coefficients a_j, b_j, and c have the simple expressions

$$a_j=1+\frac{1}{\pi}\int_{-\theta_{\text{in}}}^{\theta_{\text{in}}}|g_j(\theta)|\sin(\phi_j+\theta)K_j{}^{-1}[\,|g_j|\sin(\phi_j+\theta)]\frac{d\theta}{|g_i(\theta)|^2}\ ,$$

$$b_j=\frac{1}{2\pi}\int_{-\theta_{\text{in}}}^{\theta_{\text{in}}}\{\beta_j(\theta)K_j{}^{-1}[\,|g_j|\sin(\phi_j+\theta)]+|g_j|\sin(\phi_j+\theta)K_j{}^{-1}[\beta_j]\}\frac{d\theta}{|g_j(\theta)|^2}\ , \tag{3.4}$$

$$c=1-\sum_{i=1}^{6}\left\{\frac{w^2(0)\widetilde{\varphi}_i{}^2(0)}{S^2(0)}+\frac{1}{\pi}\int_{-\theta_{\text{in}}}^{\theta_{\text{in}}}\beta_j(\theta)K_j{}^{-1}[\beta_j]\frac{d\theta}{|g_j(\theta)|^2}\right\}$$

in terms of the new functions β_i, defined as

$$\beta_i(\theta)=\widetilde{\rho}_i(\theta)-\frac{w(0)\widetilde{\varphi}_i(0)}{S(0)}|g_i(\theta)|\sin\phi_i(\theta);\ \ i=1,\ldots,6\ .$$

From the inequality (3.3) and the relation (3.2) we obtain easily bounds upon the parameter $\varphi_j'(0)$ of interest,

$$\frac{S(0)}{w(0)}\left[\frac{b_j-(b_j^2+a_jc)^{1/2}}{a_j}-\left[\frac{w}{S}\right]'_{z=0}\varphi_j(0)\right]\le\varphi_j'(0)\le\frac{S(0)}{w(0)}\left[\frac{b_j+(b_j^2+a_jc)^{1/2}}{a_j}-\left[\frac{w}{S}\right]'_{z=0}\varphi_j(0)\right]. \tag{3.5}$$

These model-independent upper and lower bounds upon the derivative $\varphi_j'(0)$ improve the previous bounds obtained in Ref. 7 without incorporating the unitarity condition. They can be evaluated numerically without difficulties. In order to find the coefficients a_j, b_j, and c we have to solve the corresponding Fredholm integral equations of the type (2.42), built up in terms of the physical input contained in the functions $S(\theta)$, $\rho_i(\theta)$ and the parameters $\tilde{\varphi}_i(0)$. The derivatives $S'(0)$ and $w'(0)$ can be evaluated easily from definitions (2.17) and (2.45), respectively. A subsequent optimization upon the functions w, as we indicated above, allowed us to find the best results yielded by (3.5). By properly modifying within reasonable errors of 5–10% the absorptive parts $\mathrm{Im}\bar{A}_i$ and the differential cross section $\sigma(\nu^2,t)$, we estimated finally the uncertainties affecting the results. From the bounds thus obtained upon $\varphi_j'(0)$, $j=2,3$, we deduced, using Eqs. (2.19), upper and lower bounds upon the subtraction functions $F_1(t)$ and $F_2(t)$. The results of our calculations are presented in Tables II and III, respectively. Actually, in Table III we indicated directly the corresponding bounds upon the π^0-pole contribution, obtained using the estimations of the higher dispersion contributions appearing in (2.3), taken from Ref. 7. These contributions were listed for completeness in the third column of Table III, and are seen to be, in this range of t, small compared to the magnitude of the pion pole,[7] indicated in the second column of the table (the upper and lower signs correspond, respectively, to $F_\pi>0$ and $F_\pi<0$, using the sign convention of Ref. 1). The errors quoted in Table III contain, besides the effect of the experimental errors, evaluated according to the above discussion, also a theoretical uncertainty related to the model-dependent term $F_2^{\mathrm{HS}}(t)$, of about 20–30% of its value.

