Spinning fluids in the Einstein-Cartan theory

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An Eulerian variational principle for a spinning fluid in the Einstein-Cartan metric-torsion theory is presented. The variational principle yields the complete set of field equations for the system. The symmetric energy-momentum tensor is a sum of a perfect-fluid term and a spin term.

I. INTRODUCTION

An Eulerian variational principle for a perfect fluid in general relativity was presented some years ago.[1] Recently,[2] we generalized this variational principle to deal with fluids having intrinsic spin. Our main goal in this paper is to generalize the spinning-fluid variational principle to the Einstein-Cartan (EC) theory.

In the EC theory spin takes on an important role as the source of the torsion part of the gravitational field. Thus, what we derive in this paper is a fundamental theory, based on a Lagrangian, which introduces spin into the EC theory for a macroscopic spinning fluid. The torsion-spin equation of our theory leads to the Weyssenhoff convective form for the canonical spin tensor; however, the energy-momentum tensor has explicit spin-dependent terms. There have been several recent studies of spinning matter in the EC theory[3–5]; however, none of these studies deduce the field equations of the theory from an Eulerian variational principle such as in this paper.

The theories that we studied in Ref. 2 and in this paper are based on the work of Halbwachs[6] who formulated an Eulerian variational principle for a spinning fluid in special relativity. Reference 1 is a generalization of Halbwachs's theory to general relativity for fluids without intrinsic spin while Ref. 2 is for fluids with intrinsic spin.

II. LAGRANGIAN DENSITY

The Lagrangian density for a spinning fluid in general relativity that was discussed in Ref. 2 has the form

$$\begin{aligned}\mathcal{L} = {} & c_1\sqrt{-g}R + \frac{1}{c}\sqrt{-g}F(\rho,s,s_{ij}) + \sqrt{-g}\,\lambda_1(g_{ij}U^iU^j+c^2) + \sqrt{-g}\,\lambda_2(\rho U^i)_{;i} + \sqrt{-g}\,\lambda_3 X_{,i}U^i + \sqrt{-g}\,\lambda_4 s_{,i}U^i \\ & - \sqrt{-g}\frac{\rho\kappa}{c}g_{ij}a^{1i}\dot{a}^{2j} + \sqrt{-g}\,\lambda^{11}(g_{ij}a^{1i}a^{1j}-1) + \sqrt{-g}\,\lambda^{22}(g_{ij}a^{2i}a^{2j}-1) + 2\sqrt{-g}\,\lambda^{12}g_{ij}a^{1i}a^{2j} \\ & + \sqrt{-g}\frac{2}{c}\lambda^{14}g_{ij}a^{1i}U^j + \sqrt{-g}\frac{2}{c}\lambda^{24}g_{ij}a^{2i}U^j\ . \end{aligned} \tag{2.1}$$

The term involving κ in (2.1) is the spin kinetic energy density of the fluid T:

$$T = \tfrac{1}{2}S_{ij}\omega^{ij}\ , \tag{2.2}$$

where S_{ij} is the spin tensor and the angular velocity associated with the intrinsic spin is defined by[6–8]

$$\omega_{ij} = \dot{a}^{\nu}{}_{i}a_{\nu j} = a^{\nu}{}_{j}a_{\nu i;k}U^k\ , \tag{2.3}$$

where we have introduced an orthonormal tetrad $a^{\nu}{}_{i}$ to represent the spin density s_{ij} and the four-velocity U^i of the fluid via

$$S_{ij} = \rho\kappa(a^1{}_ia^2{}_j - a^1{}_ja^2{}_i) = \rho s_{ij}\ , \tag{2.4}$$

$$U^i = \frac{1}{c}a^{4i}\ , \tag{2.5}$$

where κ is a scalar function $\kappa(x)$ with the dimensions of angular momentum per unit mass and ρ is the density of the fluid. In (2.1) we have gone somewhat beyond the results of Ref. 2 by allowing the thermodynamic properties of the fluid to depend upon the spin density s_{ij},

$$s_{ij} = S_{ij}/\rho \quad , \tag{2.6}$$

through the s_{ij} dependence of F which now has the form

$$F = -\rho[c^2 + \epsilon(\rho, s, s_{ij})] \quad . \tag{2.7}$$

The thermodynamic law now takes the form

$$d\epsilon = T ds - P d(1/\rho) + \tfrac{1}{2}\omega_{ij} ds^{ij} \quad , \tag{2.8}$$

with

$$\tfrac{1}{2}\omega_{ij} = \left(\frac{\partial \epsilon}{\partial s^{ij}}\right)_{s\rho} \quad . \tag{2.9}$$

