PHYSICAL REVIEW D VOLUME 27, NUMBER 6 15 MARCH 1983

Path-integral formulation of high-energy scattering in quantum field theories

Hung Cheng and Xiquan Zhu*
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 12 October 1982)

We give a formulation of high-energy scattering via path integrals. This nonperturbative formulation allows the approximate treatment of both fixed-angle scattering as well as small-angle scattering. The only approximation made is replacing the summation of the paths of an external particle by the contribution of the classical path—a straight line for small-angle scattering and two straight lines joined at the origin at an angle for fixed-angle scattering. In this way, the scattering amplitude is factorized into a product of two amplitudes: the amplitude for the interaction between the external particles and the amplitude for vacuum-to-vacuum transition in the presence of the external fields produced classically by the external particles. We show that all of the exactly calculable factors (the eikonal formula, the Sudakov form factor, and the energy-dependent factor of multiphoton exchange for fixed-angle scattering) belong to the first amplitude and are easily produced by a semiclassical treatment. The second amplitude is fully quantum mechanical, and no justified approximation has been found. In the case of small-angle scattering, we deduce, with this formulation, the principle of the equivalence of phase space. For the case of fixed-angle scattering, we find that there are three time scales: ω^{-1}, λ, and ω, where ω is the incident c.m. energy and λ is the photon mass.

I. INTRODUCTION

In this paper, we give a formulation of high-energy scattering via path integrals.[1] This formulation has the advantage of being nonperturbative. Furthermore, it treats fixed-angle high-energy scattering just as easily as small-angle high-energy scattering.

So that there be no misunderstanding, let us state at the beginning the approximations involved. Basically, the behavior of the external particles during collision, which lasts a relatively short time compared to the time scales of these particles, is semiclassical. Therefore, instead of summing over all paths possible for these particles, we take into account only the classical paths. For example, for small-angle scattering, we make the approximation that the incident particles travel along only straight paths. For fixed-angle scattering, we make the approximation that the dominant contribution comes from the classical path which is roughly two straight lines joined at the origin at an angle θ, which is the scattering angle. It must be emphasized, however, that the process of pair creation and annihilation should be taken care of throughout the lifetime of the pair. We shall not make any approximations on that.

This semiclassical approximation treats the external particles as point particles and completely ignores the fact that a hadron is a bound state. We can only hope that if and when the bound-state problem is solved, the hadron-hadron scattering amplitude can be constructed from the particle-particle scattering amplitude, in very much the same way that the deuteron-deuteron scattering amplitude is constructed from the nucleon-nucleon scattering amplitude in nuclear physics.

We shall begin with the simplest process in the simplest gauge field theories: elastic two-body scattering in scalar QED. Scattering in QCD will be treated in a future paper. To avoid infrared divergence we give the photon a mass λ. Let the incoming and the outgoing momenta of the two charged bosons be p_i and p_i', $i=1,2$, respectively. Then the four-point Green's function is given by the path integral

$$G(x_1',x_2';x_1,x_2)=\langle 0\,|\,T(\phi^\dagger(x_1')\phi^\dagger(x_2')\phi(x_1)\phi(x_2))\,|\,0\rangle$$
$$=\int \mathscr{D}A^\mu \mathscr{D}\phi\, \mathscr{D}\phi^* \phi^*(x_1')\phi^*(x_2')\phi(x_1)\phi(x_2)\exp\left[i\int \mathscr{L}\, d^4x\right], \tag{1.1}$$

where T denotes the time-ordered product and

$$\mathscr{L}=-\tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)+\tfrac{1}{2}\lambda^2A^2+(D_\mu\phi)^*(D^\mu\phi)-m^2\phi^*\phi \tag{1.2}$$

with

$$D_\mu\equiv\partial_\mu+ieA_\mu\ . \tag{1.3}$$

Also, $|0\rangle$ denotes the state of the vacuum. Thus, (1.1) involves integrations over the initial and the final ground-state wave functions of all harmonic oscillators involved.

Next, we carry out the functional integration over ϕ and ϕ^*, obtaining

$$G(x_1',x_2';x_1,x_2)=\int \mathscr{D}A^\mu \det(-D_\mu^*D^\mu-m^2+i\epsilon)^{-1}\exp\left\{\frac{i}{2}\int[\lambda^2A^2-(\partial_\mu A_\nu)(\partial^\mu A^\nu)]d^4x\right\}$$
$$\times[\Delta_A(x_1',x_1)\Delta_A(x_2',x_2)+\Delta_A(x_2',x_1)\Delta_A(x_1',x_2)]\ , \tag{1.4}$$

where Δ_A is the Green's function for a charged boson in the electromagnetic field A_μ. Specifically,

$$(-D_\mu D^\mu-m^2)\Delta_A(x,x')=\delta^{(4)}(x-x') \tag{1.5}$$

with Feynman's boundary conditions. The S-matrix element for elastic scattering is related to G by

$$S(p_1',p_2';p_1,p_2)=\lim_{\substack{t_1,t_2\to-\infty\\ t_1',t_2'\to\infty}}\prod_{i=1}^{2}(4E_iE_i')^{1/2}\int\prod_{i=1}^{2}(d^3x_id^3x_i'e^{-ip_i\cdot x_i+ip_i'\cdot x_i'})G(x_1',x_2';x_1,x_2)\ . \tag{1.6}$$

