PHYSICAL REVIEW D VOLUME 27, NUMBER 6 15 MARCH 1983

Generalized Bohr-Sommerfeld rules for anomalies with applications to symmetry breakdown and decoupling

A. P. Balachandran, V. P. Nair, and C. G. Trahern
Physics Department, Syracuse University, Syracuse, New York 13210
(Received 5 February 1982; revised manuscript received 18 October 1982)

In the presence of anomalies, the requirement that a classical symmetry group G has a proper action on the fermion measure or in the effective Lagrangian description imposes Bohr-Sommerfeld conditions on the anomalies, and often implies that G is broken to a subgroup H as well. We show these results in this paper and apply them to QCD and SU(5). In particular, constraints on the QCD order parameter are derived, and an argument is presented which suggests that the breakdown of the chiral flavor symmetry and the emergence of some sort of generation structure in QCD may be natural.

I. INTRODUCTION

It is well known that quantum corrections can alter the symmetry group of a classical system in a dramatic way. A familiar example of this kind occurs in theories with spontaneous symmetry breaking where the symmetry group cannot be unitarily implemented at the quantum level, but instead acts only as a group of automorphisms of the algebra of observables. Another example is found in gauge theories where classically conserved currents may cease to be conserved by the presence of quantum "anomalies." The corresponding group action at the quantum level produces a change of coordinates to a physically inequivalent set and leads to a number of well-known and often dramatic consequences, for instance a perfectly reasonable classical theory may become inconsistent at the quantum level. In recent years, these anomalies have played a useful role by furnishing constraints for model builders,[1] by suggesting a resolution of the U(1) problem,[2] and by providing information on the spectrum of massless fermionic bound states.[3]

The divergences of currents have an intimate relation with the Lie algebra of the classical symmetry group and its *infinitesimal* transformations. While there have been many detailed investigations in the literature of current divergences and therefore of these infinitesimal transformations, there has been hardly any study of *finite* transformations even though the response of Green's functions to these transformations can be expressed in terms of the anomalies as well.[4] In this paper we attempt such a global study of anomalies.

The global analysis shows that unless certain constraints are satisfied, the action of the classical symmetry group cannot even be *defined* at the quantum level. The usual local analysis of anomalies does not of course reveal this important feature. However, to derive statements that go beyond such formal conclusions, we must make additional assumptions. A reasonable set of assumptions, including one of those used by 't Hooft,[3] leads to constraints on order parameters for QCD (Refs. 5 and 6), conditions on the pattern of breakdown of gauged symmetries, and conditions on sectors which can decouple. In particular we present an argument which suggests that the breakdown of the global flavor group $SU_L(N_f)\times SU_R(N_f)$ along with the emergence of a generation structure may be natural in theories like QCD.

Our investigations do not appear to be related directly to those of 't Hooft.[3] 't Hooft's conditions govern statements about the spectrum of gauge theories. Thus a gauge theory may be realized in different phases at different energy scales, and he requires that the particle spectrum in each phase reproduce the same anomaly structure of the theory. By contrast our analysis gives a judgement about the overall consistency of a theory and requires the gauge symmetry to breakdown in a specified pattern in order to accommodate the global action of anomalous symmetries.

We do not expect our results to be dependent on renormalization effects. Our analysis of the non-Abelian anomaly depends only on its tensor structure. This is known to be unique.[7] Consequently, any overall renormalization of this anomaly should not affect our conclusions. For the $U_A(1)$ anomaly the overall normalization is important, but the Adler-Bardeen theorem[8] guarantees its insensitivity to renormalization. Recently, it has been suggested that the Adler-Bardeen theorem may fail in supersymmetric gauge theories.[9] If this is true, the global implementation of the $U_A(1)$ symmetry would indicate new constraints on the theory. However, we do not analyze supersymmetric theories in this paper.

Sections II and III contain a resume of known results on the properties of quantum anomalies for a classical symmetry group G. We also show the relationship between the response of the fermion functional measure $d\mu$ and the effective action under finite group transformations. In Sec. IV we derive a generalized Bohr-Sommerfeld condition for the anomalous transformations from the requirement that the functional measure $d\mu$ be single-valued under the global (i.e., rigid) action of the group G. We also show that the same statements follow from the effective Lagrangian description.

Since the Wess-Zumino formalism[4] which we use may be unfamiliar, in Sec. V we illustrate the Bohr-Sommerfeld condition in this formalism for the $U_A(1)$ anomaly before analyzing the full non-Abelian case in later sections. Here the Bohr-Sommerfeld condition leads to constraints on the possible order parameters in QCD. The results of this section are analogous to results obtained by Harari and Seiberg[5] and Weinberg[6] who used the Bohr-Sommerfeld condition for $U_A(1)$ groups, of course without referring to it by this name, to eliminate mass terms in the effective Lagrangians of composite fermion theories.

Section VII analyzes the Bohr-Sommerfeld condition for non-Abelian anomalies in detail. We summarize the results in the form of two constraints. In particular constraint 1 describes the pattern of breaking for the group G. The surviving symmetry group H is embedded in G in a well-defined way.

These results are illustrated in Sec. VII by taking SU(5), QCD and a composite fermion model of Yamawaki and Yokota[10] as examples. The discussion includes the results mentioned in previous paragraphs.

Appendix A recalls some formulas involving the non-Abelian anomalies which are needed in Sec. VII. A lengthy Appendix B describes the proof of constraint 1.

II. EFFECTIVE LAGRANGIANS AND ANOMALIES

We review the work of Wess and Zumino[4] in this section. The construction of the effective Lagrangian which generates a given set of anomalies is outlined, and its transformation properties under finite gauge transformations are deduced.

Let G and $\underline{G}$ be the representations of the gauged group and of its Lie algebra on a subset of spin-half fermions. Let L_α be a basis for $\underline{G}$ with $L_\alpha^\dagger = -L_\alpha$. The anomalous conservation laws of the currents J_μ^α which correspond to these base elements can be expressed as

$$\partial^\mu J_\mu^\alpha = A_\alpha(W) , \tag{2.1}$$

where $W_\mu = W_\mu^\alpha L_\alpha$ is the gauge field.

(For consistency of the gauge theory, these anomalies have of course to be canceled by those due to another set of spin-half fermions.)

It is convenient to introduce the notation

$$B(\epsilon, W) = \int d^4x\, \epsilon^\alpha(x) A_\alpha(W) , \tag{2.2}$$

where

$$\epsilon = L_\alpha \epsilon^\alpha \tag{2.3}$$

is a Lie-algebra-valued function. Then if δ_η $[\eta(x) \in \underline{G}]$ generates an infinitesimal gauge transformation,

$$\delta_\eta W_\mu = [\eta, W_\mu] - \partial_\mu \eta , \tag{2.4}$$

Wess and Zumino[4] show that B must fulfill the "consistency" condition

$$\delta_\eta B(\epsilon, W) - \delta_\epsilon B(\eta, W) = B([\eta,\epsilon], W) . \tag{2.5}$$

Here, of course,

$$\delta_\eta B(\epsilon, W) = \text{linear term in } \eta \text{ in} \quad B(\epsilon, W - \delta_\eta W) - B(\epsilon, W) . \tag{2.6}$$

The effective action S_A which generates the anomalies is a functional of W and possibly of other fields (the Goldstone modes) which under a gauge variation generates the appropriate anomaly:

$$\delta_\epsilon S_A = B(\epsilon, W) . \tag{2.7}$$

The gauge variation is here performed on all the fields in S_A.

Since

$$[\delta_\eta, \delta_\epsilon] = \delta_{[\eta,\epsilon]} , \tag{2.8}$$

we see (by applying this identity on S_A) that the consistency condition is the integrability condition for the existence of S_A.

[It may be noted that in the presence of fermionic condensates which contribute to the anomalies, the anomaly expression in (2.7) is the difference of the anomaly expression in the fundamental theory and the contribution of these fermions. Thus for precision, we should distinguish the true anomalies from those which occur in (2.7), the two may or may not be the same. We do not do so since such a distinction will not be required for our needs.]

Let us introduce the field g with values in $G, g(x) \in G$. Under gauge transformations, it is transformed as follows:

$$g(x)\rightarrow s(x)g(x),\ s(x)\in G\ . \tag{2.9}$$

The action S_A will be constructed as a functional of g and W: $S_A=S_A(g,W)$. Thus

$$\begin{aligned}\delta_\epsilon S_A &\equiv \text{linear term in } \epsilon \text{ in}\\ &\quad S_A(g-\delta_\epsilon g, W-\delta_\epsilon W)-S_A(g,W)\\ &=B(\epsilon,W)\ ,\\ \delta_\epsilon g&=\epsilon g\ .\end{aligned} \tag{2.10}$$

The field g represents the Goldstone modes. If the theory depends on all the components of g in a nontrivial way, then the entire symmetry group G is spontaneously broken (in the limit $W_\mu=0$). If the breakdown is only to a subgroup H, then we have to impose the condition

$$S_A(gh,W)=S_A(g,W),\ H(x)\in H \tag{2.11}$$

on S_A since the Goldstone modes for such a symmetry breakdown have values in G/H. (This condition makes sure that the true degrees of freedom in S_A are W and these Goldstone modes.) We will defer the discussion of this condition to the end of this section.