We shall give the equations of motion only for the EC case since general relativity is included as a special case. The variables to be varied are ρ, s, X, U^i, g_{ik}, a^{1i}, a^{2i}, and the various λ's representing the Lagrange multipliers plus the torsion $S_{ij}{}^k$ which arises in an EC spacetime and is defined as the antisymmetric part of the asymmetric affine connection

$$S_{ij}{}^k = \Gamma^k_{[ij]} \quad . \tag{2.10}$$

III. FIELD EQUATIONS

This section is similar to the calculations in Ref. 2 which may be consulted for more details. Variations of $\mathfrak{L}$ with respect to a^{1i} and a^{2i} yield eventually the spin equation of motion

$$\frac{DS_{ij}}{d\tau} + \left(\frac{2}{c^2}\dot{U}_l U_i + \omega_{li}\right) s_j{}^l + \left(\frac{2}{c^2}\dot{U}_l U_j + \omega_{lj}\right) s^l{}_i = 0 \quad . \tag{3.1}$$

The precession of the spin implied by (3.1) is that for Fermi-Walker transport since

$$\omega_{li} s_j{}^l + \omega_{lj} s^l{}_i = -\left(\frac{1}{c^2}\dot{U}_l U_i s_j{}^l + \frac{1}{c^2}\dot{U}_l U_j s^l{}_i\right) \quad , \tag{3.2}$$

which, when combined with (3.1), gives

$$\frac{Ds_{ij}}{d\tau} + \frac{1}{c^2}\dot{U}_l U_i s_j{}^l + \frac{1}{c^2}\dot{U}_l U_j s^l{}_i = 0 \quad . \tag{3.3}$$

Note, however, that since this is EC theory the absolute derivatives in (3.3) are with respect to the full affine connection Γ^i_{jk}. This same result (3.3) is also valid in general relativity.

The variation with respect to torsion $S_{ij}{}^k$ yields the result

$$S^{ijk} = \frac{\rho}{4cc_1} s^{ij} U^k \quad , \tag{3.4}$$

the Weyssenhoff form, so that the canonical spin angular momentum tensor has the form

$$\tau^{ijk} = (\rho/2) s^{ij} U^k \quad . \tag{3.5}$$

Variation with respect to the metric g_{ik} leads to the EC field equation

$$G^{(ik)} - \overset{*}{\nabla}_l (T^{lik} + T^{lki}) = \frac{1}{2cc_1} T^{ik} \quad , \tag{3.6}$$

where the modified torsion tensor is given by

$$T_{ij}{}^k = S_{ij}{}^k + 2\delta^k_{[i} S_{j]x}{}^x \quad , \tag{3.7}$$

and where the symmetric energy-momentum tensor has the form

$$\begin{aligned} T^{ik} = {} & \rho(1 + \epsilon/c^2 + P/\rho c^2) U^i U^k + P g^{ik} \\ & + \frac{1}{c^2}\rho U^{(i} s^{k)j} \dot{U}_j + \nabla_j [\rho U^{(i} s^{k)j}] - \rho \omega^{l(i} s^{k)}{}_l \\ & + \rho U^{(i} s^{k)l} \omega_{lj} U^j \quad . \end{aligned} \tag{3.8}$$

The variations with respect to ρ and U^i have been used to reduce the form of T^{ik}. The energy-momentum tensor can be broken down into two pieces:

(1) a perfect-fluid part,

$$T_F^{ik} = \rho(1 + \epsilon/c^2 + P/\rho c^2) U^i U^k + g^{ik} P \quad ; \tag{3.9}$$

(2) an intrinsic-spin part,

$$\begin{aligned} T_S^{ik} = {} & \frac{1}{c^2}\rho U^{(i} s^{k)l} \dot{U}_l + \nabla_j [\rho U^{(k} s^{i)j}] - \rho \omega^{l(i} s^{k)}{}_l \\ & + \rho U^{(i} s^{k)l} \omega_{lj} U^j \quad . \end{aligned} \tag{3.10}$$

This same form for the symmetric energy-momentum tensor is also valid in general relativity. If the intrinsic spin of the matter vanishes then we are left with just the perfect fluid T_F^{ik}. We have not found a derivation of this energy-momentum tensor in the literature. Note that the last two terms in (3.10), those involving ω_{ij}, are corrections to the energy-momentum tensor due to the inclusion of spin as a thermodynamic variable in (2.8).