Similarly, the S matrix $\mathscr{S}_A$ for the Klein-Gordon equation with the external field A_μ is related to Δ_A by

$$\mathscr{S}_A(p',p)=\lim_{\substack{t\to-\infty\\ t'\to\infty}}(4EE')^{1/2}\int d^3x\,d^3x'e^{-ip\cdot x+ip'\cdot x'}\Delta_A(x',x)\ . \tag{1.7}$$

Substituting (1.4) into (1.6) and making use of (1.7), we get

$$S(p_1',p_2';p_1,p_2)=\int \mathscr{D}A^\mu \det(-D_\mu^*D^\mu-m^2+i\epsilon)^{-1}\exp\left\{\frac{i}{2}\int[\lambda^2A^2-(\partial_\mu A_\nu)(\partial^\mu A^\nu)]d^4x\right\}$$
$$\times[\mathscr{S}_A(p_1',p_1)\mathscr{S}_A(p_2',p_2)+\mathscr{S}_A(p_2',p_1)\mathscr{S}_A(p_1',p_2)]\ . \tag{1.8}$$

Extension to inelastic scattering is straightforward. For example, the amplitude for the process $\pi^++\pi^+\to\pi^++\pi^++\pi^++\pi^-$, where the momenta of the outgoing particles are p_1', p_2', p_3', and q, respectively, is (see Appendix A for the derivation)

$$S(p_1',p_2',p_3',q;p_1,p_2)=\int \mathscr{D}A^\mu \det(-D_\mu^*D^\mu-m^2+i\epsilon)^{-1}\exp\left\{\frac{i}{2}\int[\lambda^2A^2-(\partial_\mu A_\nu)(\partial^\mu A^\nu)]d^4x\right\}$$
$$\times[\mathscr{S}_A(p_1',p_1)\mathscr{S}_A(p_2',p_2)T_A(p_3',q)$$
$$+\text{all other permutations of } p_1',\ p_2',\text{ and } p_3']\ . \tag{1.9}$$

In (1.9) T_A is the amplitude for pair creation in an external field:

$$T_A(k_2,k_1)=\lim_{t,t'\to\infty}(4E_1E_2)^{1/2}\int d^3x\,d^3x'e^{ik_1x+ik_2x'}[\Delta_A(x,x')-\Delta(x,x')]\ , \tag{1.10}$$

where $\Delta(x,x')$ is equal to $\Delta_A(x,x')$ with $A_\mu=0$.

So far no approximation has been made, and (1.8) and (1.9) are exact.

II. SMALL-ANGLE SCATTERING

What simplifications occur in the high-energy limit? Let us first study elastic scattering near the forward direction. More specifically, consider the limit

$$p_{1z} = -p_{2z} = \omega \to \infty$$

with

$$\Delta \equiv p'_1 - p_1 = p_2 - p'_2$$

fixed. In this limit $\mathscr{S}_A$ takes the eikonal form[2]

$$\mathscr{S}_A(p'_1,p_1) \cong \int d^2x_\perp dx_- \exp\left[i\left[\frac{\Delta_+ x_-}{2} - \vec{\Delta}_\perp\cdot\vec{x}_\perp - \frac{e}{2}\int_{-\infty}^{\infty} A_-(x_+,x_-,\vec{x}_\perp)dx_+ \right]\right]$$

and (2.1)

$$\mathscr{S}_A(p'_2,p_2) \cong \int d^2x_\perp dx_+ \exp\left[i\left[-\frac{\Delta_- x_+}{2} + \vec{\Delta}_\perp\cdot\vec{x}_\perp - \frac{e}{2}\int_{-\infty}^{\infty} A_+(x_+,x_-,\vec{x}_\perp)dx_- \right]\right] .$$