The construction of S_A proceeds as follows.[4] Let

$$\begin{aligned}[U(s)F](g,W)&=F(s^{-1}g,s^{-1}\circ W)\ ,\\ (s^{-1}\circ W)_\mu(x)&\equiv s^{-1}(x)W_\mu(x)s(x)\\ &\quad +s^{-1}(x)\partial_\mu s(x)\end{aligned} \tag{2.12}$$

define the operator $U(s)$ on functionals F. It implements a finite gauge transformation on such functionals. Consider now a one-parameter family of fields s_t and write $s_t(x)=s(x,t)$. Since $s^{-1}(x,t)\partial_t s(x,t)\in \underline{G}$, it is evident that

$$\begin{aligned}&\{U^{-1}[s(x,t)]\partial_t U[s(x,t)]S_A\}(g,W)=B[s^{-1}\dot{s},W],\\ &\dot{s}(x,t)\equiv\partial_t s(x,t)\ .\end{aligned} \tag{2.13}$$

If s fulfills

$$\begin{aligned}s(x,0)&=e\ ,\\ s(x,1)&=g(x)\ ,\end{aligned} \tag{2.14}$$

we can write

$$\begin{aligned}U(g)S_A-S_A&=\int_0^1 dt\,\frac{d}{dt}U(s)S_A\\ &=\int_0^1 dt\,U(s)\{U^{-1}(s)\partial_t U(s)S_A\}\ .\end{aligned} \tag{2.15}$$

The evaluation of this at (g,W) thus gives

$$\begin{aligned}S_A(g,W)&=S_A(e,g^{-1}\circ W)-\int_0^1 dt\,B(s^{-1}\dot{s},s^{-1}\circ W)\\ &\equiv S_A(e,g^{-1}\circ W)-\int_e^g B(s^{-1}ds,s^{-1}\circ W)\ ,\end{aligned} \tag{2.16}$$

where a convenient notation has been introduced and the limits there indicate the limiting values of s. (Note that d implies differentiation only in t.)

Now $S_A(e,g^{-1}\circ W)$ is gauge invariant. Thus (2.16) gives $S_A(g,W)$ in terms of B up to the arbitrary gauge-invariant function $S_A(e,g^{-1}\circ W)$.

We want to deduce the transformation properties of S_A under finite gauge transformations. Let g' be a (t-independent) field with values in G [that is, $g'(x)\in G$]. Then from (2.16),

$$S_A(g'g,g'\circ W)-S_A(g,W)=\int_e^{g'^{-1}} B(s^{-1}ds,s^{-1}\circ W). \tag{2.17}$$

The integrals in (2.16) and (2.17) are path independent (that is, depend only on the terminal points of the path) provided the integrability conditions (2.5) are fulfilled. We shall discuss this point further later.

Since the Goldstone modes have values in G/H, if there are anomalies associated with H, then in the absence of fermionic contributions to anomalies, the functional $S_A(e,W)$ has to reproduce the H anomalies under gauge transformation of W. This follows from (2.11):

$$\begin{aligned}&S_A(h^{-1}h^{-1}\circ W)=S_A(e,h^{-1}\circ W)\ ,\\ &h(x)\in H\ .\end{aligned} \tag{2.18}$$

An anomaly which admits such an $S_A(e,W)$ is removable by the counterterm $-S_A(e,W)$ in the original classical action.[11] Such anomalies, perhaps, are not so interesting. However, there are cases where the anomalies in H cannot be reproduced by a counterterm depending only on W. In other words H may have nonremovable anomalies. Since the Goldstone modes which have values in G/H cannot reproduce these anomalies in the effective action, the effective theory should then contain spin-half fermions transforming nontrivially under H which reproduce the anomalies in H via one-loop diagrams. The representation content of these fermions will be constrained using the anomaly matching conditions of 't Hooft.[3]

III. THE FUNCTIONAL MEASURE

Let ψ denote the multicomponent field of spin-half fermions which transforms according to the representation G. It has been pointed out by Vergeles and Fujikawa[12] that the anomalous Ward identities are due to the anomalous transformation laws of the fermion functional measure $d\psi d\bar{\psi}$. We write

$$d\psi d\bar{\psi}=d\mu(\psi,W)\ ,$$

where the notation indicates the implicit dependence of $d\psi d\bar{\psi}$ on W (through the anomalies). Then according to their work,

$$\begin{aligned} &d\mu(s\psi,s\circ W)=d\mu(\psi,W)e^{i\beta(s,W)}\ ,\\ &s(x)\in G\ , \end{aligned} \tag{3.1}$$

where if $s=1-\epsilon+O(\epsilon^2)$,

$$\beta(s,W)=B(\epsilon,W)+O(\epsilon^2)\ . \tag{3.2}$$

It follows that $d\mu(\psi,W)$ and $\exp(iS_A)$ transform in a similar way under gauge transformations:

$$d\mu(g'\psi,g'\circ W) =d\mu(\psi,W)\exp\left[i\int_e^{g'^{-1}}B(s^{-1}ds,s^{-1}\circ W)\right]\ . \tag{3.3}$$

IV. BOHR-SOMMERFELD CONDITIONS ON ANOMALIES

The existence of a well-defined action of the group G on the measure $d\mu(\psi,W)$ requires that the anomalies fulfill certain quantization conditions of the Bohr-Sommerfeld type. These conditions come about as follows. Let us vary g' in (3.3) over a closed path C from identity to identity. Then since the left side should return to its original value when g' returns to e, we find

$$\exp\left[\int_C B(s^{-1}ds,s^{-1}\circ W)\right]=1\ . \tag{4.1}$$

If the transformation s corresponds to a nongauged current, then (4.1) should hold for the fundamental theory as well as all effective versions of it. An example of such an s is the axial $U_A(1)$ transformation which we discuss later. However, for gauged currents one often encounters the situation where the theory has two or more sectors in which the anomalies from one sector cancel against another. At certain energy scales only some of these sectors may be relevant as a result of condensation, decoupling, etc. An effective Lagrangian description of such a phase would require (4.1) as a consistency condition on the measure of the surviving fermions and the effective action of the Goldstone bosons.

An important corollary of (4.1) is that under small deformations of C,

$$\delta\int_C B(s^{-1}ds,s^{-1}\circ W)=0\ . \tag{4.2}$$

The integrability conditions (2.5) ensure this equation. However, when G is not a suitably broken symmetry, these integrability conditions are as a rule fulfilled only if $\epsilon(x)$ and $\eta(x)$ vanish at space-time time infinity. This in turn means that (4.2) may not be fulfilled for small deformations of C which do not vanish at infinity. We shall in fact prove in Sec. VI that the validity of (4.2) *for all deformations* often requires the breakdown of G (spontaneously or otherwise) to one of a class of subgroups H. The physical significance of this result will be the subject of Sec. VII.

Remark. The transformation $\psi\to g'\psi$, $W\to g'\circ W$ can be thought of as a change of variables in the functional measure. The failure of the condition (4.1) then means that $d\mu(\psi,W)$ is not single valued in its arguments. It is this lack of single-valuedness which prevents a consistent action of G on $d\mu(\psi,W)$ when (4.1) is not fulfilled.

V. THE $U_A(1)$ ANOMALY AND THE QCD ORDER PARAMETER

In this section, we illustrate the Bohr-Sommerfeld condition in the context of the axial $[U_A(1)]$ anomaly in QCD (with no electroweak interactions). It leads to restrictions on the QCD order parameter.

We shall make the conventional assumption that the fermions in the QCD effective Lagrangian (baryons) do not reproduce the gluon contribution to the $U_A(1)$ anomaly. Therefore the effective Lagrangian itself must exhibit the $U_A(1)$ anomaly.

Let us first briefly recall the QCD effective Lagrangian and the standard representation of the $U_A(1)$ anomaly therein.

In the QCD effective Lagrangian, the mesons are described by a matrix-valued field M which is a color singlet and has definite transformations laws under the flavor group $U(N_f)_L\times U(N_f)_R=\{(u,v)\}$ where N_f is the number of flavors. The Lagrangian density is invariant under this group except for a piece $\mathscr{L}_A$. This piece under the chiral U(1) transformations $(e^{i\alpha},1)$ or $(1,e^{i\alpha})$ transforms as follows[2,4]:

$$(e^{i\alpha},1):\ \mathscr{L}_A\to\mathscr{L}_A+\alpha N_f Q\ , \tag{5.1}$$

$$(1,e^{i\alpha}):\ \mathscr{L}_A\to\mathscr{L}_A-\alpha N_f Q\ , \tag{5.2}$$

$$Q=\frac{1}{16\pi^2}\mathrm{Tr}F^{\mu\nu}F^*_{\mu\nu}\ . \tag{5.3}$$

We have denoted the gluon field tensor by $F_{\mu\nu}$. This anomalous transformation law reproduces the anomalies of the associated currents. $\mathscr{L}_A$ is invariant under $SU(N_f)_L\times SU(N_f)_R$ and the vector U(1) group $\{(e^{i\alpha},e^{i\alpha})\}$.