As in the previous example, referring to the Born-pole residua, we exploited the unitarity condition below the double photoproduction threshold, $\omega_{\mathrm{in}}=2\mu+2\mu^2/m$, and also at some energies slightly above this threshold ($\omega_{\mathrm{in}}=400$ MeV). For completeness, we showed also the values obtained previously,[7] without the unitarity condition ($\omega_{\mathrm{in}}=\omega_0=\mu+\mu^2/2m$). As expected, by increasing ω_{in}, the resulting bounds become gradually stronger.

From Tables II and III it follows that at very low $|t|$ the calculated bounds upon the subtraction functions are rather weak. However, starting from $|t|\approx 2\mu^2$ the bounds become quite restrictive and impose nontrivial limitations upon their evaluations in particular models. For instance, the range allowed for $\mu^3F_1(t)$ at $|t|=2\mu^2$ is situated entirely above the value $\mu^3F_1(t)=0.0032$ obtained in the model evaluation.[1,7] In general, the range allowed for $F_1(t)$ at these values of t is seen to be shifted towards positive values. Unfortunately, this strong limitation is not obtained at very low values of $|t|$, which are of particular interest for making a precise estimation of the electromagnetic polarizabilities, according to Eq. (2.2). However, assuming a reasonably smooth variation of $F_1(t)$ in

TABLE II. Calculated upper and lower bounds upon the subtraction function $\mu^3F_1(t)$, for several values of ω_{in} and t.

	$\omega_{\mathrm{in}}=\omega_0=\mu+\mu^2/2m$		$\omega_{\mathrm{in}}=2\mu+2\mu^2/m\approx 320$ MeV		$\omega_{\mathrm{in}}=400$ MeV	
t/μ^2	Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
−0.1	−0.78 ±0.07	0.79 ±0.08	−0.18 ±0.005	0.21 ±0.001	−0.07 ±0.02	0.095 ±0.001
−0.5	−0.18 ±0.02	0.19 ±0.02	−0.028 ±0.0005	0.054 ±0.0002	−0.0035±0.0005	0.029 ±0.0003
−1.0	−0.09 ±0.01	0.10 ±0.01	−0.0065±0.0004	0.033 ±0.0001	0.0058±0.0003	0.021 ±0.0001
−2.0	−0.040±0.002	0.058±0.006	0.0056±0.0003	0.023 ±0.0001	0.0113±0.0002	0.0175±0.0001
−2.5	−0.037±0.004	0.048±0.005	0.0071±0.0003	0.021 ±0.0001	0.012 ±0.0002	0.0174±0.0001
−3.0	−0.030±0.003	0.043±0.004	0.0105±0.0002	0.0207±0.0001	0.0141±0.0002	0.0173±0.0001
−3.3	−0.027±0.003	0.039±0.004	0.0067±0.0002	0.0154±0.0001	0.01 ±0.001	0.0124±0.0001

TABLE III. Calculated upper and lower bounds upon the pion-pole contribution $\mu^4 F_2^{\pi^0\text{pole}}(t)$ to the subtraction function $\mu^4 F_2(t)$, for several values of ω_{in} and t. The second column contains the magnitude of this term (upper and lower signs correspond to $F_\pi > 0$ and $F_\pi < 0$, respectively). The model calculations of $\mu^4 F_2^{\text{HS}}(t)$ in the third column are taken from Refs. 1 and 7.