In (2.1)

$$x_\pm = t \pm z \ , \quad \vec{x}_\perp = x\vec{e}_x + y\vec{e}_y \ .$$

The amplitudes $\mathscr{S}_A(p'_2,p_1)$ and $\mathscr{S}_A(p_2,p'_1)$, being those of large momentum transfer, give little scattering and will be ignored. Substituting (2.1) into (1.8), we get

$$\begin{aligned} S(p'_1,p'_2;p_1,p_2) \cong \int dx_{1-}dx_{2+}d^2x_{1\perp}d^2x_{2\perp} \exp[\tfrac{1}{2}i(\Delta_+x_{1-} - \Delta_-x_{2+})] \exp[-i\vec{\Delta}_\perp\cdot(\vec{x}_{1\perp} - \vec{x}_{2\perp})] \\ \times \int \mathscr{D}A_\mu \det(-D^*_\mu D^\mu - m^2 + i\epsilon)^{-1} \\ \times \exp\left[i\int d^4x[\tfrac{1}{2}\lambda^2A^2 - \tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - eA_-(x)\delta(x_- - x_{1-})\delta^{(2)}(\vec{x}_\perp - \vec{x}_{1\perp}) \right. \\ \left. - eA_+(x)\delta(x_+ - x_{2+})\delta^{(2)}(\vec{x}_\perp - \vec{x}_{2\perp})] \right] . \end{aligned} \quad (2.2)$$

Note that we have interchanged the order of integrating and taking the high-energy limit. Since (2.1) is valid only in the limit $\omega \to \infty$ with A_μ independent of ω, it does not hold if A belongs to a mode with $|k_z|$ comparable to ω. This means that the fragments with nonzero fractions of the longitudinal momentum are not properly treated. The treatment of such fragments belongs to the bound-state problem and will not be covered here.

Next, we utilize translational invariance and reduce (2.2) to

$$S(p'_1,p'_2;p_1,p_2) = (2\pi)^4\delta^{(4)}(p_1 + p_2 - p'_1 - p'_2)\int d^2b\, e^{-i\vec{\Delta}_\perp\cdot\vec{b}} S_0(\vec{b}) \ , \quad (2.3)$$

where $\vec{b} = \vec{x}_{1\perp} - \vec{x}_{2\perp}$ is the impact distance and

$$\begin{aligned} S_0(\vec{b}) = \int \mathscr{D}A^\mu \det(-D^*_\mu D^\mu - m^2 + i\epsilon)^{-1} \\ \times \exp\left\{ i\int d^4x[\tfrac{1}{2}\lambda^2A^2 - \tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - eA_-(x)\delta(x_-)\delta^{(2)}(\vec{x}_\perp - \vec{b}) - eA_+(x)\delta(x_+)\delta^{(2)}(x_\perp)] \right\} . \end{aligned} \quad (2.4)$$

Recovering the integrations over ϕ and ϕ^*, we may show that (2.4) is equivalent to

$$S_0(\vec{b}) = \int \mathscr{D}A^\mu \mathscr{D}\phi\, \mathscr{D}\phi^* \exp\left[i\int d^4x[\mathscr{L} - eA_-(x)\delta(x_-)\delta^{(2)}(\vec{x}_\perp - \vec{b}) - eA_+(x)\delta(x_+)\delta^{(2)}(x_\perp)] \right] , \quad (2.5)$$

where $\mathscr{L}$ is given by (1.2). This can be proved by carrying out the integrations over ϕ and ϕ^* in (2.5). We may further simplify (2.5) by making a change of variables:

$$A_\pm(x) \to A_\pm(x) + V_\pm(x) , \tag{2.6}$$

where

$$V_+(x) = 2e(\Box+\lambda^2)^{-1}\delta(x_-)\delta^{(2)}(x_\perp - b) = e\delta(x_-)K_0(\lambda\,|\vec{x}_\perp - \vec{b}\,|)/\pi$$

and (2.7)

$$V_-(x) = 2e(\Box+\lambda^2)^{-1}\delta(x_+)\delta^{(2)}(x_\perp) = e\delta(x_+)K_0(\lambda\,|\vec{x}_\perp|)/\pi .$$

In (2.7) K_0 is the modified Bessel function. Note that V_+ and V_- are both real. With the substitution (2.6), (2.5) becomes

$$S_0(b) = e^{i\chi} \int \mathscr{D}A^\mu \mathscr{D}\phi \mathscr{D}\phi^* \exp\left[i \int d^4x\, \mathscr{L}_{\text{eff}}\right] , \tag{2.8}$$

where

$$\mathscr{L}_{\text{eff}} \equiv -\tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \tfrac{1}{2}\lambda^2 A^2 + (d_\mu\phi)^*(d^\mu\phi) - m^2\phi^*\phi \tag{2.9a}$$

with

$$d_\mu \equiv \partial_\mu + ie(A_\mu + V_\mu) \tag{2.9b}$$

and

$$\chi = -\tfrac{1}{2}\int V_\mu(\Box+\lambda^2)V^\mu d^4x = -e^2K_0(\lambda b)/2\pi . \tag{2.10}$$