It is also conventionally assumed that M_{ab} transforms like $\bar{q}_a(1+\gamma_5)q_b$ where a,b are the flavor indices of the quark fields q and their color indices are summed over.

The purpose of this section is to find restrictions on the properties of the order parameter M and to suggest that the identification

$$M_{ab} \sim \bar{q}_a(1+\gamma_5)q_b \tag{5.4}$$

is essentially unique [up to $U(N_f)_L \times U(N_f)_R$ and color-invariant factors]. For example, we exclude the following order parameters:

(i) For $N_f=1$,

$$M \sim \bar{q}(1+\gamma_5)q\bar{q}(1+\gamma_5)q \ . \tag{5.5}$$

(ii) For $N_f>1$,

$$M_{ab,cd} \sim \bar{q}_a(1+\gamma_5)q_b\bar{q}_c(1+\gamma_5)q_d \ . \tag{5.6}$$

Such restrictions are derived from the hypothesis that these order parameters are responsible for the $U_A(1)$ anomaly in the effective-Lagrangian approach.

As a prelude to the derivation of these results, we review the construction of $\mathscr{L}_A$ in greater detail.

Let $U_a(1)=\{(e^{i\alpha},1)_a\}$ denote the U(1) group which acts only on the left-handed quark of flavor a:

$$\begin{aligned}&(e^{i\alpha},1)_a : q_a \to e^{i(1+\gamma_5)\alpha/2} q_a \ , \\ &(e^{i\alpha},1)_a : q_b \to q_b \ , \ b \neq a \ .\end{aligned} \tag{5.7}$$

For such a transformation,

$$(e^{i\alpha},1)_a : \mathscr{L}_A \to \mathscr{L}_A + \alpha Q(x) \tag{5.8}$$

[which follows from setting $N_f=1$ in (5.1)].

To construct $\mathscr{L}_A$, we have to first construct a function $F(M)$ with the following properties: (i) It transforms homogenously under $U_a(1)$ for each a:

$$(e^{i\alpha},1)_a F(M) = e^{i\rho\alpha} F(M) \ . \tag{5.9}$$

Here ρ is an integer ($\neq 0$) which should be independent of a. (ii) It is invariant under $SU(N_f)_L \times SU(N_f)_R$ and the vector U(1) group (this condition is required for the invariance of $\mathscr{L}_A$ under these groups).

If a function $F(M)$ with properties (i) and (ii) can be found, then according to known results,[2,4] we can set

$$\mathscr{L}_A = \frac{i}{2}\frac{1}{\rho} Q(x) \ln F(M) + \text{H.c.} \tag{5.10}$$

and reproduce (5.8). It is easily checked that direct integration of Eq. (2.16) for S_A in the case where s is a $U_A(1)$ transformation leads to this equation; $\ln F(M)$ can be identified with the Goldstone mode.

We now show that if the color group is SU(3) and the quarks are in its triplet representation, ρ must in fact be ± 1. This follows from requiring that $\exp(iS_A)$ be a single-valued function of M Here

$$S_A = \int d^4x \, \mathscr{L}_A \ . \tag{5.11}$$

The proof is as follows. Since

$$\int d^4x \, Q(x) = 2\pi p, \ \ p=0,\pm 1, \pm 2, \ldots \tag{5.12}$$

(cf. Ref. 13), the choice of a different branch of $\ln F(M)$ changes S_A to

$$S_A + \frac{2\pi p}{\rho} \ , \tag{5.13}$$

where p can take any integer value. Hence $|\rho|=1$.

The condition $|\rho|=1$ is also required if the group $U_A(1)$ is to act properly on $\exp(iS_A)$. This follows because

$$(e^{i2\pi/\rho},1)_a : F(M) \to F(M) \ . \tag{5.14}$$

Thus $(e^{i2\pi/\rho},1)_a$ should not affect e^{iS_A}. Since

$$(e^{i\alpha},1)_a : e^{iS_A} \to e^{iS_A} \exp\left(i \int d^4x \, \alpha Q\right) \tag{5.15}$$

because of (5.8), we find

$$(e^{i2\pi/\rho},1)_a : e^{iS_A} \to e^{iS_A} \exp\left[i\frac{2\pi}{\rho} \int d^4x \, Q \right] . \tag{5.16}$$

Hence the condition $|\rho|=1$.

The result $|\rho|=1$ can be deduced in a third way from the properties of the θ vacuums [and the requirement that the theory should be insensitive to the choice of the branch of $\ln F(M)$]. Thus, from (5.12) and (5.13), we see that a change in the branch of $\ln F(M)$ corresponds to the change

$$\theta \to \theta + \frac{2\pi p}{\rho} \tag{5.17}$$

of the vacuum parameter θ of QCD. Since any choice $\theta + 2\pi p$ gives an identical theory, we see that if $|\rho|=1$, the theory is insensitive to the definition of $\ln F(M)$.

For the choice (5.4), we can take $F(M)=\det M$ and fulfill the conditions (i), (ii), and $|\rho|=1$. For (5.5), the choice $F(M)=M$ leads to $\rho=2$ and is not allowed. [Since

$$\ln M^{1/2} = \tfrac{1}{2} \ln M \ (\text{mod} \, 2\pi i) \ ,$$

the choice $F(M)=M^{1/2}$ also leads to trouble.] For (5.6), neither Tr $M^T M$ nor det M has the requisite properties.

If the quarks are not in $\underline{3}$ of SU(3), but in some other representation Γ, we have to change (5.12) to

$$\int d^4x \, Q(x) = 2\pi\nu \frac{\text{Tr}_\Gamma \left[\sum \lambda_a{}^2 \right]}{\text{Tr}_3 \left[\sum \lambda_a{}^2 \right]} \ . \tag{5.18}$$

The subscript on Tr indicates the representation in which the trace of the quadratic Casimir operator $\sum \lambda_a{}^2$ is taken. Thus in general, we have to change the condition on ρ depending on Γ. For $\Gamma=\underline{6}$,

$$\frac{\mathrm{Tr}_6\left[\sum\lambda(a)^2\right]}{\mathrm{Tr}_3\left[\sum\lambda(a)^2\right]}=5\;, \tag{5.19}$$

and ρ can be 1 or 5.

VI. GLOBAL TRANSFORMATIONS AND THE EXISTENCE OF FUNCTIONAL MEASURE

In this section we examine the restrictions which follow from the Bohr-Sommerfeld condition [(4.1) and (4.2)] for a non-Abelian group G acting on the measure $d\mu(\psi,W)$. We assume in our discussions that the entire group G is gauged. The case where only a subgroup $\widetilde{G}$ of G is gauged can be easily obtained by setting the gauge fields in the quotient $G/\widetilde{G}$ equal to zero.

The gauge theory consisting of the fields ψ and W alone is not consistent; the anomalies generated by ψ have to be canceled by the anomalies due to other fermion fields L. The purpose of examining the ψ sector by itself is to see if it can decouple from the L sector or form condensates with little admixture from the L sector, etc.

The gauged group can in general contain invariant continuous subgroups which act trivially on ψ. Such subgroups are irrelevant when discussing the properties of $d\mu(\psi,W)$. Thus it is the quotient of these two groups that we call G. The group G is thought of concretely as a group of matrices acting on ψ.

The presentation of the results is much simplified by using the convention where the right chiral projections are singlets under G (cf. Ref. 14). We adopt this convention. Thus ψ is a left-handed field.

Let

$$G=\bigotimes_{\nu=1}^{P} G_S^{(\nu)}\otimes G_A\;, \tag{6.1}$$

where $G_S^{(\nu)}$ is *simple.* For later purposes, we shall also regard them as *simply connected.* The group G_A is Abelian.

Of course, the group which acts faithfully on the space of fields (ψ,W) may not be G but G/Z where Z is a discrete subgroup. Since the arguments leading to the first constraint are not affected by the presence of Z, it is ignored for the moment.

We denote the components of the field ψ by $\psi_{\alpha_1\alpha_2\cdots\alpha_P}$ where the index α_ν is transformed only by $G_S^{(\nu)}$.