IV. DISCUSSIONS AND CONCLUSIONS

In order to arrive at the EC Lagrangian density (2.1) it was necessary to extend the results of Ref. 2 to the metric-torsion geometry. However, there is not always a unique extension of a metric Lagrangian to a metric-torsion geometry.[9] For example, we can write the conversion of mass constraint in two forms in general relativity:

$$\sqrt{-g}\,(\rho U^i)_{;i} = 0 \tag{4.1}$$

or

$$(\sqrt{-g}\,\rho U^i)_{,i}=0 \ . \tag{4.2}$$

These two expressions are identical in the Riemannian geometry of general relativity. If we extend these to a metric-torsion geometry we have

$$\nabla_i(\rho U^i)=0 \tag{4.3}$$

and

$$(\sqrt{-g}\,\rho U^i)_{,i}=\overset{*}{\nabla}_i(\sqrt{-g}\,\rho U^i)=0$$
$$\equiv\sqrt{-g}\,\nabla_i(\rho U^i)+2\sqrt{-g}S_{ij}{}^j\rho U^i \ , \tag{4.4}$$

respectively, where we have used ∇_i to represent the covariant derivative with respect to the asymmetric affine connection Γ^i_{jk} of the EC spacetime

$$\Gamma^i_{jk}=\{^{\,i}_{jk}\}+S_{jk}{}^i-S_k{}^i{}_j+S^i{}_{jk} \ , \tag{4.5}$$

and $\{^{\,i}_{jk}\}$ is the Christoffel connection. The two extended constraints (4.3) and (4.4) now differ by the term $2S_{ij}{}^j\rho U^i$, which is not a divergence. These extensions therefore lead to different, inequivalent, theories. We have studied the theory with constraint (4.3) for nonspinning matter in Ref. 10. The mass constraint for this theory has the form

$$\nabla_i(\rho U^i)=(\sqrt{-g}\,\rho U^i)_{,i}-2\sqrt{-g}S_{ij}{}^j\rho U^i=0 \ . \tag{4.6}$$

Applying Gauss's law to (4.6) for the region Σ between two spacelike hypersurfaces σ_1 and σ_2 enclosing all the matter we arrive at

$$M(\sigma_2)-M(\sigma_1)=2\int_\Sigma\sqrt{-g}S_{ij}{}^j\rho U^i d^4X \ , \tag{4.7}$$

where $M(\sigma)=\int_\sigma\sqrt{-g}\,\rho U^i dV_i$ is the mass of the system on σ. This theory then implies that the torsion is associated with the change of mass of the system and not with spin. Although such theories would probably not be interesting in astrophysical settings, they might be models for Dirac's[11] mass-creation cosmological theories: The geometrical torsion field creating mass through the law (4.6). This seems to us to be a novel interpretation of torsion.

The second extension, Eq. (4.4), leads to the more conventional interpretation of the mass constraint as conservation of mass:

$$(\sqrt{-g}\,\rho U^i)_{,i}=0 \ , \tag{4.8}$$

which we have chosen for our anlaysis.

In order to avoid confusion we point out that the intrinsic spin discussed in this paper is not associated with quantum-mechanical spin, as is sometimes done in discussions of the EC theory. Here the intrinsic spin of the fluid is associated with the particles of the fluid, which in the case of cosmology are galaxies or clusters of galaxies. Thus the field variables occurring in our variational principle are the fluid variables ρ and s_{ij} that are normally employed. This theory should be useful in studying the influence of spin on the dynamics of the early universe. At early enough times we must, of course, switch to a description in terms of quantum mechanics. The theory discussed herein is then valid back to a time when quantum processes become important.

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[1]J. R. Ray, J. Math. Phys. 13, 1451 (1972).
[2]J. R. Ray and L. L. Smalley, Phys. Rev. D 26, 2619 (1982).
[3]G. Cognola, R. Soldati, L. Vanzo, and S. Zerbini, Phys. Rev. D 25, 3109 (1982).
[4]P. B. Yasskin and W. R. Stoeger, Phys. Rev. D 21, 2081 (1980).
[5]S. Hojman, Phys. Rev. D 18, 2741 (1978).
[6]F. Halbwachs, *Theorie Relativiste Des Fluides A Spin* (Gauthier-Villars, Paris, 1960).
[7]I. Bailey, Ann. Phys. (N.Y.) 119, 76 (1979).
[8]I. Bailey and W. Israel, Commun. Math. Phys. 42, 65 (1975).
[9]L. L. Smalley, Phys. Rev. D 18, 3896 (1978).
[10]J. R. Ray and L. L. Smalley, Phys. Rev. D 26, 2615 (1982).
[11]*On the Measurement of Cosmological Variations of the Gravitational Constant,* edited by L. Halpern (University Presses of Florida, Gainesville, 1978).