We note that the factor $\exp(i\chi)$ on the right side of (2.8) is the well-known amplitude for multiphoton exchanges between the two high-energy colliding particles,[3] while the functional integral is the vacuum to vacuum amplitude in the presence of the external field V_μ. The scattering amplitude is *factorized* into a product of these two amplitudes. The multiphoton amplitude is given in closed form, while the functional integral remains to be evaluated. In this integral, the effects of the external particles are represented by an external field generated by two classical particles traveling with the velocity of light in the positive-z direction and the negative-z direction, respectively. Actually, the velocity v of the incident particles is close to c but not equal to c. This can be remedied by cutting off the Fourier components of $V_\mu(x)$ with longitudinal momenta larger than ω. Thus, the modes of the boson field and the photon field with $|k_z|$ larger than ω are essentially unaffected by the external field V_μ. Alternatively, we may retain the form (2.7) for V_μ and exclude all harmonic oscillators of $|k_z| > \omega$ from the path integral in (2.8).

The formulas (2.3) and (2.8), valid for elastic scattering, are easily extended to inelastic scattering. For example, for the process $\pi^+ + \pi^+ \to \pi^+ + \pi^+ +$one pair, the S-matrix element is equal to

$$(2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2'-k_1-k_2) \times \int d^2b\, e^{-i\vec{\Delta}_\perp\cdot\vec{b}} S_1(\vec{b}) , \tag{2.11}$$

where k_1 and k_2 are the momenta of the particles created and $S_1(b)$ is equal to $e^{i\chi}$ times the amplitude for vacuum to one pair in the presence of the external field V_μ.

The amplitude $S_0(b)$ given by (2.8) includes the contributions of diagrams independent of V_+ and V_-. They are the vacuum diagrams in the original system and should be eliminated. Also, the diagrams involving V_+ (V_-), but not V_- (V_+), describe the interaction of one of the external particles with itself but not with the other external particle. They contribute to the bound-state effects which we do not take into account. Therefore, we want to eliminate them as well. We note that since in all these diagrams the two external particles are disconnected, their contributions are independent of $\vec{b}$. Furthermore, since no intermediate state in these diagrams are on the mass shell and also since V_μ is real, these diagrams alter only the overall phase angle of the scattering amplitude but not the cross sections. This contribution to the phase angles may be eliminated if we divide the amplitudes by

$$S_0\,(V_+=0)S_0\,(V_-=0)/S_0\,(V_+=V_-=0) .$$

III. PHYSICAL PICTURE AND THE EQUIVALENCE OF PHASE SPACE

The results in the preceding section can be interpreted in a physical way. As a consequence of relativistic time dilation, the time scale of the external particles is ω/m, while the lifetime of a created pair is of the order of E/m where E is the energy of the pair. If the energy E is such that

$$E/\omega \ll 1 , \tag{3.1}$$

then to the external particles the time interval of the creation process is very short.

During such a short time the behavior of the external particles is essentially *classical.* This can be understood from the viewpoint of the path-integral formulation: the classical path is the collection of stationary points of the path integral. In a very small time interval contributions to the path integral come mostly from the neighborhood of the stationary point. Therefore, during the time span of the existence of pairs with energies satisfying (3.1), the two incident particles behave like two classical particles traveling in opposite directions with the velocity of light and can be replaced by the classical electromagnetic potential generated by such classical currents. This potential is given by (2.7), with its Fourier components with $|k_z| > \omega$ cut off. Alternatively, we may consider the physical system to be approximated by a collection of harmonic oscillators with $|k_z| < \omega$ coupled to the classical electromagnetic potential (2.7).

The classical approximation for the external particles fails if the created particles have energies comparable to ω. The treatment of these particles, called fragments, belongs to the bound-state problem and is beyond the scope of this paper.

The picture described above defines the simplification as well as the limitation of the high-energy approximation. To obtain the high-energy elastic scattering amplitude, we have to obtain the vacuum to vacuum amplitude for a system of coupled harmonic oscillators in an external field. In the latter problem, we can no longer use approximations valid for short-time collisions. This is because we must take care of the behavior of the pairs throughout the time span of their existence. Therefore, we are facing a fully quantum-mechanical problem. The large parameter ω still remains, but its only role is restricting the size of the relevant phase space. As the energy increases, more and more harmonic oscillators enter into the picture. The asymptotic behavior of the high-energy amplitude is, therefore, related to the ultraviolet divergence for the system of harmonic oscillators coupled to the external field V_μ as the cutoff goes to infinity. This appears to be a formidable problem. Since the Fourier component of V_μ is approximately independent of k_z for $|k_z| \ll \omega$, the scattering amplitude is approximately independent of k_z for $|k_z| \ll \omega$. In other words, two points in the restricted phase space with the same transverse momenta are completely equivalent. We shall call this the *equivalence of the phase space.*

We give two examples of the consequences of this equivalence.