As constraint 1 below will make precise, G must break to a subgroup H if G does not satisfy the Bohr-Sommerfeld condition. The subgroup H can be written in the form

$$H=H_S\otimes H_A\;, \tag{6.2}$$

where H_S is a *semisimple* and H_A is an Abelian Lie group. (The actual surviving symmetry would in general be $[H_S\otimes H_A\times D]/Z_H$ where D and Z_H are discrete groups. We shall say nothing about D, it is ignored hereafter. For our current reasoning, Z_H is also not important and is ignored.) The index α_ν will transform in a definite way under H_S:

$$\psi_{\cdots\alpha_\nu\cdots}\to\cdots D^{(\nu)}_{\alpha_\nu\alpha'_\nu}(h)\cdots\psi_{\cdots\alpha'_\nu\cdots}\;,\qquad h\in H_S\;. \tag{6.3}$$

The matrices $\{D^{(\nu)}(h)\}$ form a representation of H_S which we call $H_S^{(\nu)}$.

Constraint 1 can now be stated as follows. *Suppose $H_S^{(\nu)}$ is nontrivial, $H_S^{(\nu)}\neq\{e\}$. Then the Bohr-Sommerfeld condition [or equivalently, the existence of a well-defined action of G on $d\mu(\psi,W)$] implies one of the following:*

(a) *$G_S^{(\nu)}$ is anomaly free (that is, $G_S^{(\nu)}$ is an anomaly-free representation) or* (b) *$G_S^{(\nu)}$ is a representation of SU(M) for some M and $H_S^{(\nu)}$ is a representation of an SU(2) or SO(3) subgroup of SU(M). In the defining representation of SU(M), by a choice of basis, this subgroup can be brought to the form*

$$\left\{\begin{vmatrix} g & & 0\\ & g & \\ & & \ddots & \\ 0 & & & g\end{vmatrix}\right\}, \tag{6.4}$$

where $\{g\}$ is one of the (2j+1)-dimensional irreducible representations of SU(2).

A preliminary version of this result was presented in Ref. 15. It is a consequence of the condition (4.2). The latter is fulfilled only if the consistency condition (2.5) is fulfilled. We shall deduce the above result from the consistency condition when η and ϵ are space-time independent.

Note the following: we can write up to discrete subgroups,

$$H_S=H_S(1)\otimes H_S(2)\otimes\cdots\;, \tag{6.5}$$

where $H_S(k)$ are simple. Correspondingly,

$$H_S^{(\nu)}=H_S^{(\nu)}(1)\otimes H_S^{(\nu)}(2)\otimes\cdots\;, \tag{6.6}$$

where $H_S^{(\nu)}(k)$ is simple or trivial. Now, for those ν for which $G_S^{(\nu)}$ has anomalies, the existence of $d\mu(\psi,W)$ requires $H_S^{(\nu)}$ to be either trivial or a representation of SU(2). In the latter case, only one $H_S^{(\nu)}(l)$ is thus nontrivial; further it is homomorphic to SU(2). This in turn means that $H_S(l)$ is homomorphic to SU(2). In summary, if $G_S^{(\nu)}$ has anomalies, at most one $H_S^{(\nu)}(l)$ is nontrivial and if there is such a nontrivial $H_S^{(\nu)}(l)$, then $H_S(l)$ and $H_S^{(\nu)}(l)$ are homomorphic to SU(2).

The expression for the anomalies A_α we shall use will be that where (cf. Appendix A)

$$\begin{aligned} A_\alpha(W) &= d_{\alpha\beta\gamma} a_{\beta\gamma}(W) , \\ d_{\alpha\beta\gamma} &= \mathrm{Tr} L(\alpha)(L(\beta)L(\gamma)+L(\gamma)L(\beta)) \\ &\equiv \mathrm{Tr} L(\alpha)\{L(\beta),L(\gamma)\} . \end{aligned} \tag{6.7}$$

This expression is not unique, but the nonuniqueness does not affect the conclusions as we shall indicate when we conclude this discussion (see Remark 3).

When ϵ is x independent,

$$B(\epsilon,W)=\epsilon^\alpha \int d^4x\, A_\alpha(x) . \tag{6.8}$$

The explicit expression (A1.1) for $A_\alpha(W)$ also shows that for global transformations, $A_\alpha(W)$ transforms like the component of a vector in the adjoint representation. Thus if η is also x independent,

$$\delta_\eta B(\epsilon,W)=B([\eta,\epsilon],W) . \tag{6.9}$$

This means, by (2.5), that

$$B([\eta,\epsilon],W)=0 . \tag{6.10}$$

The simplicity of $G_S^{(\nu)}$ implies that any element of the Lie algebra $\underline{G}$ of G with nonzero components only in the Lie algebra $\underline{G}_S^{(\nu)}$ of $G_S^{(\nu)}$ can be written as $[\eta,\epsilon]$ for some η and ϵ. Thus

$$B(\eta,W)=0 \tag{6.11}$$

for every $\eta\in\underline{G}_S^{(\nu)}$.

Now since A_α is a total divergence [cf. (A1)], we can write

$$\begin{aligned} \int d^4x\, A_\alpha(W) &= \int_{S^3} d_{\alpha\beta\gamma}\hat{a}^\mu_{\beta\gamma}(W)dS_\mu , \\ a_{\beta\gamma}(W) &= \partial_\mu \hat{a}^\mu_{\beta\gamma}(W) , \end{aligned} \tag{6.12}$$

where S^3 is the sphere at space-time infinity. On this S^3, only those gauge bosons associated with the unbroken subgroup H do not vanish, since the remaining gauge bosons are massive. Consequently, expression (6.11) along with (6.12) will imply restrictions on the subgroup H. Further analysis of these equations is carried out in Appendix B. It is shown there that these restrictions are the same as those stated in constraint 1. The sufficiency and necessity of these restrictions are also proven there.

We would like to emphasize that constraint 1 follows from the fact that the Bohr-Sommerfeld condition is a topological invariant; namely, the integral in Eq. (4.1) is invariant under small deformations of the curve C in the group G. Beyond the stability requirement for such deformations, Eq. (4.1) constrains the value of the integral since the phase factor is set equal to unity. We call these further conditions constraint 2, and reformulate them below for facility in computations.

It was remarked earlier that the group which acts faithfully on the space of fields (ψ,W) may not be $\otimes G_S^{(\nu)}\otimes G_A$, but rather its quotient by a discrete subgroup Z. To emphasize this distinction, we introduce the notation

$$\begin{aligned} \bar{G} &= \otimes G_S^{(\nu)}\otimes G_A , \\ G &= \bar{G}/Z . \end{aligned} \tag{6.13}$$

The elements of Z will be denoted by

$$(z_S,z_A),z_S\epsilon\otimes G_S^{(\nu)},\ z_A\in G_A . \tag{6.14}$$

A curve $\bar{g}(t)$ $[0\le t\le 1]$ in $\bar{G}$ becomes a curve $C[\bar{g}(t)]$ in G under the homomorphism $\bar{G}\to G$. It is closed in G if $\bar{g}(1)$ differs from $\bar{g}(0)$ by an element of Z, $\bar{g}(1)\bar{g}(0)^{-1}\in Z$. Further

$$C[\bar{g}(t)(z_S,z_A)]=C[\bar{g}(t)] . \tag{6.15}$$

The Bohr-Sommerfeld condition in this notation reads

$$\begin{aligned} &\int_{C[\bar{g}(t)]} B(s^{-1}ds,s^{-1}\circ W)=2\pi p , \\ &p=0,\pm1,\pm2,\ldots \end{aligned} \tag{6.16}$$

for

$$\begin{aligned} &\bar{g}(0)=\text{ identity} , \\ &\bar{g}(1)\in Z . \end{aligned}$$

The following consequence of (6.15) is useful: If

$$\bar{g}(0)=\text{ identity},\ \bar{g}(1)=(z_S,z_A) , \tag{6.17}$$

then

$$\begin{aligned} \left[\int_{C[\bar{g}(t)]}+\int_{C[\bar{g}(t)]}+\cdots\right]B(s^{-1}ds,s^{-1}\circ W) &= \left[\int_{C[\bar{g}(t)]}+\int_{C[\bar{g}(t)\bar{g}(1)]}+\int_{C[\bar{g}(t)\bar{g}(1)^2]}+\cdots\right]B(s^{-1}ds,s^{-1}s\circ W) \\ &= \int_{C[\bar{m}(t)]} B(s^{-1}ds,s^{-1}\circ W) . \end{aligned} \tag{6.18}$$

Here $\overline{m}(t)$ is a single curve in $\overline{G}$ from e to $\overline{g}(1)^n$ where n is the number of terms in the starting expression. The explicit form of $\overline{m}(t)$ is

$$\begin{aligned}\overline{m}(t)&=\overline{g}(nt),\ 0\leq t\leq 1/n\\ &=\overline{g}(nt-1)\overline{g}(1),\ \frac{1}{n}\leq t\leq\frac{2}{n}\\ &\ldots\\ &=\overline{g}(nt-p)\overline{g}(1)^p,\ \frac{p}{n}\leq t\leq\frac{p+1}{n}\ ,\\ p&=0,1,2,\ldots,(n-1)\ .\end{aligned} \tag{6.19}$$

The purpose of the second constraint is the simplification of (6.16). It may be stated as follows: *In order that (6.16) is fulfilled, it is necessary and sufficient to fulfill constraint 1 and*

$$\int_{C[\overline{g}(t)]}B(s^{-1}ds,s^{-1}\circ W)=2\pi p\ ,\quad p=0,\pm1,\pm2,\ldots\ , \tag{6.20}$$

for every curve of the form

$$\overline{g}(0)=\text{identity in }\overline{G}\ ,\qquad \overline{g}(1)=(z_s=\text{identity},z_A)\in Z\ . \tag{6.21}$$

Further we can regard $\overline{g}(t)$ in (6.20) as confined entirely to the Abelian part G_A of $\overline{G}$. (That is, on this curve, only the Abelian factors of the group element need vary.)