t/μ^2	$\mu^4 F_2^{\pi^0\text{pole}}(t)$	$\mu^4 F_2^{\text{HS}}(t)$	$\omega_{\text{in}}=\mu+\mu^2/2m$		$\omega_{\text{in}}=2\mu+2\mu^2/m \approx 320$ MeV		$\omega_{\text{in}}=400$ MeV	
			Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
−0.1	±0.0133	−0.00076	−4.7 ±0.2	5.26 ±0.2	−1.28 ±0.01	1.25 ±0.02	−0.6 ±0.01	0.45 ±0.03
−0.5	±0.00978	−0.000772	−0.45 ±0.01	0.57 ±0.04	−0.12 ±0.002	0.12 ±0.004	−0.059 ±0.002	0.033 ±0.05
−1.0	±0.00734	−0.000788	−0.16 ±0.008	0.23 ±0.01	−0.037 ±0.001	0.04 ±0.002	−0.019 ±0.002	0.011 ±0.002
−2.0	±0.0048	−0.0008	−0.056±0.01	0.093±0.01	−0.009 ±0.0013	0.016 ±0.002	−0.0039±0.016	0.0049±0.0024
−2.5	±0.0041	−0.00085	−0.039±0.002	0.07 ±0.005	−0.0042±0.001	0.01 ±0.001	−0.001 ±0.0009	0.0046±0.002
−3.0	±0.00367	−0.0009	−0.028±0.002	0.056±0.003	−0.0014 ±0.0008	0.014 ±0.0014	0.0007±0.0006	0.0046±0.0017
−3.3	±0.0034	−0.00095	−0.02 ±0.005	0.04 ±0.003	−0.00004±0.0002	0.0097±0.002	0.0018±0.0009	0.0046±0.002

the present range of t, we obtain from the bounds given in Table II an almost model-independent indication that the ordering $\alpha-\beta>0$ holds for the electric and magnetic polarizabilities of the proton. This conclusion is of considerable interest in connection with the experimental determinations[3,24] and the previous theoretical studies[1–3] of these quantities.

Referring now to the pion pole contribution to $F_2(t)$ given in Table III, we notice first that at low $|t|$ the range allowed for it is consistent with both signs of the pion decay constant F_π. When $|t|$ is increased, this range remains no more symmetrical around zero, but is shifted systematically towards positive values. As a consequence, in a number of cases, the negative values of $F_2^{\pi^0\text{pole}}$ given in column 3, which correspond to $F_\pi<0$, remain outside this allowed range, which, on the other hand, is always compatible with the positive values yielded by $F_\pi>0$. These results might be interpreted as an almost unbiased indication that the s-channel physical data on the proton-Compton scattering, when exploited in an optimal way, seem to reject the alternative $F_\pi<0$ and to favor the positive sign $F_\pi>0$. We notice that this is actually the sign suggested in the past by the Goldberger-Treiman calculation of the $\pi^0\to 2\gamma$ decay rate in the $\bar{N}N$ model[25] and by the Primakoff effect.[26] Of course, the above conclusion is not completely model independent, but the effect of the model calculated part, in the range of t considered, is very low. It could receive further support when accurate data on the reaction $\gamma\gamma\to\pi\pi$, required for the evaluation of this part, will become available. Our results prove anyway that the present formalism is a powerful tool for providing a decisive answer to the question of the sign of F_π.

IV. CONCLUDING REMARKS

In the present paper we derived model-independent bounds upon the subtraction functions of the proton Compton dispersion relations, which exploit optimally the physical quantities of the s channel, together with the fixed-t analyticity, the s-u crossing symmetry, and the gauge invariance of the scattering amplitudes. The formalism developed here generalizes in a manifest way the previous treatments,[4–8] which succeeded in incorporating only partially this physical information.

Our results evidence the considerable constraining power of the physical s-channel information, which is able to impose severe limitations upon the magnitude of the t-channel model-dependent contributions. In particular, our results provide a serious indication towards the inequality $\alpha-\beta>0$, involving the generalized electromagnetic polarizabilities of the proton, and favor the sign $F_\pi>0$ for the pion decay constant which is actually the sign suggested in Refs. 25 and 26.

We mention, finally, that by suitable developments the formalism presented here can be applied in order to study correlations between the two subtraction functions $F_1(t)$ and $F_2(t)$, to incorporate data on the differential cross section on γ-nucleon scattering below the pion photoproduction threshold, and to investigate higher derivatives of the scattering amplitudes below this threshold.

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