Color-singlet confinement in chromostatics

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By constructing explicit projection operators for the SU(n) configurations of quark-antiquark ($q\bar{q}$) and quark-quark (qq) systems, we demonstrate that confinement occurs in the color-singlet state, while the (n^2-1)-plet of ($q\bar{q}$) and both the $(1/2)n(n+1)$-plet and the $(1/2)n(n-1)$-plet of qq have infinite energy and are hence decoupled from the spectrum.

Recently, a promising formalism has been developed[1,2] describing heavy quark-antiquark ($q\bar{q}$) confinement based on an algebraic representation of static quark sources[3,4] together with the use of the one-loop, renormalization-group improved Lagrangian as an effective Lagrangian.[5–7] The principal justification for use of the latter is that it implies the correct trace anomaly.[8] The above formalism apparently implies the following[1,2]: confinement via flux-tube formation (presumably a linear potential) in the color-singlet channel, whereas color-nonsinglet channels [$\underline{8}$ for $q\bar{q}$, $\underline{\bar{3}}$ or $\underline{6}$ for qq in SU(3)] have positive-infinite energy.

Here we wish to fill in a major gap in the previous argument by explicitly demonstrating that it is the color-singlet, and only the color-singlet, channel in which confinement occurs. We do this through a modified action expression based on use of the singlet projection operator. In so doing it will also become clear that the SU(2)$\times$U(1) "pseudocolor" symmetry has in fact no bearing on the confinement issue since the singlet projector annihilates the U(1) algebra. (It has already been noticed[2] that pseudocolor rotations fail to preserve the current commutation relations.)

We start by recapitulating the pseudocolor algebra. This algebra is based on the SU(n) outer product

$$[u,v]^A=\frac{i}{2}f^{ABC}(u^Bv^C+v^Cu^B)\,, \tag{1}$$

where $A,B,C,\ldots=1,2,3,\ldots,n^2-1$. In terms of the orthonormal set $\{\tilde{e}_i^A\}$,

$$\begin{aligned}
\tilde{e}_1^A&=\frac{2}{n}(Q_1^A\times 1+1\times\bar{Q}_2^A)\,,\\
\tilde{e}_2^A&=\frac{4}{n}d^{ABC}Q_1^B\times\bar{Q}_2^C-\frac{4}{n^2}(Q_1^A\times 1-1\times\bar{Q}_2^A)\,,\\
\tilde{e}_3^A&=\frac{4}{n}f^{ABC}Q_1^B\times\bar{Q}_2^C\,,\\
\tilde{e}_4^A&=-(2/n^2)(n^2-4)^{1/2}(Q_1^A\times 1-1\times\bar{Q}_2^A)\\
&\quad-(8/n)(n^2-4)^{-1/2}d^{ABC}Q_1^B\times\bar{Q}_2^C\,,
\end{aligned} \tag{2}$$

where ($a=0,1,2,\ldots,n^2-1$)

$$Q^a=\frac{\lambda^a}{2},\quad \bar{Q}^a=-\frac{\lambda^{a*}}{2},\quad \lambda^0=\left[\frac{2}{n}\right]^{1/2}1 \tag{3}$$

represent the two static quarks, we find[2,9]

$$\begin{aligned}
&[\tilde{e}_i,\tilde{e}_j]^A=i\epsilon_{ijk}\tilde{e}_k^A,\quad \{i,j,k\}=\{1,2,3\}\\
&[\tilde{e}_i,\tilde{e}_4]^A=0\,.
\end{aligned} \tag{4}$$

The qq algebra $\{e_i^A\}$ is obtained by the replacements

$$\begin{aligned}
&Q_1^A\times\bar{Q}_2^B\to Q_1^A\times Q_2^B\,,\\
&1\times\bar{Q}_2^A\to 1\times Q_2^A\,,\\
&Q_1^A\times 1\to -Q_1^A\times 1\,,
\end{aligned} \tag{5}$$

as well as a reversal of sign of $\tilde{e}_1^A$. The normalization of these vectors is given by

$$\mathrm{Tr}\,\tilde{e}_i^A\tilde{e}_j^B=(4/n)\delta_{ij}\delta^{AB}\,. \tag{6}$$