The necessity of this constraint is trivial. To prove sufficiency, consider a curve $\overline{g}(t)$ from identity of $\overline{G}$ to

$$\overline{g}(1)=(\hat{z}_S,\hat{z}_A)\in Z\ . \tag{6.22}$$

Let k be the period of $\hat{z}_s$ so that $\hat{z}_s^k$ is the identity. Then we can replace (6.16) by the equation where $C[\overline{g}(t)]$ is traversed k times and use (6.18),

$$2\pi pk=\left[\int_{C[\overline{g}(t)]}+\int_{C[\overline{g}(t)]}+\cdots\right]B(s^{-1}ds,s^{-1}\circ W)=\int_{C[\overline{m}(t)]}B(s^{-1}ds,s^{-1}\circ W)\ . \tag{6.23}$$

Here $\overline{m}(1)=(\hat{z}_S^k,\hat{z}_A^k)=(\text{identity},\hat{z}_A^k)$. Thus the projection into $\otimes G_S^{(\nu)}$ of the curve $\overline{m}(t)$ is closed. As $\otimes G_S^{(\nu)}$ is simply connected, we can deform $\overline{m}(t)$ so that this projection shrinks to a point. In this process, (i) $C[\overline{m}(t)]$ becomes a curve confined to the Abelian part, (ii) the value of the last integral in (6.23) does not change due to constraint 1 (which guarantees the integrability condition). Thus we arrive at constraint 2.

Restrictions on the QCD order parameter which we derived in Sec. V are a direct application of constraint 2 [cf. Eq. (5.13)]. In the next section we discuss several examples illustrating the constraints. However, since constraint 1 is a necessary condition for constraint 2, we have not found new consequences coming from constraint 2 for these examples.

Remark 1. Suppose the ψ sector condenses. Let S_A be the part of the effective action which represents the interaction of the ψ condensates with W due to the presence of the anomalies [cf. Sec. II]. Let $d\mu^c$ be the measure for the fermions in the condensed phase which contribute to the anomalies. Then $d\mu(\psi,W)$ and $d\mu^c\exp(iS_A)$ are supposed to transform in the same way. Therefore in the absence of constraint 1, the global group G does not act consistently on $d\mu^c\exp(iS_A)$. Further in the absence of such fermions, constraint 2 implies quantization rules of the Bohr-Sommerfeld type on S_A.

Remark 2. We can now clarify in what way the theory can be inconsistent if the Bohr-Sommerfeld condition is not satisfied. When the constraints are not fulfilled, the correlation functions of ψ in a given external field W are not well defined. For instance, the propagator is

$$\langle T[\psi(x)\overline{\psi}(y)]\rangle_W=\int d\mu(\psi,W)\psi(x)\overline{\psi}(y)e^{iS(\psi,W)} \tag{6.24}$$

up to a normalization. So

$$\begin{aligned}\langle T[\psi(x)\overline{\psi}(y)]\rangle_{g\circ W}&=\int d\mu(\psi,g\circ W)\psi(x)\overline{\psi}(y)e^{iS(\psi,g\circ W)}\\ &=\int d\mu(g\psi,g\circ W)(g\psi)(x)\overline{(g\psi)(y)}e^{iS(g\psi,g\circ W)}\\ &=e^{i\beta(g,W)}\int d\mu(\psi,W)(g\psi)(x)\overline{(g\psi)(y)}e^{iS(\psi,W)}\end{aligned} \tag{6.25}$$

since the classical action S is gauge invariant. Now $(g\psi)(x)$ is just the matrix $g(x)$ applied to the vector $\psi(x)$; the last integral is thus single valued in g. The single-valuedness of the propagator in W is thus controlled by $\exp[i\beta(g,W)]$. When g is varied over any closed loop, the propagator returns to its original value only if the constraints are fulfilled.

Remark 3. The expression for the anomalies is not unique. But if B and B' are two expressions for the anomalies, Bardeen[11] has shown that the latter can be obtained from the former by changing the measure $d\mu(\psi,W)$ to

$$d\mu'(\psi,W)=d\mu(\psi,W)e^{iS_C}\ ,$$
$$S_C=\int d^4x\,\mathscr{L}_C(W)\ , \tag{6.26}$$

for a suitable choice of $\mathscr{L}_C$. This is equivalent to adding the counterterm S_C to the canonical action. The important point for us is that $\mathscr{L}_C(W)$ is a polynomial function of W and its derivatives.[11] Thus $\exp(iS_C)$ is single valued and the group has a well-defined action on it. We therefore conclude that our results are not sensitive to the choice A_α for the anomalies.

VII. APPLICATIONS

A. The standard model

In the standard model (cf. Ref. 1), let us regard ψ as quarks and L as leptons. The gauged group is $SU(3)_C\times SU(2)\times U(1)$ and the unbroken subgroup is $SU(3)_C\times U(1)_Q$, where $U(1)_Q$ is generated by electric charge. In the quark sector, there is only one $H_S^{(\nu)}$, and it is the $\underline{3}+\underline{3}^*$ representation of $SU(3)_C$; it is also equal to the associated $G_S^{(\nu)}$. Since the latter is anomaly-free, constraint 1 is fulfilled. The $U(1)_Q$ anomaly in the ψ sector due to SU(2) gauge bosons has zero topological charge since SU(2) is broken, while there is no $U(1)_Q$ anomaly due to electromagnetism. Thus constraint 2 is also fulfilled. This means that there is a consistent effective Lagrangian description of ψ condensates in interaction with $SU(2)\times U(1)$ gauge bosons.

B. The first two hypotheses

Let us divide the fermions into two sectors ψ and L each transforming under a definite (possibly reducible) representation of the gauged group. The anomalies cancel between the two sectors.

We now make the following two hypotheses.

H1: *The ψ sector can decouple from the L sector only if the measure $d\mu(\psi,W)$ exists.*

If ψ condenses and L does not, then it is reasonable to expect that the condensed sector admits a consistent effective Lagrangian description. We turn this expectation into a second hypothesis.

H2: *If the ψ sector condenses and the condensates have little admixture from the fermions L, then there should exist a consistent effective Lagrangian description for the ψ condensates in interaction with gauge bosons which correctly reproduces the anomalies.*

If there are fermions in the condensed ψ sector, their contributions to the anomalies are of course to be included in the effective Lagrangian approach (cf. Remark 1, Sec. VI).

The existence of the effective-Lagrangian description and the existence of $d\mu(\psi,W)$ are equivalent. Both require the fulfillment of the two constraints.

We note that H2 is supposed to be valid even if the ψ condensates have admixtures from the gauge bosons (and perhaps other bosonic fields like Higgs fields). What is not allowed are admixtures from other fermionic sectors.

We now discuss QCD and the SU(5) model under these two hypotheses.

C. QCD and SU(5)

We consider QCD in the absence of weak and electromagnetic interactions. According to our conventions, the left quarks $q_L=\frac{1}{2}(1+\gamma_5)q$ transform as $\underline{3}+\underline{3}^*$ of $SU_C(3)$ while $q_R=\frac{1}{2}(1-\gamma_5)q$ are singlets. We denote the fields in $\underline{3}$ by ψ and the fields in $\underline{3}^*$ by L. There are anomalies in the ψ and L sectors, but they mutually cancel. The group G acting on ψ is the $\underline{3}$ representation of $SU(3)_C$.

We now consider the case where the ψ sector condenses. This is perhaps not very realistic, but it provides an illustration of the general ideas. In this case, by constraint 1, either H_S is trivial or it is SO(3), ψ and L transforming under the three-dimensional irreducible representation of this SO(3). (A model with such an unbroken symmetry group has been considered in the literature.[16]) If H_S is SO(3), then H_A is absent since SO(3) is a maximal subgroup of SU(3). If on the other hand H_S is trivial, H_A is contained in the maximal Abelian subgroup $U(1)\times U(1)$ of $SU(3)_C$.

