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Lie-Bäcklund symmetries for the Harry-Dym equation

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A recursion operator (strong symmetry) for the Harry-Dym equation is found. It is also hereditary, and can be used to generate infinitely many Lie-Bäcklund symmetries.

Among the nonlinear evolution equations solvable by means of the inverse spectral transform, the so-called Harry-Dym equation[1]

$$u_t = \lambda u^3 u_{xxx} \quad , \tag{1}$$

where λ is a constant, deserves special attention.

In fact, Eq. (1) is not quasilinear, unlike the most popular completely integrable nonlinear evolution equations, and admits soliton solutions expressed by implicit functions.[2] Furthermore, performing a prolongation calculation,[3] we have found that one can associate with Eq. (1) a (presumably) infinite-dimensional non-Abelian Lie algebra whose closure can be forced to give an $SL(2,R)$ algebra.

We remember that closed algebras can be generally used, within the prolongation scheme, to obtain Bäcklund transformations.[4,5] However, for the Harry-Dym equation the usual procedure fails, in the sense that this leads to the trivial Bäcklund transformation only.

In order to overcome this difficulty, we have made a symmetry approach[6–12] to Eq. (1). Precisely, we have shown that Eq. (1) possesses infinitely many Lie-Bäcklund symmetries[13] which can be generated each from the other through a *recursion operator,*[11] which enjoys the property of being hereditary[6,12] (see later). This feature might be related to the existence of N-soliton solutions[7,10] of Eq. (1) and, it is hoped, to the possibility of writing down Bäcklund transformations.

We recall that[6] a given evolution equation of the type

$$u_t = K(u) \tag{2}$$

admits a Lie-Bäcklund symmetry of the form $\eta(u,u_1,\ldots,u_N)$, N arbitrary, where

$$u_j \equiv \left(\frac{\partial}{\partial x}\right)^j u, \quad j=0,1,\ldots,N \tag{3}$$

if and only if

$$X(\eta)K = X(K)\eta \quad , \tag{4}$$

where $X(\eta)$ is the Lie-Bäcklund operator associated with (2), defined by

$$X(\eta) = \eta\frac{\partial}{\partial u} + (D_t\eta)\frac{\partial}{\partial u_t} + (D\eta)\frac{\partial}{\partial u_1} + (D^2\eta)\frac{\partial}{\partial u_2} + \cdots \quad . \tag{5}$$

Here D_t and D denote the total derivative with respect to t and x, respectively.

Taking

$$K(u) \equiv \lambda u^3 u_3 \quad , \tag{6}$$

condition (4) gives

$$3u^2u_3\eta + u^3D^3\eta = \eta_u u^3 u_3 + \eta_{u_1} D(u^3u_3) + \eta_{u_2} D^2(u^3u_3) + \cdots \quad , \tag{7}$$

where

$$\eta_{u_j} = \partial\eta/\partial u_j \quad (j=0,1,2,\ldots) \quad .$$

One sees immediately that Eq. (7) is satisfied by

$$\eta^{(1)} = u_1 \tag{8}$$

and

$$\eta^{(2)} = u^3 u_3 \quad , \tag{9}$$

which correspond to the invariance under space and time translation of Eq. (1), respectively.

Now we look for a generalized symmetry of Eq. (1) of the form

$$\eta^{(3)} \equiv \eta^{(3)}(u,u_1,u_2,u_3,u_4,u_5) \quad . \tag{10}$$

To this end, substituting (10) in (7) and putting the coefficients of powers of u_8 and u_7 equal to zero, we get

$$\eta^{(3)}_{u_5} = u^5 \quad , \tag{11}$$

from which one has

$$\eta^{(3)} = u^5 u_5 + F(u,u_1,u_2,u_3,u_4) \quad , \tag{12}$$

where F is a function of integration. Repeating this procedure, i.e., inserting the quantity (12) in Eq. (7) and equating coefficients of powers of u_5, u_6, and so on to zero, after cumbersome but straightforward calculations one finally obtains

$$\eta^{(3)}=u^5u_5+5u^4u_1u_4+5u^4u_2u_3+\tfrac{5}{2}u^3u_1{}^2u_3 \ . \tag{13}$$

At this point, we have looked for a *recursion operator* for Eq. (1), namely an operator which satisfies the relation[6]

$$\Delta'[K]=[K',\Delta]\equiv K'\Delta-\Delta K' \ , \tag{14}$$

where K' and Δ' denote the Fréchet derivatives of the function (6) and the operator-valued function Δ, defined by

$$K'(u)[v]=\frac{d}{d\epsilon}K(u+\epsilon v)\bigg|_{\epsilon=0} \tag{15}$$

and

$$\Delta'(u)[v]w=\frac{d}{d\epsilon}\Delta(u+\epsilon v)w\bigg|_{\epsilon=0} \ , \tag{16}$$

respectively.

To this end, we have first checked that the operator

$$\Delta=u^2D^2-uu_1D+uu_2+u^3u_3D^{-1}\frac{1}{u^2} \ , \tag{17}$$

where D^{-1} is defined by

$$(D^{-1}f)(x)=\int_{-\infty}^{x}f(\xi)d\xi \ , \tag{18}$$

generates both symmetry $\eta^{(3)}$ from $\eta^{(2)}$ and $\eta^{(2)}$ from $\eta^{(1)}$. We have assumed that asymptotically $(x\to\pm\infty)$ $u\to\text{const}\neq 0$ and $u_j\to 0$ $(j=1,2,\ldots)$.