The dynamical effects of the static charges are described in the development of Refs. 1–4 by the effective Euclidean action

$$W=\int d^3x\left[\mathscr{L}_{\rm eff}(F^2)-\mathscr{L}_{\rm eff}(\kappa^2)\right]-\frac{1}{n^2}\mathrm{Tr}\int d^3x\, c_\mu^A(x)j_\mu^A(x)\,, \tag{7}$$

where[5–7]

$$\mathscr{L}_{\rm eff}(F^2)=\tfrac{1}{8}b_0F^2(\ln F^2/\kappa^2-1)\,, \tag{8}$$

b_0 being the first-order β-function coefficient, and

$$F^2=\frac{1}{n^2}\mathrm{Tr}[E^{A,j}E^{A,j}+B^{A,j}B^{A,j}] , \tag{9}$$

the spatial index j running from 1 to 3. In terms of the potentials, the field strengths are

$$\begin{aligned} E^{A,j}&=-\partial_j c^{A,0}+i[c^j,c^0]^A , \\ B^{A,j}&=\epsilon^{jkl}\left[\partial_k c^{A,l}-\frac{i}{2}[c^k,c^l]^A\right] . \end{aligned} \tag{10}$$

(Time derivatives do not appear since we are describing statics.) Here all colored quantities are to be expanded in the pseudocolor basis, $\{\tilde{e}_i\}$ for $q\bar{q}$, $\{e_i\}$ for qq.

However, solving the resulting equations of motion for the prescribed source distribution,

$$\begin{aligned} j_\mu^A=\delta_{\mu 0}[Q_1^A\times 1\delta(x-x_1) \\ +1\times\bar{Q}_2^A\delta(x-x_2)] , \end{aligned} \tag{11}$$

does not describe either the $\underline{1}$ or the (n^2-1)-plet of the $n\times\bar{n}$ $q\bar{q}$ state. SU(2)$\times$U(1) gauge transformations, which are certainly not invariances of the underlying SU(n) theory, mix these representations in some nontrivial way. In order to be able to discuss the singlet-state configuration, the presumed physical state, separately, let us consider for a moment how the quark states enter the problem in this formalism. The outer product space of two quarks or of a quark and an antiquark forms the basis for the operators, e.g.,

$$\begin{aligned} \langle q_i\bar{q}_k\,|\,j^0\,|\,q_j\bar{q}_l\rangle &= \delta(x-x_1)\frac{\lambda_{ij}}{2}\times\delta_{kl} \\ &\quad+\delta(x-x_2)\delta_{ij}\times\left[-\frac{\lambda_{kl}^*}{2}\right] \\ &=j^0_{ij;kl} . \end{aligned} \tag{12}$$

The trace operation represents contraction with δ_{ij} and δ_{kl}, and a sum over *all* states. The mean-field potential is determined by the configuration that minimizes W. It is not obvious, in this formalism, what the group-theoretic nature of that configuration is. What we suggest here is an alternate, less ambiguous procedure for identifying the various physical states by inserting projection operators in the traces of (7).[10] It is easy to construct these operators: In the $q\bar{q}$ space the singlet projector is

$$P=-(2/n)Q_1^a\times\bar{Q}_2^a . \tag{13}$$

This operator has the required properties, which easily follow from the algebra given in Ref. 3:

$$\begin{aligned} &P^2=P , \\ &P(Q_1^a\times 1+1\times\bar{Q}_2^a)=0 , \\ &\mathrm{Tr}P=1 . \end{aligned} \tag{14}$$

Its complement, $1-P$, evidently projects out the (n^2-1)-plet.