These considerations are also valid if ψ is to decouple from L. There is however no simple way to introduce a large mass scale for ψ and attempt such a decoupling when the group is $SU(3)_C$. Let us therefore consider the case where the color group is SO(3). Then a color-invariant Majorana mass μ can be introduced for ψ and we can discuss the limit $\mu\to\infty$. Since SO(3) is anomaly-free, the ψ sector can very well decouple from the L sector without a breakdown of the symmetry (in so far as the two constraints are concerned).

In the SU(5) grand unification model (cf. Ref. 1), the left fermions ψ and L transform as $\underline{5}^*$ and $\underline{10}$ of SU(5), respectively, and the anomalies cancel between ψ and L. In this model, there can be no decoupling of the two sectors or condensation of either sector without a breakdown of SU(5). H_S can be trivial or isomorphic to SO(3), where under this SO(3), $\underline{5}^*$ reduces to the five-dimensional irreducible representation. If H_S is trivial, H_A is contained in

$$U(1)\times U(1)\times U(1)\times U(1)\subset SU(5)\ .$$

In the contrary case, its generators are a commuting set which commute with any element in the Lie algebra of SO(3).

Veneziano[17] has considered condensation of both sectors together in an SU(5) tumbling scheme. The Bohr-Sommerfeld condition is satisfied for this process.

D. Hypothesis 3

This third hypothesis resembles the one introduced by 't Hooft.[3]

Consider an anomaly-free gauge theory of fermion fields ψ with gauged group G', gauge bosons W' and flavor symmetry group G with no anomalies. The third hypothesis amounts to requiring a sort of stability of this theory when G is gauged as well and may be stated as follow:

H3. *The gauge theory of ψ and W' is a candidate for a physical theory only if the following is true: (i) The fermion functional measure is well defined when the flavor group G is gauged. (ii) If there are condensates in this theory, then there is a consistent effective Lagrangian description which correctly exhibits the anomalies when G is gauged.*

As noted by 't Hooft,[3] spectator fermions L can be added to cancel the G anomalies (which may arise when G is gauged) if it is so desired. Further the coupling constant e associated with G can be made arbitrarily small. The effects of the new interaction on the symmetry-breaking patterns of the undisturbed theory (with $e=0$) is therefore expected to be marginal.

We now reexamine QCD in the absence of electroweak interactions, assuming it fulfills (i) and (ii) above. The results are striking.

E. QCD once more

The unbroken flavor group G is

$$[\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R]/\mathrm{U}_A(1)$$

if we assume zero bare quark masses. Thus

$$G = G_S^{(L)} \otimes G_S^{(R)} \otimes \mathrm{U}(1)/Z \ , \tag{7.1}$$

where Z is a discrete group and

$$G_S^{(\nu)} = \mathrm{SU}(N_f)_\nu, \ \nu = L,R \ . \tag{7.2}$$

When G is gauged, there are anomalies associated with $G_S^{(\nu)}$. By constraint 1, it follows that *in the undisturbed theory ($e=0$), the flavor group $SU(N_f)_L \times SU(N_f)_R$ must be broken.*

Constraint 1 also gives information on the unbroken subgroup

$$H = H_S \otimes H_A / Z_H \ , \tag{7.3}$$

where Z_H is a possible discrete group.

To be concrete, let us discuss the case $N_f = 6$. Then there are three possible SU(2) [or SO(3)] subgroups of $\mathrm{SU}(6)_\nu$ of interest to us; we denote them by $\mathrm{SU}(2)_\nu^j$ ($j=\frac{1}{2}$, 1, $\frac{5}{2}$). The subgroup $\mathrm{SU}(2)_\nu^j$ is identified by the requirement that the $\underline{6}$ representation of $\mathrm{SU}(6)_\nu$ splits into a direct sum of spin-j representations under this $\mathrm{SU}(2)_\nu^j$. Now we can have (i) a trivial H_S or (ii) $H_S = \mathrm{SU}(2)_L^j \otimes \mathrm{SU}(2)_R^k$, or (iii) $H_S =$ an SU(2) or SO(3) subgroup of $\mathrm{SU}(2)_L^j \otimes \mathrm{SU}(2)_R^k$. [That is, H_S can be $\mathrm{SU}(2)_L^j$, $\mathrm{SU}(2)_R^k$, or the group generated by the sums of the corresponding generators of these two groups. These generators of $\mathrm{SU}(2)_L^j$ and $\mathrm{SU}(2)_R^k$ must of course fulfill similar commutation relations.]

The presence of H_S endows the quarks with some sort of generation structure. If $j=k=\frac{1}{2}$ in (ii), this generation structure is the usual one.

The possible H_A can be classified by routine methods once H_S is fixed. The generalization of these considerations to arbitrary N_f is also straightforward.

Note that the generations can be distinguished, for example, by the presence of a U(1) subgroup in H_A, the corresponding U(1) charge being distinct for each generation. (Such symmetries have been considered before.[18])

It is interesting that some sort of generation structure emerges from such formal considerations as ours. Unfortunately, these do not suggest any technique for the computation of mass differences between generations.

F. Composite models

't Hooft[3] has suggested that the Wigner realization of chiral symmetries may be used to guarantee the masslessness of composite fermions in gauge theories. Independently of the anomaly-matching conditions which he imposes, the Bohr-Sommerfeld condition provides further restrictions on possible models. As an example we consider the model proposed by Yamawaki and Yokota.[10]

In this model hypercolor (HC), color (C), and weak (W) gauge forces are grouped into the category of "flavor" interactions with respect to a subcolor (SC) gauged group $\mathrm{SU}(3)_{\mathrm{SC}}$. These forces are characterized by energy scales $\Lambda_{\mathrm{SC}}, \Lambda_{\mathrm{HC}}, \Lambda_{\mathrm{C}}$, and Λ_{W} which are assumed to fulfill $\Lambda_{\mathrm{SC}} \gg \Lambda_{\mathrm{HC}} \gg \Lambda_{\mathrm{C}}, \Lambda_{\mathrm{W}}$. The basic fermions all belong to a $\underline{3}$ representation of subcolor and in addition each of these fermions is assigned to a representation of the HC $=\mathrm{SU}(2)_{\mathrm{HC}}$, $C = \mathrm{SU}(3)_{\mathrm{C}}$, and $W = \mathrm{SU}(2)_{\mathrm{W}}$ groups, respectively. In an obvious notation the three representations are

(2,0,0), (0,3*,0), and (0,0,2).

At energy scales $\Lambda \gg \Lambda_{HC}$ one expects that the gauge interactions associated with $H = SU(2)_C \times SU(3)_C \times SU(2)_W$ will be weak and an approximate (classical) flavor symmetry $G = SU_L(7) \times SU_R(7) \times U_A(1)$ will emerge. In this case one may ask if the breaking $G \to H$ is consistent with the Bohr-Sommerfeld condition for G. Constraint 1 implies that as there are anomalies associated with global transformations in G and H is not of the required form, such a breaking is inconsistent with the global action of G.

ACKNOWLEDGMENTS

We thank P. P. Divakaran, G. Marmo, and S. Rajeev for discussions. This work was supported by the U.S. Department of Energy under Contract No. DE-AC02-76ER03533.

APPENDIX A

We assume without loss of generality that the field ψ is left handed. Let $\{L(\alpha)\}$ be a basis for the Lie algebra $\underline{G}$ of G, they are matrices which act on ψ. The anomalies can be written as[11,19]

$$A_\alpha(W) = \frac{1}{24\pi^2} \mathrm{Tr} L(\alpha) \epsilon^{\mu\nu\lambda\rho} \partial_\mu (W_\nu \partial_\lambda W_\rho + \tfrac{1}{2} W_\nu W_\lambda W_\rho) , \tag{A1}$$

where $W_\mu = W_\mu^\alpha L(\alpha)$. It is evident from this expression that $A_\alpha(W)$ transforms like the component of a vector in the adjoint representation when W_μ is transformed globally.

It is trivial to manipulate (A1) to the form

$$A_\alpha(W) = d_{\alpha\beta\gamma} a_{\beta\gamma}(W) ,$$

where

$$\begin{aligned} &d_{\alpha\beta\gamma} = \mathrm{Tr} L(\alpha) \{ L(\beta), L(\gamma) \} , \\ &a_{\beta\gamma} = \partial_\mu \hat{a}^\mu_{\beta\gamma} , \\ &\hat{a}^\mu_{\beta\gamma} = \frac{1}{48\pi^2} \mathrm{Tr} \epsilon^{\mu\nu\lambda\rho} \partial_\mu \{ W_\nu^\beta \partial_\lambda W_\rho^\gamma + \tfrac{1}{4} W_\nu^\beta [W_\lambda, W_\rho]^\gamma \} , \\ &[W_\lambda, W_\rho] \equiv [W_\lambda, W_\rho]^\gamma L(\gamma) . \end{aligned} \tag{A2}$$

Equations (B2) and (B3) in Appendix B follow easily from (A2).

APPENDIX B

Following the discussion of Sec. VI, we prove constraint 1 in this appendix.