(i) Consider the creation of a photon of momentum $\vec{k}$. The scattering amplitude is independent of k_z; thus, the differential cross section $\sigma(\vec{k})$ times the kinematic factor E is independent of k_z:

$$E\sigma(\vec{k}) = f(\vec{k}_\perp) .$$

Thus, we have

$$\sigma(\vec{k}) d^3k = f(\vec{k}_\perp)\frac{d^3k}{E} . \tag{3.2}$$

The factor d^3k/E is the relativistically invariant phase space, and we may write

$$\frac{d^3k}{E} = d\tau d^2k_\perp , \tag{3.3}$$

where $\tau = \frac{1}{2}\ln(k_+/k_-)$ is the rapidity. This shows that the statement of equivalence is more appropriately applied in the rapidity space.

Equations similar to (3.2) hold for processes in which more than one particle are created as well as for inclusive processes.

(ii) We have made the approximation of cutting off the phase space at $|k_z| > \omega$. One may estimate the error involved with such a cutoff. The rapidity space is of width $\ln\omega$. The phase space from $k_z = \omega$ to $k_z = \omega/e$, say, is of width unity in the rapidity space. In other words, the width of the region where our approximation fails is of the order of $(\ln\omega)^{-1}$ smaller than the width of the region where our approximation holds. Since all points in the phase space contribute equally, we believe that our approximation is a good one. Furthermore, since the dynamics is generated by an effective Lagrangian which is Hermitian, the asymptotic amplitudes satisfy unitarity—a difficult condition to observe for high-energy approximations.

In summary, we have made use of the short-time nature of the collision to make classical approximations for the external particle. The vacuum to vacuum amplitude in the presence of external fields corresponds to the elastic scattering amplitude in the original system, while the vacuum to n-pair amplitude in the presence of external fields corresponds to the amplitude for n-pair creation in the original system. We believe that additional approximations made in the literature, such as summing leading terms or utilizing eikonal forms or Regge behaviors, are models, but not field theory.

IV. FIXED-ANGLE SCATTERING

In this section, we study the two-body elastic scattering amplitude in the limit $\omega \to \infty$ with the scattering angle fixed, i.e., s and t both large, with the ratio s/t fixed. We begin with (1.4) and (1.6), which are exact. Now $\Delta_A(x',x)$ can be expressed by the path integral

$$\Delta_A(x',x) = -\frac{i}{2}\int_0^\infty d\tau \int_{\substack{x(0)=x\\ x(\tau)=x'}} \mathscr{D}x^\mu \exp\left\{\frac{-i}{2}\int_0^\tau du\,[\dot{x}^2 + m^2 + 2e\dot{x}\cdot A(x)]\right\} . \tag{4.1}$$

In (4.1) x is a shorthand notation for $x(u)$, the path of the boson as a function of u and $\dot{x} \equiv dx/du$. Then, (1.6) can be written as

$$S(p_1',p_2';p_1,p_2) = \lim_{\substack{t_1,t_2\to-\infty \\ t_1',t_2'\to\infty}} (E_1E_2E_1'E_2)^{1/2} \times \int \prod_{n=1}^{2} d^3x_n d^3x_n' e^{-ip_n\cdot x_n + ip_n'\cdot x_n'} \times \int_0^\infty d\tau_n \int_{\substack{x_n(0)=x_n \\ x_n(\tau_n)=x_n'}} \mathscr{D}x_n^\mu \exp\left[-\frac{i}{2}\int_0^{\tau_n} du\,(\dot{x}_n^{\,2}+m^2)\right] \times \mathscr{D}A^\mu \det(-D_\mu^* D^\mu - m^2 + i\epsilon)^{-1} \times \exp\left\{\frac{i}{2}\int [\lambda^2 A^2 - (\partial_\mu A_\nu)(\partial^\mu A^\nu) - 2A\cdot(J_1+J_2)]d^4x\right\} + \text{preceding term with } p_1 \leftrightarrow p_2 . \tag{4.2}$$

In (4.2),

$$J_n^\mu(x) = e\int_0^{\tau_n} du\, \delta^{(4)}(x - x_n(u))\dot{x}_n^\mu , \quad n=1,2 \tag{4.3}$$

is the current generated by a particle traversing the path $x_n(u)$. Similar to (2.6), we make the change of variables