Using (17), we notice that Eq. (1) can be written in the form

$$u_t=\Delta u_1 \ , \tag{19}$$

where we have put $u(-\infty)=1/\lambda\neq 0$.

The operator (17) verifies condition (14), i.e., it is a recursion operator (or a *strong symmetry*[10]) for Eq. (1). To prove this formally, let us explicitly calculate the commutator on the right of (14).

One has

$$\begin{aligned}K'\Delta&=(3u^2u_3+u^3D^3)\left(u^2D^2-uu_1D+uu_2+u^3u_3D^{-1}\frac{1}{u^2}\right)\\&=u^5D^5+5u^4u_1D^4+(3u^3u_1{}^2+4u^4u_2)D^3+6u^4u_3D^2+(4u^3u_1u_3+5u^4u_4)D\\&\quad+4u^4u_5+14u^3u_2u_3+15u^3u_1u_4+6u^2u_1{}^2u_3+[3u^5u_3{}^2+u^3D^3(u^3u_3)]D^{-1}\frac{1}{u^2}\end{aligned} \tag{20}$$

and

$$\begin{aligned}\Delta K'&=\left(u^2D^2-uu_1D+uu_2+u^3u_3D^{-1}\frac{1}{u^2}\right)(3u^2u_3+u^3D^3)\\&=u^5D^5+5u^4u_1D^4+(3u^3u_1{}^2+4u^4u_2)D^3+4u^4u_3D^2+(8u^3u_1u_3+6u^4u_4)D\\&\quad+10u^3u_2u_3+9u^3u_1u_4+3u^4u_5+2u^3u_3D^{-1}u_3 \ ,\end{aligned} \tag{21}$$

where the relations

$$Dv=vD+v_1,\quad D^2v=vD^2+2v_1D+v_2,\quad D^3v=vD^3+3v_1D^2+3v_2D+v_3$$

have been used.

In virtue of (20) and (21), one sees that the expression for $[K',\Delta]$ coincides with the Fréchet derivative

$$\begin{aligned}\Delta'[K]&=2u^4u_3D^2-(4u^3u_1u_3+u^4u_4)D+6u^2u_1{}^2u_3+4u^3u_2u_3+6u^3u_1u_4+u^4u_5\\&\quad+[3u^5u_3{}^2+u^3D^3(u^3u_3)]D^{-1}\frac{1}{u^2}-2u^3u_3D^{-1}u_3\end{aligned} \tag{22}$$

of the operator (17).

Furthermore, we have checked that the operator (17) is also a *hereditary* operator,[12] i.e., it satisfies

$$[\Delta,\Delta'][v]w=[\Delta,\Delta'][w]v \ , \tag{23}$$

where

$$[\Delta,\Delta'][v]w=\Delta(\Delta'[v]w)-\Delta'[\Delta v]w \ , \tag{24}$$

and v, w are arbitrary functions of $u, u_1, \ldots, u_N$.

To point out the importance of the hereditary operators in the study of nonlinear evolution equations, we recall that[10] if an operator $\phi(u)$ is hereditary and is a strong symmetry for a given evolution equation (2), then it is also a strong symmetry for any of the equations

$$u_t = \phi^n(u) K(u), \quad n = 1, 2, \ldots \quad (25)$$

Moreover, if $\phi(u)$ is invariant under x translation, it is a strong symmetry for $u_t = u_1$ and for any of the equations

$$u_t = \phi^n(u) u_1, \quad n = 1, 2, \ldots \quad (26)$$

In the case of the Harry-Dym equation one has that the recursion operator (strong symmetry) given by (17) is also a strong symmetry for any of the equations

$$u_t = \Delta^n(u) u_1, \quad n = 1, 2, \ldots \quad (27)$$

In other words, Eq. (1) possesses infinitely many Lie-Bäcklund symmetries which can be obtained each from the other by means of the recursion operator (17). The first few of these are given by

$$\eta^{(1)} = u_1 , \quad (28)$$

$$\eta^{(2)} = u^3 u_3 , \quad (29)$$

$$\eta^{(3)} = u^5 u_5 + 5u^4 u_1 u_4 + 5u^4 u_2 u_3 + \tfrac{5}{2} u^3 u_1^2 u_3 , \quad (30)$$

$$\eta^{(4)} = u^7 u_7 + \tfrac{105}{2} u^5 u_1^2 u_5 + 21 u^6 u_2 u_5 + 14 u^6 u_1 u_6 + \tfrac{105}{2} u^4 u_1^3 u_4 + 105 u^5 u_1 u_2 u_4 + 21 u^6 u_3 u_4 + \tfrac{147}{2} u^4 u_1^2 u_2 u_3 + \tfrac{63}{2} u^5 u_2^2 u_3 + 42 u^5 u_1 u_3^2 + \tfrac{63}{8} u^3 u_1^4 u_3 . \quad (31)$$

To conclude this Communication, we note that Eq. (27) could be exploited to find multisoliton solutions of Eq. (1). In fact, concerning this it has been shown[7,12] that, if a relation of the type (27) holds, then the soliton solutions of (27) are given by those functions $u(x,t)$ which can be expressed by

$$u_1(x,t) = \sum_{j=1}^{N} w_j(x,t) , \quad (32)$$

where the quantities w_j are eigenvectors, with time-independent eigenvalues, of the recursion operator Δ.

This property could be useful in looking for Bäcklund transformations of Eq. (1). At present this problem is under investigation.

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