If $\{t(a)\}$ is a basis for the Lie algebra $\underline{H}$ of H, then combining (6.11) and (6.12) [cf. (A2)] we get

$$B(\eta, W) = \int_{S^3} \mathrm{Tr} \eta \{ t(a), t(b) \} \hat{a}^\mu_{ab} dS_\mu . \tag{B1}$$

The form of $\hat{a}^\mu_{ab}$ on this S^3 is [cf. (A2)]

$$\hat{a}^\mu_{ab} = \mathrm{const} \epsilon^{\mu\nu\rho\sigma} \{ (h^{-1}\partial_\nu h)_a [h^{-1}\partial_\rho h, h^{-1}\partial_\sigma h]_b + a \leftrightarrow b \} , \tag{B2}$$

where $h(x) \in H$ and we have assumed that W_μ becomes a pure gauge $h^{-1}\partial_\mu h$ at infinity. The remaining notation is explained by

$$\begin{aligned} &h^{-1}\partial_\mu h \equiv t(a)(h^{-1}\partial_\mu h)_a , \\ &[h^{-1}\partial_\rho h, h^{-1}\partial_\sigma h] \equiv t(b)[h^{-1}\partial_\rho h, h^{-1}\partial_\sigma h]_b . \end{aligned} \tag{B3}$$

[Thus, in this notation, γ_a is the component of $\gamma \in \underline{H}$ in the direction $t(a)$.]

If h is Abelian, $\hat{a}^\mu_{ab}$ is zero. Therefore let $h \in H_S$. In that case, we find, from (6.11) and (A2.1),

$$\int_{S_3} \mathrm{Tr} \eta \{ T(a), T(b) \} \hat{a}^\mu_{ab} dS_\mu = 0 , \tag{B4}$$

where $\eta \in \underline{G}_S^{(\nu)}$ and $\{T(a)\}$ is a basis for the Lie algebra $\underline{H}_S$ of H_S.

To simplify (B4), we write

$$T(a) = \oplus T^{(\sigma)}(a) , \tag{B5}$$

where $\{T^{(\sigma)}(a)\}$ act only on the index α_σ in $\psi_{\alpha_1 \alpha_2 \cdots \alpha_\sigma \cdots}$ and span the Lie algebra $\underline{H}_S^{(\sigma)}$ of $H_S^{(\sigma)}$. Then, for $\sigma, \sigma' \neq \nu$, we have

$$\mathrm{Tr}\eta\{T^{(\sigma)}(a),T^{(\sigma')}(b)\}=\mathrm{Tr}\eta\,\mathrm{Tr}\{T^{(\sigma)}(a),T^{(\sigma')}(b)\} \tag{B6}$$

and this is zero since $\mathrm{Tr}\eta=0$. Similarly if $\sigma\neq\nu$.

$$\begin{aligned}\mathrm{Tr}\eta\{T^{(\sigma)}(a),T^{(\nu)}(b)\}&=\mathrm{Tr}T^{(\sigma)}(a)\{T^{(\nu)}(b),\eta\}\\&=\mathrm{Tr}T^{(\sigma)}(a)\mathrm{Tr}\{T^{(\nu)}(b),\eta\}\\&=0\ ,\end{aligned} \tag{B7}$$

since $\mathrm{Tr}T^{(\sigma)}(a)=0$ (for $H_S^{(\sigma)}$ is a representation of the semisimple H_S). Thus, (B4) becomes

$$\int_{S^3}\mathrm{Tr}\eta\{T^{(\nu)}(a),T^{(\nu)}(b)\}\hat{a}^{\mu}_{ab}dS_\mu=0\ . \tag{B8}$$

If $G_S^{(\nu)}$ is anomaly free, this equation is empty. Let us therefore consider the case where $G_S^{(\nu)}$ has anomalies. In this case, $G_S^{(\nu)}$ is a representation of SU(M) for some $M\geq 3$.[20,21] Further, the tensor structure of the anomalies and hence of the trace in (B8) is governed by its expression in the fundamental representation.[21,22] Thus let $\{L^{(\nu)}(\alpha)\}$ be a basis for the Lie algebra $\underline{G}_S^{(\nu)}$ of $G_S^{(\nu)}$ and let $\hat{L}^{(\nu)}(\alpha)$, $\hat{T}^{(\nu)}(a)$, and $\hat{\eta}$ be the representatives of $L^{(\nu)}(\alpha)$, $T^{(\nu)}(a)$, and η in the defining M-dimensional representation of SU(M). Then

$$\begin{aligned}&\mathrm{Tr}L^{(\nu)}(\alpha)\{L^{(\nu)}(\beta),L^{(\nu)}(\gamma)\}\\&\qquad=\xi\,\mathrm{Tr}\hat{L}^{(\nu)}(\alpha)\{\hat{L}^{(\nu)}(\beta),\hat{L}^{(\nu)}(\gamma)\}\end{aligned} \tag{B9}$$

(ξ being independent of α, β, γ) and so

$$\mathrm{Tr}\eta\{T^{(\nu)}(a),T^{(\nu)}(b)\}=\xi\,\mathrm{Tr}\hat{\eta}\{\hat{T}^{(\nu)}(a),\hat{T}^{(\nu)}(b)\}\ . \tag{B10}$$

By assumption, $G_S^{(\nu)}$ has anomalies which by (B9) means $\xi\neq 0$. Therefore

$$\int_{S^3}\mathrm{Tr}\hat{\eta}\{\hat{T}^{(\nu)}(a),\hat{T}^{(\nu)}(b)\}\hat{a}^{\mu}_{ab}dS_\mu=0\ . \tag{B11}$$

The analysis of this equation can proceed as follows: $\hat{\eta}$ is any traceless anti-Hermitian matrix and therefore

$$\int_{S^3}\epsilon^{\mu\nu\rho\sigma}\hat{h}^{-1}\partial_\nu\hat{h}\,\hat{h}^{-1}\partial_\rho\hat{h}\hat{h}^{-1}\partial_\sigma\hat{h}dS_\mu = \text{multiple of } \mathbb{1}, \tag{B12}$$

where we have used (B2) and $\hat{h}$ is the representative of h in the defining representation of SU(M). Now let $\mathcal{N}$ be an arbitrarily small neighborhood of a point p on S^3. We can choose $\hat{h}$ such that $\hat{h}^{-1}d\hat{h}$ has support in $\mathcal{N}$ and such that up to leading terms

$$\hat{h}^{-1}d\hat{h}=\sum_{i=1}^{3}\hat{T}^{(\nu)}(i)d\epsilon^i \tag{B13}$$

in $\mathcal{N}$. Here ϵ^i are arbitrary functions with support in $\mathcal{N}$ and $\hat{T}^{(\nu)}(i)$ $(i=1,2,3)$ are three *arbitrarily* selected linearly independent generators. For this choice, (B12) is equivalent to

$$\epsilon_{ijk}\hat{T}^{(\nu)}(i)\hat{T}^{(\nu)}(j)\hat{T}^{(\nu)}(k)=\Delta\mathbb{1}\ , \tag{B14}$$

where ϵ_{ijk} is the Levi-Civita symbol which comes from the volume on S^3 and Δ is a number.

The sufficiency of constraint 1 should now be obvious. For with constraint 1, there are only three linearly independent $\hat{T}^{(\nu)}$'s and they can be written as

$$\hat{T}^{(\nu)}(i)=\zeta_{ij}L_j\ , \tag{B15}$$

where

$$\det\zeta\neq 0\ , \tag{B16}$$

$$[L_i,L_j]=\epsilon_{ijk}L_k\ . \tag{B17}$$

Thus

$$\begin{aligned}&\epsilon_{ijk}\hat{T}^{(\nu)}(i)\hat{T}^{(\nu)}(j)\hat{T}^{(\nu)}(k)\\&\qquad=\tfrac{1}{2}\epsilon_{ijk}\hat{T}^{(\nu)}(i)[\hat{T}^{(\nu)}(j),\hat{T}^{(\nu)}(k)]\\&\qquad=\det\zeta L_iL_i\ .\end{aligned} \tag{B18}$$

Equation (B18) fulfills (B14) due to constraint 1.

We now prove the necessity of constraint 1. We first point out that it is in fact quite plausible. For (B14) implies that its left-hand side is an invariant for the group generated by $\{T^{(\nu)}(i)\}$, but it involves only three of the generators. This suggests that there are altogether only three $T^{(\nu)}(i)$, in which case $H_S^{(\nu)}$ is homomorphic to SU(2).