$$A^\mu \to A^\mu + \mathscr{A}^\mu , \tag{4.4}$$

where

$$\mathscr{A}^\mu = (\Box + \lambda^2)^{-1} J^\mu \tag{4.5}$$

with

$$J^\mu = J_1^\mu + J_2^\mu ;$$

then (4.2) becomes

$$S(p_1',p_2';p_1,p_2) = \lim_{\substack{t_1,t_2\to-\infty \\ t_1',t_2'\to\infty}} (E_1E_2E_1'E_2')^{1/2} \times \int \prod_{n=1}^{2} d^3x_n d^3x_n' e^{-ip_n\cdot x_n + ip_n'\cdot x_n'} \times \prod_{n=1}^{2} \int_0^\infty d\tau_n \int_{\substack{x_n(0)=x_n \\ x_n(\tau_n)=x_n'}} \mathscr{D}x_n^\mu \exp\left[-\frac{i}{2}\int_0^{\tau_n} du\,(\dot{x}_n^{\,2}+m^2)\right] \times \exp\left[-\frac{i}{2}\int d^4x\, J^\mu \frac{1}{\Box+\lambda^2} J_\mu\right] S_0 + \text{preceding term with } p_1 \leftrightarrow p_2 , \tag{4.6}$$

where

$$S_0 = \int \mathscr{D}A^\mu \mathscr{D}\phi\, \mathscr{D}\phi^* \exp\left[i\int \mathscr{L}'_{\text{eff}} d^4x\right] \tag{4.7}$$

with

$$\mathscr{L}'_{\text{eff}} = -\tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \tfrac{1}{2}\lambda^2 A^2 + [\partial_\mu\phi^* - ie(A_\mu + \mathscr{A}_\mu)\phi^*][\partial^\mu\phi + ie(A^\mu + \mathscr{A}^\mu)\phi] - m^2\phi^*\phi .$$

In (4.7), we have recovered the functional integrations over ϕ and ϕ^*.

Equation (4.6) is still exact. We shall now make high-energy approximations. Since the collision lasts a very short period in the time scale of the two external particles, we shall treat these particles semiclassically. Therefore, we shall make stationary phase expansions around the classical paths of the two external particles.

The classical path for high-energy fixed-angle scattering is very different from that for high-energy small-angle scattering. In classical mechanics, the path of a high-energy particle with a given impact distance $\vec{b}\neq 0$ is essentially a straight line. This is because if the particle is of very high kinetic energy, the direction of its momentum is not altered appreciably by its interaction with the other particle. Such a straight path of $\vec{b}\neq 0$ has a very small scattering angle. On the other hand, if the impact distance $\vec{b}\approx 0$, the interaction during the close encounter may alter the direction of the momentum by a fixed angle. More specifically, for a particle in a Coulomb potential, the impact distance of the classical path of fixed-angle scattering is of the order of Δ^{-1}. For each scattering angle θ, there corresponds but one classical path associated with this small value of b. Roughly, this path resembles two straight lines joined at the origin at an angle θ. The high-energy fixed-angle scattering amplitude is given by the WKB expansion around this path.[2]

In a stationary-phase calculation, factors of the integrand not sensitive to the variation of the path can be taken out of the integral. For example, in the limit $\delta\to 0$,

$$\int \exp[if(x)/\delta]g(x)dx \cong g(x_0)\int \exp[if(x)/\delta]dx \;,$$

where x_0 is the stationary point defined by

$$f'(x_0)=0 \;.$$

The quantity S_0 in (4.7), describing the interaction of the fields with the potential $\mathscr{A}^\mu$, is such a factor. This is because a small variation of the path only leads to a small variation of $\mathscr{A}^\mu$ and hence of S_0. Therefore, we may, in the evaluation of S_0, replace x_μ by the classical potential generated by the paths

$$\begin{aligned}\vec{x}_1&=t\vec{e}_3 \;, \quad t<0 \;,\\ &=t\vec{n} \;, \quad t>0 \;,\end{aligned} \tag{4.9}$$

with

$$\vec{n}=\cos\theta\vec{e}_3+\sin\theta\vec{e}_1$$

for the first external particle and

$$\vec{x}_2=-\vec{x}_1 \tag{4.10}$$

for the second external particle.

The electromagnetic potential generated by the path (4.9) is determined to be

$$V_0=U+W \;, \tag{4.11}$$

and

$$\vec{V}=U\vec{e}_z+W\vec{n} \;, \tag{4.12}$$

where

$$U(x)=ie\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{(k^2-\lambda^2+i\epsilon)(k_0-k_3-i\epsilon)} \tag{4.13}$$

and

$$W(x)=ie\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{(k^2-\lambda^2+i\epsilon)(-k_0+\vec{n}\cdot\vec{k}-i\epsilon)} \;. \tag{4.14}$$

Similarly, the classical electromagnetic potential generated by the classical particle which traverses the path $\vec{x}_2$ is

$$V_0'=U'+W' \;, \tag{4.15}$$

$$\vec{V}'=-U'\vec{e}_z-W'\vec{n}\,' \;, \tag{4.16}$$

where

$$\begin{aligned}U'(\vec{x},t)&=U(-\vec{x},t) \;,\\ W'(\vec{x},t)&=W(-\vec{x},t) \;.\end{aligned} \tag{4.17}$$

We mention that V_μ and V_μ' are complex.