For qq, the projection operator for the $\frac{1}{2}n(n+1)$-plet is

$$P'=\tfrac{1}{2}1+Q_1^a\times Q_2^a , \tag{15}$$

with the properties

$$\begin{aligned} &P'^2=P' , \\ &\mathrm{Tr}P'=\tfrac{1}{2}n(n+1) , \\ &\mathrm{Tr}(1-P')=\tfrac{1}{2}n(n-1) . \end{aligned} \tag{16}$$

To describe the chromostatics of $q\bar{q}$ in the singlet state, we insert P into the trace defining F^2, Eq. (9), and into the source term in (7) and remove the now superfluous $1/n^2$ factors. In the source term we must understand symmetric multiplication to maintain Hermiticity:

$$\frac{1}{n^2}\mathrm{Tr}\int d^3x\, c_\mu^A j_\mu^A\to\mathrm{Tr}\int d^3x\, c_\mu^A\tfrac{1}{2}(Pj_\mu^A+j_\mu^A P) . \tag{17}$$

The effect of the singlet projector on the basis vectors is easily seen to be

$$\begin{aligned} &P(\tilde{e}_2+i\tilde{e}_3)^A=(\tilde{e}_2+i\tilde{e}_3)^A , \\ &P(\tilde{e}_2-i\tilde{e}_3)^A=P\tilde{e}_1^A=P\tilde{e}_4^A=0 . \end{aligned} \tag{18}$$

[This apparently demonstrates the irrelevance of the SU(2)$\times$U(1) pseudocolor gauge symmetry to the confinement issue, since $\tilde{e}_4^A$ is the U(1) algebraic element.] Then from the expansion of the charges,

$$\begin{aligned} Q_1^A\times 1&=\frac{n}{4}\tilde{e}_1^A-\tfrac{1}{2}\tilde{e}_2^A-\tfrac{1}{4}(n^2-4)^{1/2}\tilde{e}_4^A , \\ 1\times\bar{Q}_2^A&=\frac{n}{4}\tilde{e}_1^A+\tfrac{1}{2}\tilde{e}_2^A+\tfrac{1}{4}(n^2-4)^{1/2}\tilde{e}_4^A , \end{aligned} \tag{19}$$

we find for the source term (17)

$$-\tfrac{1}{4}Nc_2^0(x)[\delta(x-x_1)-\delta(x-x_2)] , \tag{20}$$

where

$$N=4(n^2-1)/n . \tag{21}$$

The field part of the action involves

$$\begin{aligned} F^2\to\tilde{F}^2&=\mathrm{Tr}P(E^2+B^2) \\ &=\frac{N}{2}(E_2^jE_2^j+E_3^jE_3^j+B_2^jB_2^j+B_3^jB_3^j) . \end{aligned} \tag{22}$$

A symmetrical set of variables is

$$c^{\mu}_{\pm}=c^{\mu}_{2}\pm ic^{\mu}_{3}\ , \tag{23}$$

and similarly for E and B. Then the field equations obtained by extremizing W are

$$\begin{aligned}&(\partial_j \pm ic^j_1)\epsilon E^j_{\pm}=-\tfrac{1}{2}[\delta(x-x_1)-\delta(x-x_2)]\ ,\\ &\epsilon^{jkl}(\partial_k\pm ic^k_1)\epsilon B^l_{\pm}=\pm ic^0_1\epsilon E^j_{\pm}\ ,\\ &E^j_+c^j_- - E^j_-c^j_+=0\ ,\\ &E^j_+c^0_- - E^j_-c^0_+=\epsilon^{jkl}(c^k_-B^l_+ - c^k_+B^l_-)\ ,\end{aligned} \tag{24}$$

where

$$\begin{aligned}&E^j_{\pm}=-\partial_j c^0_{\pm}\pm ic^0_1c^j_{\pm}\mp ic^j_1c^0_{\pm}\ ,\\ &B^j_{\pm}=\epsilon^{jkl}(\partial_k c^l_{\pm}\pm ic^k_1c^l_{\pm})\ ,\end{aligned} \tag{25}$$

and

$$\begin{aligned}\epsilon&=\partial\mathscr{L}_{\rm eff}(\tilde F^2)/\partial(\tfrac{1}{2}\tilde F^2)\\ &=\tfrac{1}{4}b_0\ln\left[\frac{N}{2}(E^j_+E^j_- + B^j_+B^j_-)\Big/\kappa^2\right]\ .\end{aligned} \tag{26}$$