Let $\hat{H}_S^{(\nu)}$ denote the group generated by $\{\hat{T}^{(\nu)}(a)\}$ and let $\underline{\hat{H}}_S^{(\nu)}$ be its Lie algebra. If we write

$$[\hat{T}^{(\nu)}(a),\hat{T}^{(\nu)}(b)]=f_{abc}\hat{T}^{(\nu)}(c)\ , \tag{B19}$$

it follows from (B19) that

$$\epsilon_{ijk}f_{jka}\hat{T}^{(\nu)}(i)\hat{T}^{(\nu)}(a)=2\Delta\mathbb{1}\ . \tag{B20}$$

In what follows, i,j,k, will run over 1,2,3, and a,b,c will run over 1,2,... dimension of $\underline{\hat{H}}_S^{(\nu)}$. Below we introduce indices r,s,t whose values are restricted by $r,s,t\geq 4$. One of our tasks is to show that dimension of $\underline{\hat{H}}_S^{(\nu)}=3$ and therefore that there are no generators with indices r,s,t. We will also hereafter drop the superscript ν on the generators since it plays no role in the following.

Let

$$g_{ia}=\epsilon_{ijk}f_{jka}\ . \tag{B21}$$

Then

$$g_{ia}\hat{T}(i)\hat{T}(a)=2\Delta\mathbb{1}\ . \tag{B22}$$

This expression is also valid (with a possibly different Δ) if $\hat{T}(a)$ are replaced by

$$\hat{T}(a)+\delta_{ar}\hat{T}(r)\ , \text{ no summation on } r\ . \quad \text{(B23)}$$

Therefore,

$$g_{ir}\hat{T}(i)\hat{T}(r)=2\Delta'\mathbb{1}, \text{ no summation on } r \quad \text{(B24)}$$

for some Δ'. Replace a by i and r by j in (B23) and substitute the new $\hat{T}(i)$ in (B24) to find

$$g_{jr}\hat{T}(j)\hat{T}(r)=2\Delta''\mathbb{1}\ , \text{ no summation on } j \text{ and } r\ . \quad \text{(B25)}$$

Since we can assume without loss of generality that

$$\mathrm{Tr}\hat{T}(a)\hat{T}(b)=\delta_{ab}\ , \quad \text{(B26)}$$

we also have

$$\Delta''=0\ . \quad \text{(B27)}$$

There are three ways to fulfill (B25) and (B27): (a) $\hat{T}(j)\hat{T}(r)=0$, (b) $g_{jr}=0$, or (c) there is no generator with index r.

As for (a), along with its Hermitian conjugate equation, it implies that

$$[\hat{T}(j),\hat{T}(r)]=0\ . \quad \text{(B28)}$$

Further

$$[\hat{T}(i),\hat{T}(j)]=f_{ijk}\hat{T}(k)\ , \quad \text{(B29)}$$

$$[\hat{T}(r),\hat{T}(s)]=f_{rst}\hat{T}(t)\ , \quad \text{(B30)}$$

since the trace of the left-hand side of (B29) [(B30)] with $\hat{T}(r)$ [$\hat{T}(i)$] is zero by (a). Thus

$$\underline{\hat{H}}_S^{(\nu)}=K_1\oplus K_2\ , \quad \text{(B31)}$$

where the Lie algebras K_1 and K_2 have bases $\{\hat{T}(j)\}$ and $\{\hat{T}(r)\}$, respectively. But a product like $\hat{T}(j)\hat{T}(r)$ where the factors are in distinct Lie algebras cannot be zero unless one of the factors is zero. We chose $\hat{T}(i)$ in Sec. V so that they are linearly independent and hence not zero. So (a) means $\hat{T}(r)=0$ which is equivalent to (c).

Next, to analyze (b), introduce the 3×3 antisymmetric matrices $L(i)$, $F(r)$ where

$$L(i)_{jk}=\epsilon_{ijk}\ , \quad \text{(B32)}$$

$$F(r)_{jk}=f_{jkr}\ . \quad \text{(B33)}$$

Then (b) says

$$\mathrm{Tr}L(i)F(r)=0\ . \quad \text{(B34)}$$

Since $L(i)$ is a basis for 3×3 antisymmetric matrices,

$$F(r)=0\ . \quad \text{(B35)}$$

We thus conclude from (B19) that $\{\hat{T}(i)\}$ is a basis for a subalgebra. Since the choice of these three $\hat{T}(i)$ was arbitrary, (b) implies that any three linearly independent elements in $\underline{\hat{H}}_S^{(\nu)}$ is a basis for a subalgebra. This is not possible without (c). For without (c), the rank of $\underline{\hat{H}}_S^{(\nu)}$ is larger than 1 and in the Cartan notation, the generators E_α, $E_{-\alpha}$, and $H'\neq \text{const}\times\alpha^i H_i$ do not form a basis for a subalgebra.

Thus we are left with only the choice (c). $\underline{\hat{H}}_S^{(\nu)}$ is therefore of dimension 3. Since it is also semisimple, it is necessarily isomorphic to $\underline{\mathrm{SU}(2)}$, the Lie algebra of SU(2).

Now $\mathrm{Tr}\hat{T}(i)[\hat{T}(j),\hat{T}(k)]$ is totally antisymmetric in i,j,k and hence proportional to ϵ_{ijk}. Since it is also proportional to f_{ijk} by (B26), (B(22) reduces to

$$\hat{T}(i)\hat{T}(i)=\text{const}\mathbb{1} \quad \text{(B36)}$$

[in view of (B21)] and this is true in the whole representation space of SU(M). It follows that on reduction, the fundamental representation of SU(M) contains the same irreducible representation of $\hat{H}_S^{(\nu)}$ with a suitable multiplicity.

---

[1]P. Langacker, Phys. Rep. 72, 185 (1981).

[2]C. Rosenzweig, J. Schechter, and C. G. Trahern, Phys. Rev. D 21, 3388 (1980); P. Di Vecchia and G. Veneziano, Nucl. Phys. B171, 253 (1980); P. Nath and R. Arnowitt, Phys. Rev. D 23, 473 (1981); E. Witten, Ann. Phys. (N.Y.) 128, 363 (1980); A. Aurilia, Y. Takahashi, and P. Townsend, Phys. Lett. 95B, 265 (1980).

[3]G. 't Hooft, in *Recent Developments in Gauge Theories,* proceedings of the NATO Advanced Study Institute, Cargèse, 1979, edited by G. 't Hooft *et al.* (Plenum, New York, 1980); T. Banks, S. Yankielowicz, and A. Schwimmer, Phys. Lett. 96B, 67 (1980); Y. Frishman, A. Schwimmer, T. Banks, and S. S. Yankielowicz, Nucl. Phys. B177, 157 (1981); S. Coleman and B. Grossman, *ibid.* B203, 205 (1982).

[4]J. Wess and B. Zumino, Phys. Lett. 37B, 95 (1971).

[5]H. Harari and N. Seiberg, Phys. Lett. 102B, 263 (1981).

[6]S. Weinberg, Phys. Lett. 102B, 401 (1981).

[7]H. P. W. Gottlieb and S. Marculescu, Nucl. Phys. B49, 633 (1972).

[8]S. L. Adler and W. A. Bardeen, Phys. Rev. 182, 1515 (1969); W. A. Bardeen, Nucl. Phys. B75, 246 (1974).

[9]D. R. T. Jones and J. P. Leveille, Phys. Lett. 109B, 449 (1982).

[10] K. Yamawaki and T. Yokota, Phys. Lett. 113B, 293 (1982).

[11] W. A. Bardeen, Phys. Rev. 184, 1848 (1969); A. P. Balachandran, G. Marmo, V. P. Nair, and C. G. Trahern, Phys. Rev. D 25, 2713 (1982).

[12] S. N. Vergeles, quoted in A. A. Migdal, Phys. Lett. 81B, 37 (1979); K. Fujikawa, Phys. Rev. Lett. 44, 1733 (1980); Phys. Rev. D 21, 2848 (1980); A. P. Balachandran, G. Marmo, V. P. Nair, and C. G. Trahern, Phys. Rev. D 25, 2713 (1982).

[13] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 571 ff.

[14] S. Raby, S. Dimopoulos, and L. Susskind, Nucl. Phys. B169, 373 (1980).

[15] A. P. Balachandran, V. P. Nair, and C. G. Trahern, in Proceedings of the Third Annual Montreal-Rochester-Syracuse-Toronto Meeting, 1981 (unpublished), p. 81.

[16] R. Slansky, T. Goldman, and G. L. Shaw, Phys. Rev. Lett. 47, 887 (1981); A. P. Balachandran, V. P. Nair, and C. G. Trahern, Nucl. Phys. B196, 413 (1982).

[17] G. Veneziano, Phys. lett. 102B, 139 (1981).

[18] A. Davidson, M. Koça, and K. C. Wali, Phys. Rev. Lett. 43, 92 (1979); Phys. Rev. D 20, 1195 (1979).

[19] D. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972).

[20] H. Georgi and S. L. Glashow, Phys. Rev. D 6, 429 (1972); F. Gürsey, P. Ramond, and P. Sikivie, Phys. Lett. 60B, 77 (1975).

[21] S. Okubo, Phys. Rev. D 16, 3528 (1977).

[22] J. Banks and H. Georgi, Phys. Rev. D 14, 1159 (1976).