As we have discussed, we may replace $\mathscr{A}_\mu$ by $V_\mu+V_\mu'$ in (4.7). Then, S_0 becomes independent of the variables $x_n(u)$ and τ_n, and (4.6) is simplified into

$$S(p_1',p_2';p_1,p_2)\simeq\bar{S}(p_1',p_2';p_1,p_2)\bar{S}_0+\text{preceding term with } p_1\leftrightarrow p_2 \;, \tag{4.18}$$

$$\bar{S}(p_1',p_2';p_1,p_2)=\prod_{n=1}^{2}\lim_{\substack{t_n\to-\infty\\ t_n'\to\infty}}(E_nE_n')^{1/2}\int d^3x_n d^3x_n' e^{-ip_nx_n+ip_n'x_n'} \times\int_0^\infty d\tau_n\int_{\substack{x_n(0)=x_n\\ x_n(\tau_n)=x_n'}}\mathscr{D}x_n^\mu \exp\left[-\frac{i}{2}\int_0^{\tau_n}du\,(\dot{x}_n{}^2+m^2)\right] \times\exp\left[-\frac{i}{2}\int d^4x(J_1+J_2)^\mu\mathscr{A}_\mu\right] \quad (4.19)$$

and

$$\bar{S}_0=\int\mathscr{D}A^\mu\mathscr{D}\phi\,\mathscr{D}\phi^*\exp\left[i\int\mathscr{L}'d^4x\right], \quad (4.20)$$

with

$$\mathscr{L}'=-\tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)+\tfrac{1}{2}\lambda^2A^2 +[\partial_\mu\phi^*-ie(A+V+V')_\mu\phi^*][\partial^\mu\phi+ie(A+V+V')^\mu\phi]-m^2\phi^*\phi\,. \quad (4.21)$$

We note from (4.19) that $\bar{S}(p_1',p_2';p_1,p_2)$ is the elastic scattering amplitude without charged-particle loops. The contributions of the charged loops are contained in $\bar{S}_0$, the vacuum to vacuum amplitude in the presence of the external fields V_μ and V_μ'. We also note that just as in the case of small-angle scattering, the fixed-angle scattering amplitude is factorized into a product of those two amplitudes.

To calculate $\bar{S}(p_1',p_2';p_1,p_2)$, let us call

$$\chi\equiv-\tfrac{1}{2}\int d^4x(J_1+J_2)_\mu\mathscr{A}^\mu =\chi_1+\chi_2+\chi_3\,, \quad (4.22)$$

where

$$\chi_1\equiv-\tfrac{1}{2}\int d^4x\,J_1^\mu(\Box+\lambda^2)^{-1}J_{1\mu}\,, \quad (4.23)$$

$$\chi_2\equiv-\tfrac{1}{2}\int d^4x\,J_2^\mu(\Box+\lambda^2)^{-1}J_{2\mu}\,, \quad (4.24)$$

and

$$\chi_3\equiv-\int d^4x\,J_1^\mu(\Box+\lambda^2)^{-1}J_{2\mu}\,. \quad (4.25)$$

The quantity χ_1 (χ_2) describes the effects of the emission of photons for $t<0$ and the reabsorption of these photons for $t>0$ by the first (second) electron, while χ_3 describes the exchange of photons between the two external electrons.

We shall make the approximation of replacing J_1 and J_2 in (4.23)–(4.25) by their classical currents. Such an approximation for $\bar{S}$ is not entirely justified, and we shall discuss afterwards the corrections for the approximation. We get

$$\chi_1\simeq-e^2\int\frac{d^4k}{(2\pi)^4}\frac{p_1\cdot p_1'}{(k^2-\lambda^2+i\epsilon)(p_1\cdot k-i\epsilon)(p_1'\cdot k-i\epsilon)}\,. \quad (4.26)$$

The integral in (4.26) is divergent at infinity. However, since the energy of the incident particle is ω, which is large but not infinite, we may introduce an ultraviolet cutoff $|\vec{k}|<\omega$ in (4.26). Alternatively, we may modify (4.26) into

$$\chi_1\simeq-e^2\int\frac{d^4k}{(2\pi)^4}\frac{4p_1\cdot p_1'}{(k^2-\lambda^2+i\epsilon)(2p_1\cdot k-k^2-i\epsilon)(2p_1'\cdot k-k^2-i\epsilon)}\,, \quad (4.27)$$

which is a convergent integral. Similarly,

$$\chi_2\simeq-e^2\int\frac{d^4k}{(2\pi)^4}\frac{4p_2\cdot p_2'}{(k^2-\lambda^2+i\epsilon)(2p_2\cdot k-k^2-i\epsilon)(2p_2'\cdot k-k^2-i\epsilon)}\,. \quad (4.28)$$

Since U and W are complex, it is found that χ_1 and χ_2 are dominantly imaginary. Thus, $e^{i(\chi_1+\chi_2)}$ is small in magnitude.