Now it is straightforward to adapt Adler's argument[1,2] to this description of the static singlet system. We first minimize W with respect to B variations,

$$\begin{aligned}\tilde{\mathscr{L}}_{\rm eff}\left[\frac{N}{2}E^j_+E^j_-\right]&=\min_{B^2}\left\{\mathscr{L}_{\rm eff}\left[\frac{N}{2}(E^j_+E^j_- + B^j_+B^j_-)\right]-\mathscr{L}_{\rm eff}(\kappa^2)\right\}\\ &=\begin{cases}0 \ \text{ if } \dfrac{N}{2}E^j_+E^j_-\le\kappa^2,\\ \mathscr{L}_{\rm eff}\left[\dfrac{N}{2}E^j_+E^j_-\right]-\mathscr{L}_{\rm eff}(\kappa^2) \ \text{ if } \dfrac{N}{2}E^j_+E^j_-\ge\kappa^2,\end{cases}\end{aligned} \tag{27}$$

since in the first case $B^j_+B^j_-$ fills in to bring $\mathscr{L}_{\rm eff}$ down to its minimum value, which occurs at

$$\frac{N}{2}(E^j_+E^j_- + B^j_+B^j_-)=\kappa^2\ , \tag{28}$$

and in the second, $\mathscr{L}_{\rm eff}$ is minimized by setting

$$B^j_+B^j_-=0\ . \tag{29}$$

Thus we see "bag" formation—the electric field is automatically self-quenching.

Inside the bag where (29) holds, (24) implies $c^0_1=0$. Now combining (24) and (25) we find

$$E^j_+E^j_-=\partial_jc^0_+\partial_jc^0_- - (c^j_1)^2c^0_+c^0_-\ . \tag{30a}$$

It is consistent with the field equations to further set

$$c^j_1=0\ ; \tag{30b}$$

that is, this restriction forms an upper bound to the action, which is at least a local extremum:

$$\begin{aligned}W_{\min}[E^j_+E^j_-]&\le W_{\min}[E^j_+E^j_-]_{c^j_1=0}\\ &\equiv W_{\min}[\partial_jc^0_+\partial_jc^0_-]\ ,\end{aligned} \tag{31a}$$

which in turn implies ($V_{\rm MF}=-W_{\min}$)

$$V_{\rm MF}[E^j_+E^j_-]\ge V_{\rm MF}[\partial_jc^0_+\partial_jc^0_-]\ . \tag{31b}$$

Minimizing $W[\partial_jc^0_+\partial_jc^0_-]$ leads to Gauss's law

$$\partial_jD_j=j^0\ , \tag{32a}$$

where

$$D_j=\frac{N}{2}\tilde\epsilon E_j,\ \ E_j=-\partial_jc^0_2\ , \tag{32b}$$

$$\tilde\epsilon=\begin{cases}\tfrac{1}{4}b_0\ln\dfrac{N}{2}E^2/\kappa^2, & \dfrac{N}{2}E^2\ge\kappa^2,\\ 0, & \dfrac{N}{2}E^2\le\kappa^2,\end{cases} \tag{32c}$$

and

$$j^0=-\frac{N}{4}[\delta(x-x_1)-\delta(x-x_2)]\ . \tag{32d}$$

Since it may be easily shown that[1]

$$\tilde{\mathscr{L}}_{\rm eff}\left[\frac{N}{2}E^2\right]-ED\le-(2/N)^{1/2}\kappa D\ , \tag{33}$$

the flux conservation argument of 't Hooft[11] leads to the following estimate for the mean-field potential:

$$V_{\rm MF}\ge\tfrac{1}{2}\kappa(N/2)^{1/2}R\ , \tag{34}$$

R being the quark separation. Presumably, exact integration of (32a) leads to a linear potential similar to (34).[12] Note that if we take κ as given by recent estimates for the gluon condensate,[13]

$$\begin{aligned}\kappa^2=\langle F^2\rangle&=2\pi^2(0.014\ {\rm GeV}^4)\\ &=0.28\ {\rm GeV}^4\ ,\end{aligned} \tag{35}$$

we find for the bound (34)

$$V_{\rm MF}\ge(0.6\ {\rm GeV}^2)R\ , \tag{36}$$

to be compared with the experimentally determined[14] slope of 0.17–0.24 GeV^2. The comparison here is not particularly striking; however, we remind the reader that K here may differ significantly from the condensate value, and that the linear region may not be relevant to charmonium spectroscopy.[12]

We have shown then that this model, which we hope embodies the essential physics, implies confinement in the singlet channel. On the other hand, the (n^2-1)-plet for $q\bar{q}$ and both channels for qq are unphysical because they are infinite-energy configurations. For example, for the (n^2-1)-plet up to some multiplicative factor,

$$E^2 \to \frac{1}{n^2}\mathrm{Tr}(1-P)E^{A,j}E^{A,j} = N[E_1{}^2+E_4{}^2+\tfrac{1}{2}E_+E_-] \equiv E_8{}^2 , \tag{37}$$

while the effective charges at the two quarks are (dot denotes symmetric multiplication)

$$(1-P)\cdot Q_1^A = \frac{n}{4}\tilde{e}_1^A - \tfrac{1}{4}\tilde{e}_2^A - \tfrac{1}{4}(n^2-4)^{1/2}\tilde{e}_4^A ,$$
$$(1-P)\cdot \bar{Q}_2^A = \frac{n}{4}\tilde{e}_1^A + \tfrac{1}{4}\tilde{e}_2^A + \tfrac{1}{4}(n^2-4)^{1/2}\tilde{e}_4^A , \tag{38}$$

which are neither parallel nor antiparallel. The total flux at infinity, as a consequence, does not vanish, and hence the canonical energy density[5]

$$\theta_{00} = \epsilon(E_8{}^2)E_8{}^2 - \mathscr{L}_{\mathrm{eff}}(E_8{}^2) \tag{39}$$

is infrared divergent when integrated over all space.[2] As for the two SU(n) configurations of qq, selected by P', Eq. (15), we find

$$P'(e_1+ie_3)^A = (e_1+ie_3)^A ,$$
$$P'(e_1-ie_3)^A = 0 , \tag{40}$$
$$P'\left[\left[\frac{n+2}{n-2}\right]^{1/2} e_2+e_4\right]^A = \left[\left[\frac{n+2}{n-2}\right]^{1/2} e_2+e_4\right]^A ,$$
$$P'\left[e_2-\left[\frac{n+2}{n-2}\right]^{1/2} e_4\right]^A = 0 ,$$

from which it follows that

$$P'\cdot Q_1^A = \frac{n}{8}e_1^A + \frac{n+2}{8}e_2^A + \tfrac{1}{8}(n^2-4)^{1/2}e_4^A , \tag{41}$$
$$P'\cdot Q_2^A = -\frac{n}{8}e_1^A + \frac{n+2}{8}e_2^A + \tfrac{1}{8}(n^2-4)^{1/2}e_4^A ,$$

for the $\frac{1}{2}n(n+1)$-plet, and

$$(1-P')\cdot Q_1^A = \frac{n}{8}e_1^A - \frac{n-2}{8}e_2^A + \tfrac{1}{8}(n^2-4)^{1/2}e_4^A ,$$
$$(1-P')\cdot Q_2^A = -\frac{n}{8}e_1^A - \frac{n-2}{8}e_2^A + \tfrac{1}{8}(n^2-4)^{1/2}e_4^A , \tag{42}$$

for the $\frac{1}{2}n(n-1)$-plet. Again, neither pair of charges is either parallel or antiparallel, and the previous argument indicates decoupling of these states by virtue of their infrared-infinite energy.

This algebraic, effective-action approach thus seems effective in describing the statics of the two-quark system. Our next challenge is to apply it to the three-quark system. There, does confinement indeed occur in the singlet channel? The affirmative answer is given in Ref. 15.

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