Similarly, substituting (4.11), (4.12), (4.15), and (4.16) into (4.25), we get, after some algebra,

$$\chi_3=4e^2\int\frac{d^4k}{(2\pi)^4}\left[\frac{p_1}{2p_1\cdot k+k^2+i\epsilon}-\frac{p_1'}{2p_1'\cdot k-k^2-i\epsilon}\right]\left[\frac{p_2}{2p_2\cdot k-k^2-i\epsilon}-\frac{p_2'}{2p_2'\cdot k+k^2+i\epsilon}\right], \tag{4.29}$$

where, as before, we have modified each of the denominator factors by adding k^2 to $i\epsilon$.

How good is the semiclassical approximation we have made? Basically, there are, for *e-e* scattering at high energy and fixed angle, *three,* not two, time scales: ω, λ^{-1}, and ω^{-1}. The scale ω is the Lorentz dilated time scale for a particle of energy ω. The formation of the wave functions of the external particles is of this time scale. The dynamics involved is essentially quantum mechanical, and we have not taken care of it in the present approach. The scale ω^{-1} is the time scale for hard scattering—scattering which leads to momentum transfer of the order of ω. The semiclassical method also fails during this time period, although the scale is extremely short. This is because the hard scattering always occurs near the turning points, where the WKB method breaks down. (For an example, see the treatment of potential scattering in Ref. 2.) Relatively unknown is the scale λ^{-1}, which is the range of interaction between the two external particles. During the time $\omega^{-1} \ll t \ll \lambda^{-1}$, the interaction between two charged particles gives the scattering amplitude a factor strongly dependent on ω/λ. A familiar example of this is Coulomb scattering (for which $\lambda=0$), where there is an infinite phase shift. Another familiar example is the Sudakov form factor, which is contributed by the interaction within this time scale. This time scale, although long compared to that of hard scattering, is short compared to that of the external particles. Thus, its effect can be calculated by the semiclassical approach.

The high-energy and fixed-angle electron-electron scattering amplitude has been calculated by a diagrammatic approach.[4] A comparison of the results shows that $\exp(i\chi_3)$ gives precisely the energy dependence of the multiphoton exchange amplitude, while $\exp[i(\chi_1+\chi_2)]$ gives the product of the Sudakov form factors[5]

$$\exp(i\chi_1)=\exp(i\chi_2)\cong\exp\left[-\frac{e^2}{8\pi^2}\ln^2 s\right].$$

Indeed, the expression $\exp[i(\chi_1+\chi_2+\chi_3)]$ contains all of the exactly calculable factors of these amplitudes. [To take care of charge renormalization, we need to replace e by the running coupling constant $e(k^2)$.] The other factors obtained in the diagrammatic approach are (i) the wave-function renormalization constants, (ii) a function of θ and the running coupling constant. The first factor represents the contribution of fragments not taken into account in the present treatment. The latter function is equal to $\bar{S}_0$ defined in (4.20) and the rest of the factors of the path integral (4.19) not included in the semiclassical approximation, representing the amplitude of hard scattering. Thus function cannot be calculated in closed form. Fortunately, quite unlike the case of small-angle scattering, $\bar{S}_0$ (as well as the hard scattering amplitude) does not have infinities as the length of the rapidity space goes to infinity, since the equivalence of the phase space no longer holds. Thus, it is a function of θ and the running coupling constant but not of ω. The strong energy dependence of the scattering amplitude is therefore given by $\exp[i(\chi_1+\chi_2+\chi_3)]$ and the wave functions.

ACKNOWLEDGMENT

X.Z. was supported in part by the National Science Foundation under Grant No. PHY-81-07394.

---

*Present address: Physics Department, University of Toronto, Toronto, Canada.

[1]For discussion of path integrals, see R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965). More modern treatments can be found in E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973); L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981); T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood, Chur, Switzerland, 1981), and the references cited therein.

[2]H. Cheng, D. Coon, and X. Zhu, Phys. Rev. D 26, 896 (1982).

[3]H. Cheng and T. T. Wu, Phys. Rev. 186, 1611 (1969); S. Chang and S. Ma, Phys. Rev. Lett. 22, 1334 (1969).

[4]H. Cheng, Z. Radulovic, E. C. Tsai, and X. Zhu, Phys. Rev. D 27, 421 (1983).

[5]V. Sudakov, Zh. Eksp. Teor. Fiz. 30, 87 (1956) [Sov. Phys.—JETP 3, 65 (1956)].